Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Computation of Berry phase for spin s in the state m in magnetic field

Bachelor thesis

Bratislava, 2015

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Samuel Beznák

Study program	:	Physics
Brunch of study	:	1160 Physics
Supervisor	:	doc. RNDr. Marián Fecko, PhD.
Department	:	Theoretical Physics and Didactics od Physics

70179568

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Samuel Beznák
Študijný program: fyzika (Jednoodborové štúdium, bakalársky I. st., denná forma)
Študijný odbor: 4.1.1. fyzika
Typ záverečnej práce: bakalárska
Jazyk záverečnej práce: slovenský
Sekundárny jazyk: anglický

Názov: Výpočet Berryho fázy pre spin s v stave m v magnetickom poli

Cieľ: V učebniciach sa bežne počíta Berryho fáza pre spin ½ v magnetickom poli, ktoré koná veľmi pomalý rotačný pohyb okolo fixnej osi. Toto sa dá skomplikovať (a potom aj zrátať) na spin s v stave m v takom istom magnetickom poli (pôvodná úloha má s=m=½).

Literatúra: R.Shankar: Principles of Quantum Mechanics, 1994, Plenum Press
M.Fecko: Differential Geometry and Lie Groups for Physicists, Cambridge University Press, 2006

Anotácia: Majme kvantovo-mechanickú sústavu, ktorej hamiltonián obsahuje nejaké parametre, napríklad vonkajšie magnetické pole. Ak sa tieto parametre nemenia, riešime bezčasovú Schrödingerovu rovnicu, t.j. nájdeme vlastné vektory príslušného hamiltoniánu atď. Ak sa tieto parametre menia, ale veľmi pomaly (adiabaticky), už to, prísne vzaté, nie je bezčasový prípad. Pomalosť zmeny parametrov však umožňuje usudzovať takto: keby sme vedeli riešenia (vlastné stavy a ich energie) bezčasovej Schrödingerovej rovnice postupne pre všetky hodnoty parametrov, cez ktoré pri svojom (veľmi pomalom) vývoji prechádzajú, časový vývoj sústavy by zrejme spočíval v tom, že by sa postupne darwinovsky prispôsobovala novým hodnotám parametrov. Špeciálne ak by začala v n-tom stacionárnom stave pôvodného hamiltoniánu, zostávala by stále v n-tom stacionárnom stave, ale vždy momentálneho hamiltoniánu. Týmto sa zaoberá adiabatická veta v kvantovej mechanike a v podstate tvrdí, že to tak (s istou presnosťou) je.
Vektor prislúchajúci stavu je však v kvantovej mechanike daný len s presnosťou na fázový faktor. A výber fázového faktora sa dá (v uvažovanom n-tom stave) urobiť pre každú hodnotu parametrov zvlášť. Tak urobíme. Zaujímavá situácia teraz nastane, keď je vývoj parametrov cyklický, t.j. keď sa po istom (veľmi dlhom) čase ich hodnota vráti k hodnote, ktorú mali na začiatku. Výber fázy v tomto koncovom bode už totiž nie je ľubovoľný, lebo (pre tých, čo nedávali pozor pripomíname, že) sa už urobil v čase, keď sa tento koncový bod ešte volal začiatočný. To umožňuje porovnať fázu n-tého stavu na začiatku a na konci a vysloviť výrok, k akej fázovej zmene došlo počas uvažovaného cyklu.
Toto sa vedelo dávno a ľudia sa jej výpočte obmedzovali na časť, dnes zvanú dynamická fáza.
Až pokým si Michael Berry v roku 1984 neuvedomil, že ten výpočet bol doteraz vždy tak trochu odfláknutý, zameditoval, sústredil sa viac a zistil, že tam pristupuje navyše ešte tzv. geometrická fáza, ktorá sa, ako naznačuje názov

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(tejto bakalárskej, nie jeho) práce, odvtedy volá aj Berryho fáza. A venovali sa jej ťažké stovky článkov. A vstúpila aj do bežných učebníc kvantovej mechaniky. A z Michaela Berryho sa stal Sir Michael Berry.

Kľúčové slová: Berryho fáza, forma konexie, forma krivosti

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Dátum zadania: 13.10.2014

Dátum schválenia: 13.10.2014 prof. RNDr. Jozef Masarik, DrSc.
garant študijného programu

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študent

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vedúci práce

Acknowledgement

I would like to show my gratitude to the supervisor Marián Fecko for sharing his pearls of wisdom, that helped me understand problems and their solutions and for giving valuable pieces of advice on writing this thesis. I am also thankful to my family for constant encouragement.

Declaration

I declare, that I wrote this thesis on my own under the guidance of my supervisor and using the literature stated in references.

Abstract

Author	:	Samuel Beznák
Title	:	Computation of Berry phase for spin s in the state m in magnetic field
Type of thesis	:	bachelor thesis
University	:	Comenius University in Bratislava
Faculty	:	Faculty of Mathematics, Physics and Informatics
Department	:	Department of Theoretical Physics and Didactics of Physics
Supervisor	:	doc. RNDr. Marián Fecko, PhD.
Bratislava	:	FMFI UK, 2015

46 pages

Crucial object of this thesis is the Berry's phase. If we consider quantum Hamiltonian with external parameters (such as magnetic field), that varies in time, we can solve the time-dependent Schrödinger equation for arbitrary n-th stationary state of the Hamiltonian. In this case, one does not garuantee, that the final state will be the n-th stationary state too. However, if we assume, that the change of parameters will be done very slowly (adiabatically), thanks to the adiabatic theorem we stay forever in the n-th stationary state, if we started in it. This state (as every state in quantum mechanics) is given up to a phase factor. If we choose this factor and then change external parameters cyclically (e.g. their values return after a long time to the initial values), the state gets an extra phase consisting of the well-known dynamical phase and a new geometric phase, that is the Berry phase. In our thesis, we compute the Berry phase for spin s in state m in magnetic field in various ways (following the Berry's original paper from 1984, using differential geometry for particular cases $s = 0, 1/2, 1$, rotating eigenstates of general spin operator $\hat{s}$, coupling spins and using canonical 1-form on $SU(2)$).

Keywords: geometric phase, connection 1-form, curvature 2-form, spin, rotation, canonical 1-form

Abstrakt

Autor	:	Samuel Beznák
Názov	:	Výpočet Berryho fázy pre spin s v stave m v magnetickom poli
Typ záverečnej práce	:	bakalárska práca
Univerzita	:	Univerzita Komenského v Bratislave
Fakulta	:	Fakulta matematiky, fyziky a informatiky
Katedra	:	Katedra teoretickej fyziky a didaktiky fyziky
Školiteľ	:	doc. RNDr. Marián Fecko, PhD.
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46 strán		

Kľúčovým objektom predkladanej práce je Beryho fáza. Uvažujme Hamiltonián s vonkajšími parametrami (ako je napr. magnetické pole), ktoré sa menia v čase. Potom vieme vyriešiť časovú Schrödingerovu rovnicu pre n-tý stacionárny stav Hamiltoniánu. Nemáme ale zaručené, že výsledný stav bude taktiež n-tým stacionárnym stavom. Ak však budeme predpokladať, že parametre sa budú meniť veľmi pomaly (adiabaticky), vďaka platnosti adiabatickej teorémy ostaneme navždy v n-tom stacionárnom stave, ak sme v ňom začali. Tento stav (tak, ako každý kvantovo-mechanický stav) je daný až na fázu. Vyberme teda na začiatku nejakú fázu. Ak teraz zmeníme vonkajšie parametre cyklicky (t.j. po veľmi dlhom čase sa ich hodnoty vrátia k východiskovým hodnotám), tak samotný stav získa fázový faktor pozostávajúci zo známej dynamickej fázy a z novej geometrickej fázy, ktorá je Berryho fázou. V našej práci vypočítame Berryho fázu pre spin s v stave m v magnetickom poli rôznymi spôsobmi (postupujúc podľa Berryho originálneho článku z roku 1984, využívajúc diferenciálnu geometriu v konkrétnych prípadoch spinu $s = 0,\ 1/2,\ 1$, rotovaním vlastných stavov všeobecného operátora spinu $\hat{s}$, skladaním spinov a nakoniec za pomoci kanonickej 1-formy na $SU(2)$).

Kľúčové slová: geometrická fáza, 1-forma konexie, 2-forma krivosti, spin, rotácia, kanonická 1-forma

Preface

If there would be someone, who would like to count, how many problems in physics there are, he could probably not manage to do this in his entire life. However, there are classes of problems, that have the same mathematical concept and physical basis. That means, if we understand this concept in one situation, we can solve it there and then the other situations from this class are allready solved for free (that could spare his time).[1]

With no doubt, holonomy as a basic property of space is such a class, involving plenty of beautiful and fascinating situations, where the holonomy comes to the game (especially when the space is curved). One of them is the Berry phase being a relatively new object (discovered in 1984 by Sir Michael Berry) in quantum theory.

The topic about the Berry phase is twice as interesting, because we will play with exponential factors- the Berry phase itself is in exponent of such a factor (strong connection to gauge transformations - another class of important physical situations). We will palpate rotations in space of quantum states (mediated by angular momentum operators - yet another wide class) ... We could continue in this spirit and cause, that this preface woul be infinite, because one can barely imagine, how many insights for one topic there are.

That is why, in my opinion, this topic *is* worth studying- it offers us to associate (seemingly distant) theoretical fields. It brings us the possibility to use one theoretical concept to prove a statement written in the language of another concept. And I personally think, that this is beauty of physics - everything is related to everything. Let us therefore learn something about this challenging, but yet fascinating topic!

[1] Let us give an example from secondary school:
Standard oscillation of a spring (mass m and spring constant k) can be characterised by its undamped resonant frequency $f_{spring} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. This formula comes from the fact, that when we displace the spring (of mass m) from its equilibrium position by x, the spring shows us its will to come back to its equilibrium position by the restoring force proportional to this displacement against its direction, $\vec{F} = -k\vec{x}$. Using this result, we can now solve *any* problem, where the restoring force of some action (after the object is being displaced) is proportional to this displacement with a minus sign. The only thing we have to do, is to find the corresponding "mass" and "spring constant" of this system, even though we deal with problems without springs - for example series RLC circuit, where the "mass" is its inductance L and the "spring constant" is the inverse capacitance $1/C$, so $f_{RLC} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$.

Contents

Introduction

The aim of this thesis is to present the Berry's phase in its beauty seen from many points of view. We will strictly deal with theoretical concepts of this phenomenon. You can object to this decision, but the reason is simple: even the situation of spin s in magnetic field (being only a particular choice of the external parameter of system) can be seen from such a bunch of approaches, that it is wise to dedicate the whole thesis to this single case.

In order to capture the context of giving birth to *the* phase, we will start with a chapter about holonomy, because the Berrys' one is only a "special case". However, aim of this part is not to give the reader an exhaustive explanation of holonomy with formulae from differential geometry[2]. We only want to give motivation and to activate reflexes, that should our reader has, when we say something, that is the reader familiar with.

Next chapter is dedicated to the original approach to the Berry's phase, that Sir Michael Berry underwent to obtain the result[3]. We will follow his way to get formula for geometric phase for an adiabatic procces, when an external parameter of the examined system closes a loop in a parametric space (it may seem a bit incomprehensible now, but everything will explained later). In addition, we will see, how Berry computed his phase for a spin s in magnetic field $\vec{B}$.

At the same time, we will think of other possible formulations of this problem - we will try to implement differential geometry. To this purpose, a brief summarisation of basic concepts of differential geometry is needed. Consequently, when reading second chapter again, we will rewrite formulae in the differential-geometric language. This all will be done in chapter 3.

Fourth chapter will present computations of the phase for particular (small) values of spin s in magnetic field. We will start with spin $s = 1/2$ and then continue to spins $s = 0, 1$. The reason is, that we can (thanks to addition of angular momenta) check results for $s = 0, 1$ using the result for $s = 1/2$. On this occation, we will sketch the procedure of spin coupling as well.

To move on, the aim of the next chapter is to prove the result for the Berry's phase using methods introduced when dealing with particular situations. First part will describe rotations of spin- eigenstates and consecutive computations and manipulations with these states. Second part is based on spin coupling itself. However, this concept is a bit too serious- that is why we will only sketch the idea.

The last chapter is dedicated to yet another approach. We will implement group theory to our problem and then use the canonical 1-form as a crucial object to prove the Berry's result for the last time.

In the whole thesis, we abandon explicit citation of every single equation, that can be found in the literature. We will, however, give the reader information about this literature, that we follow computations from. In many situation, masses of articles

[2]You can find this kind of information in chapter 15 in [8].

[3]According to the spirit of this concept, where the original approach is treated as the first and then another approaches (not mentioned in the original paper from 1984) as the second, we named chapters in a bit provocative way. We do *not* claim, that Sir Berry was not aware of the physics described in them - we would only like to lighten the whole thesis, that is sometimes a bit demanding.

describe the same object in a similar way (for example Wigner matrices, that will be used in 5.1). In these cases, we will give the appropriate piece of information only in a passive way - not studying whether the sources vary from each other. We will, of course, bibliograph there sources in References. This decision stems from the hope, that it can make the whole text more fluent and coherent.

This thesis should be understandable to students of the third classes of bachelor study in physics. We take therefore basic knowledge in quantum mechanics for granted[4]. However, from the third chapter on, differential geometry is used. Even though we explain its fundamental concepts, without proper understanding it would be difficult to follow our steps. Therefore a basic knowledge in differential geometry is appreciated, or better necessary. Moreover, in the last chapter we use differential geometry on Lie groups. That is why it would be desirable to have knowledge in this theoretical field too.

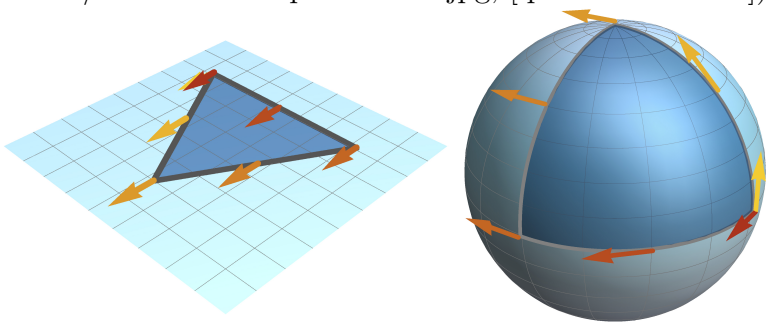
At this point, nothing constrains us from undertaking an adventurous journey into the world of theoretical physics. Let us go!

[4]Lectures on quantum mechanics can be found in [16] and [17], where in [17], R. Shankar provides us with a lot of practical applications of explained theory (Berry's phase as an example!).

1 Menu: testing sample of the geometric phase

Before we start serious and sometimes nasty computations about the Berry's phase, we would like to say something about holonomy, that is in great connection with the phase.[5] Just imagine you are a member of a team of researchers, whose aim is to explore the whole Earth. It is natural, that your team wants to be recognizable for everyone. To gain some inspiration on how to do it, you read a lot of books with historical content and consequently you suggest to carry a flag. As you came with this idea, you are assigned to be the colour-bearer. The journey can finally start.

Figure 1: Exploring the Earth (sketch of journey holding a flag; North pole is to the right) (http://www.quantum-munich.de/fileadmin/media/media/Aharonov-Bohm/Parallel-Transport-HiRes.jpg, [quoted 13.5.2015])



Your team starts at the North pole (see Figure 1, to the right) and asks you (for the sake of perfection) to hold the flag to the south. You want to explore changes in vegetation, settlement etc. in dependence on geographical latitude. That is why your team walks to the south- you walk down the meridian. Walking in every moment *forth, directly and straight* you come to the equator (flag faces still south). The next goal is to investigate mentioned properties in dependence on geographical longitude. It is clear, that you traverse along the equator. In order to be a perfect colour-bearer, you *do not* change the direction (!) of the flag. If you are done with this part of investigation (flag still to the south), you want to come back to the North pole, where your base camp is. As you come there, something is weird- as you hit for the journey, you held the flag in different direction! Your team accuses you of incorrect flag holding, but you persuade them, you were the whole time rigorous. How is it then possible?

The answer lies in geometrical properties of the space you walked on. Namely, one aspect is importat- the curvature of the space. To sum it up, if you walk on a surface, that has non-zero curvature, vectors (sympolized by the flag pointing everytime to the south) can change their direction by walking down an enclosed circuit, even though you held the flag directly (transferred the vector in a parallel way). In contrast, surfaces with zero curvature (Figure 1, to the left) cause no change of vector's direction. The described phenomenon is called *holonomy*. It depends on the curvature of the surface, of the way you transport the vectors (is encoded in *connetion*) and, of course, on the shape of the circuit enclosed on this surface.

Even though this concept is perfectly imaginable and after introduction of some mathematical topics easily computable too, its consequences in physics are far beyond any attempt of imagination. For example, if we replace the flag by a spin and the walk round the Earth by an enclosed cirtuit in magnetic field, we get the Berry's phase for a spin in magnetic field. That would be the aim of our thesis, so let us do it properly!

[5] In order to quicken your imagination, because the concept of the Berry's phase is a bit abstract.

2 How Sir Berry approached his phase?

In this paragraph, we will reconstruct the way Berry did to his famous phase[6]. We will use standard bra-ket notation to derive a general formula for an extra phase factor in solution of the Schrödinger equation for a system adiabatically changed by an external parameter (that is the Berry's phase). Later on, we will compute this phase for a system with spin in external magnetic field.

2.1 Derivation of the formula

Let us now start with the Hamiltonian $\hat{H}(\vec{R}(t))$. Here $\vec{R}(t)$ denotes set of external parameters of Hamiltonian, that cause Hamiltonian varying with time (magnetic field $\vec{B}$ can be taken as an example of such a parameter). Note, that if system with such a Hamiltonian undergoes an excursion between $t = 0$ and $t = T$, it can be depicted as a curve $\vec{R} = \vec{R}(t)$ in parametric space. Moreover, if $\vec{R}(T) = \vec{R}(0)$, then the curve is closed (from now on, let us name this circuit C).

A state $|\psi(t)\rangle$ of the system evolves, of course, in agreement with the Schrödinger equation as follows

$$\hat{H}(\vec{R}(t))|\psi(t)\rangle = i\,\hbar\,\partial_t\,|\psi(t)\rangle \tag{2.1}$$

Taking linearity of the Hamiltonian into account, one can express this solution as a linear combination of eigenstates of the Hamiltonian operator

$$|\psi(t)\rangle = \sum_n a_n(t)\,|n(\vec{R}(t))\rangle \tag{2.2}$$

where

$$\hat{H}(\vec{R}(t))\,|n(\vec{R}(t))\rangle = E_n(\vec{R}(t))\,|n(\vec{R}(t))\rangle \tag{2.3}$$

However, nothing has been said about the relation between eigenstates in different times. To be more precise - if (2.3) holds for some arbitrary time t_0, what does the state look like in any other time $t \neq t_0$? To answer this, a crucial assumption is needed: *adiabatic approximation*. It states, that if the parameters of such a system change *slowly* (= adiabacitally), then the particle will occur in the n-th eigenstate $|n(\vec{R}(t))\rangle$ of $\hat{H}(\vec{R}(t))$ at any time t, if it started in the n-th eigenstate of the initial $\hat{H}(\vec{R}(0))$.

To move further, let us try to solve (2.1). We will focus on the initial n-th eigenstate of system (due to the expansion of state into eigenstates) $|\psi(0)\rangle = |n(\vec{R}(0))\rangle$ and we are asking, how it will change in time. A naive approach would be taking standard solution

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t E_n(\tau)d\tau}|n(\vec{R}(t))\rangle \tag{2.4}$$

where the exponential factor is well-known dynamical phase factor (because of time-dependence of Hamiltonian). We used the adiabatic approximation- if one starts in the state $|n(\vec{R}(0))\rangle$, one comes to the n-th state in different time $|n(\vec{R}(t))\rangle$, gaining the only thing it can gain - an extra phase factor.

However, this is not true - if you try to verify correctness of this solution, you come very quickly to contradiction. Many people tried to fix this problem and Sir Berry

[6]We will follow the original Berry's paper from 1984, [3].

came with an elegant modification. He added an extra phase factor to the solution in (2.4):

$$|\psi(t)\rangle = e^{i\,\gamma_n(t)}\, e^{-\frac{i}{\hbar}\int_0^t E_n(\tau)d\tau}\, |n(\vec{R}(t))\rangle \tag{2.5}$$

Inserting this ansatz into Schödinger equation one gets:

$$\begin{aligned}\Big(i\,\hbar\,\partial_t - \hat{H}(\vec{R}(t))\Big)\,|\psi(t)\rangle =& \, i\hbar\, i\,\dot{\gamma_n}(t)|\psi(t)\rangle + i\hbar\,\left(-\frac{i}{\hbar}\right)\left(\partial_t \int_0^t E(\tau)d\tau\right)|\psi(t)\rangle \\ &+ i\hbar\, e^{i\gamma_n(t)}\, e^{-\frac{i}{\hbar}\int_0^t E_n(\tau)d\tau}\,\partial_{\vec{R}}|n(\vec{R}(t))\rangle\cdot\dot{\vec{R}}(t) - \hat{H}(\vec{R}(t))|\psi(t)\rangle \\ &\stackrel{!}{=} 0\end{aligned} \tag{2.6}$$

Second term is equal to $E(t)|\psi(t)\rangle$, which is to be cancled by the last term (it is Schödinger equation for $|\psi(t)\rangle$). The rest gives as condition on $\gamma_n(t)$:

$$\dot{\gamma_n}(t)|n(\vec{R}(t))\rangle = i\,\partial_{\vec{R}}|n(\vec{R}(t))\cdot\dot{\vec{R}}(t) \tag{2.7}$$

and multiplying both sides by $\langle n(\vec{R}(t))|$ yields (n-th eigenstate is normalised)

$$\dot{\gamma_n}(t) = i\,\langle n(\vec{R}(t))|\,\partial_{\vec{R}}\,|n(\vec{R}(t))\rangle\cdot\frac{d\vec{R}(t)}{dt} \tag{2.8}$$

Finally, integration over a closed loop C gives the birth of the Berry's phase in the form

$$\gamma_n(C) = \oint_C \dot{\gamma_n}(t)\,dt = i\,\oint_C \langle n(\vec{R}(t))|\,\partial_{\vec{R}}\,|n(\vec{R}(t))\rangle\cdot d\vec{R} \tag{2.9}$$

and the solution of (2.1) for the system in the n-th state is

$$|\psi(t)\rangle = e^{i\gamma_n(C)}\, e^{-\frac{i}{\hbar}\int_0^t E_n(\tau)d\tau}\,|\psi(0)\rangle \tag{2.10}$$

Note, that the phase does *not* depend on how the circuit was travelled, because it is given by an integral in the space of $\vec{R}$. Moreover, it cannot be altered by slowing down the rate of change of parameters.

This result may seem like a pointless shaking with the Schrödinger equation by forcing the solution to have a special form and then revealing condition for when this can be done. To disprove this argument, let us do some computation.

It is generally known, that if an extra phase factor comes to the game, we can choose a new basis (note, that also eigenkets are defined only up to a phase factor) and get rid of this phase. In this section, we will not compute the time derivative of function composition $n(\vec{R}(t))$ explicitly, but we will be satisfied by $\partial_t\,|n(\vec{R}(t))\rangle \equiv \partial_t\,|n(t)\rangle$.

Let us therefore start once more from (2.8) and construct a new basis[7]

$$|n'(t)\rangle = \, e^{i\chi(t)}\,|n(t)\rangle \tag{2.11}$$

Hence (2.8) is of the form (new basis is normalised too, e.g. $\langle n'(t)|n'(t)\rangle = 1$)

$$\dot{\gamma_n}(t) = i\,\langle n'(t)|\,\partial_t\,|n'(t)\rangle = i\,\langle n(t)|\,\partial_t\,|n(t)\rangle - \partial_t\chi(t) \tag{2.12}$$

[7]We follow the proof from [17], p. 594.

If we forget the fact, that C is closed loop, then eq. (2.12) can be solved for $\chi(t)$, such that the phase factor in the new basis vanishes. However, we deal with closed one, so $\hat{H}(\vec{R}(T)) = \hat{H}(\vec{R}(0))$, which yields

$$\gamma_n(C,\chi) = \oint_C i \,\langle n'(t)|\, \partial_t \,|n'(t)\rangle \, dt = \oint_C i \,\langle n(t)|\, \partial_t \,|n(t)\rangle \, dt - (\chi(T) - \chi(0)) \tag{2.13}$$

Now comes the most important fact - assumption of a *single-valued* basis (at least for the region of the closed loop), that is quite natural to request. If we assume, that $|n(t)\rangle$ is single-valued, then $|n'(t)\rangle$ is single-valued too and so must be the exponential factor $e^{\chi(T)-\chi(0)}$ - it should not change the geometric phase factor $e^{i\,\gamma_C}$. However, this can be done only if $\chi(T) - \chi(0) \stackrel{!}{=} 2k\pi$, $k \in \mathbb{Z}$, what means, that the function $\chi(t)$ is not arbitrary.

To sum it up, there is no such a change of basis (2.11), that results in eliminating of the phase factor.

It is great, that we have a closed formula for the Berry's phase (2.9), but evaluating the gradient with respect to the external parameter could be awkward. This can be fixed by using the Stokes theorem. Berry did it for 3 dimensions and so will we do. The generalization will be done in the next chapter and will be very simple.

$$\gamma_n(C) = i \oint_C \langle n(\vec{R}(t))|\, \partial_{\vec{R}} \,|n(\vec{R}(t))\rangle \cdot d\vec{R} = i \iint\limits_{int\, C} \left(\vec{\nabla} \times \langle n|\nabla_{\vec{R}}\, n\rangle\right) \cdot d\vec{S}_{\vec{R}} \tag{2.14}$$

where $int\, C$ stands for the region enclosed by closed curve C and we used the nabla-notation for gradient of function $n(t)$: $\partial_{\vec{R}}\, n(t) \equiv \nabla_{\vec{R}}\, n(t)$.

To move further, $\vec{V}_n(\vec{R}) \equiv i\,\vec{\nabla} \times \langle n|\nabla_{\vec{R}}\, n\rangle = i\,\langle \nabla\, n| \times |\nabla\, n\rangle$ (using the Leibnitz rule and the fact, that $\nabla \times (\nabla\phi) = 0$ for a differentiable function ϕ) and inserting an unity operator $\hat{1} = \sum_m |m\rangle\,\langle m|$ gives

$$\vec{V}_n(\vec{R}) = i \sum_m \langle \nabla_{\vec{R}}\, n|m\,\rangle \times \langle\, m|\nabla_{\vec{R}}\, n\rangle \tag{2.15}$$

Let us now evaluate $\nabla_{\vec{R}}\,\langle m\,|\hat{H}(\vec{R})|\,n\rangle$ in two ways:

$$\begin{aligned} \nabla_{\vec{R}}\,\langle m\,|\hat{H}(\vec{R})|\,n\rangle &= E_n \nabla_{\vec{R}}\,\langle m\,|\,n\rangle = E_n \nabla_{\vec{R}}\,\delta_{mn} = 0 \\ &= \langle \nabla_{\vec{R}}\, m|\hat{H}(\vec{R})|\,n\rangle + \langle\, m|\nabla_{\vec{R}}\,\hat{H}(\vec{R})|\,n\rangle + \langle\, m|\hat{H}(\vec{R})|\nabla_{\vec{R}}\, n\rangle \end{aligned} \tag{2.16}$$

Note, that $\hat{H}(\vec{R})$ is hermitian- it can acts both to the left and to the right. Moreover, $|n(\vec{R})\rangle$ is orthonormalised, that gives:

$$\begin{aligned} \nabla_{\vec{R}}\,\langle m|n\rangle &= \nabla_{\vec{R}}\,\delta_{mn} = 0 \\ &= \langle \nabla_{\vec{R}}\, m|n\rangle + \langle m|\nabla_{\vec{R}}\, n\rangle \end{aligned} \tag{2.17}$$

This all yields

$$0 = \,(E_n - E_m)\,\langle \nabla_{\vec{R}}\, m|\,n\rangle + \langle\, m|\nabla_{\vec{R}}\,\hat{H}(\vec{R})|\,n\rangle \tag{2.18}$$

Note, that the term $m = n$ in the sum in (2.14) is equal to zero - thanks to (2.16) it is the cross product of a vector with itself (with extra minus sign). Therefore we can modify the sum by omitting this term, which is very usefull:

$$\vec{V}_n(\vec{R}) = i \sum_{m \neq n} \frac{\langle n \,|\nabla_{\vec{R}}\, \hat{H}|m\rangle \times \langle m \,|\nabla_{\vec{R}}\, \hat{H}|n\rangle}{(E_n - E_m)^2} \tag{2.19}$$

There is no need for panic - we are dealing with *non-degenrated states*[8], so the fraction is always well-defined (I should have mentioned it earlier). Thus we finally come to the result we were searching for, namely a formula for the Berry's phase in terms of gradient of the Hamiltonian itself:

$$\gamma_n(C) = \iint\limits_{int\, C} \vec{V}_n(\vec{R}) \cdot d\vec{S}_{\vec{R}} = i \iint\limits_{int\, C} \sum_{m \neq n} \frac{\langle n \,|\nabla_{\vec{R}}\, \hat{H}|m\rangle \times \langle m \,|\nabla_{\vec{R}}\, \hat{H}|n\rangle}{(E_n - E_m)^2} \cdot d\vec{S}_{\vec{R}} \tag{2.20}$$

2.2 Berry's phase for spin in magnetic field

We mentioned in the introduction to this chapter, as soon as we build up the formula for the Phase, we will try it for a particle with spin s being in magnetical field $\vec{B}$. Let us do it! [9]

Hamiltonian of such a system is

$$\hat{H} = \hat{\vec{\mu}} \cdot \hat{\vec{B}} \tag{2.21}$$

where $\hat{\vec{\mu}} = g\frac{q\,\hbar}{2m}\hat{\vec{s}}$, g being numerical g-factor, q and m standing for the charge and mass of particle respectivelly and $\hat{\vec{s}}$ being the spin operator with eigenvalues $n \in \{-s, \ldots, +s\}$ (note, thas s can be integer or half-integer only)[10]. This implies, that eigenvalues of Hamiltonian are

$$E_n = \kappa\, \hbar\, n\, B \tag{2.22}$$

following the Berry's notation with $\kappa = g\frac{q}{2m}$.

As we indicated in the previous text, $\vec{B}$ acts as *the* parameter of parametric space, that should be changed adiabatically. Therefore

$$\nabla_{\vec{R}}\, \hat{H}(\vec{R}) = \nabla_{\vec{B}}\, \hat{\vec{\mu}} \cdot \hat{\vec{B}} = \hat{\vec{\mu}} = \kappa\, \hbar\, \hat{\vec{s}} \tag{2.23}$$

Insterting this expression into (2.20) we get

$$\vec{V}_n(\vec{B}) = \frac{i}{B^2} \sum_{m \neq n} \frac{\langle n, s(\vec{B})|\hat{\vec{s}}|m, s(\vec{B})\rangle \times \langle m, s(\vec{B})|\hat{\vec{s}}|n, s(\vec{B})\rangle}{(m-n)^2} \tag{2.24}$$

[8] Here could be an extensive discussion about what should we do, if the Hamiltonian had a degenerated spectrum. The approach could be even more advanced, if we would forget about adiabatic approximation. However, this is not aim of our thesis. You can find the appropriate information for example in the original Berry's paper [3]. Moreover, we can abandon the Abelian gauge (concept, where the Lagrangian of the system remains invariant after action of some continous group of local transformations). This is an advanced topic, that can be found in [10] for example.

[9] If there is anyone who objects to this union, please speak now or forever hold your peace.

[10] In this section, we will use n instead of m, which stands for the mass of the particle. In other sections we will return to the standard notation, where the spinor state is denoted $|s, m\rangle$.

The only thing, we need to know, is the matrix element $\langle n, s(\vec{B}) \mid \hat{\vec{s}} \mid m, s(\vec{B})\rangle$. A standard way to do this is to rotate the axes so that $\vec{B} = (0, 0, B)$ (e.g. z-axis into direction of $\vec{B}$). Then acting of operators $\hat{s}_+, \hat{s}_-, \hat{s}_z$ is known from the basic course of quantum mechanics:

$$\begin{aligned}
\hat{s}_+|n,s\rangle &= (\hat{s}_x + i\hat{s}_y)\,|n,s\rangle = \sqrt{s(s+1) - n(n+1)}\,|n+1,s\rangle = C_n^+\,|n+1,s\rangle \\
\hat{s}_-|n,s\rangle &= (\hat{s}_x - i\hat{s}_y)\,|n,s\rangle = \sqrt{s(s+1) - n(n-1)}\,|n-1,s\rangle = C_n^-\,|n-1,s\rangle \\
\hat{s}_z\,|n,s\rangle &= n\,|n,s\rangle
\end{aligned} \tag{2.25}$$

We can see, that only a few of the matrix elements are non-zero: $\langle n+1, s \mid \hat{s}_+| \, n, s\rangle$, $\langle n-1, s \mid \hat{s}_-| \, n, s\rangle$ and $\langle n, s| \, \hat{s}_z \mid n, s\rangle$. Note, that x- and y- component of the result is directly zero. The cross product results in product of matrix elements of $\hat{s}_z$ with $s_\pm$. This is always zero, because ich we try to get non-zero element of $\hat{s}_\pm$, then $m = n \pm 1$, but the matrix of $\hat{s}_z$ is diagonal, and therefore $\langle n+1, s|\hat{s}_z|n, s\rangle = 0$. The only surviving component of the $\vec{V}_n(\vec{B})$ is the z- axial one. On this purpose, let us write

$$\begin{aligned}
\langle n \pm 1, s(\vec{B})|\hat{s_x}|n, s(\vec{B})\rangle &= \langle n \pm 1, s(\vec{B})| \left(\frac{\hat{s_+} + \hat{s_-}}{2}\right) |n, s(\vec{B})\rangle = \\
&= \frac{1}{2}\sqrt{s(s+1) - n(n \pm 1)} \\
\langle n \pm 1, s(\vec{B})|\hat{s_y}|n, s(\vec{B})\rangle &= \langle n \pm 1, s(\vec{B})| \left(\frac{\hat{s_+} - \hat{s_-}}{2i}\right) |n, s(\vec{B})\rangle = \\
&= \mp \frac{i}{2}\sqrt{s(s+1) - n(n \pm 1)}
\end{aligned} \tag{2.26}$$

and then finally

$$\begin{aligned}
(V_n)_z = &\frac{i}{B^2} \sum_{m \neq n} \frac{\langle n, s(\vec{B})|\hat{s_x}|m, s(\vec{B})\rangle\langle m, s(\vec{B})|\hat{s_y}|n, s(\vec{B})\rangle}{(m-n)^2} \\
&- \frac{i}{B^2} \sum_{m \neq n} \frac{\langle n, s(\vec{B})|\hat{s_y}|m, s(\vec{B})\rangle\langle m, s(\vec{B})|\hat{s_x}|n, s(\vec{B})\rangle}{(m-n)^2}
\end{aligned} \tag{2.27}$$

From the whole sum only term with $m = n \pm 1$ is non-zero (we used the same argument at (2.24)). Hence we get the result

$$\begin{aligned}
(V_n)_z &= \frac{i}{B^2}(\langle n \mid \hat{s_x}|\, n+1\rangle\langle n+1 \mid \hat{s_y}|\, n\rangle - \langle n \mid \hat{s_y}|\, n+1\rangle\langle n+1 \mid \hat{s_x}|\, n\rangle + \\
&\quad + \langle n \mid \hat{s_x}|\, n-1\rangle\langle n-1 \mid \hat{s_y}|\, n\rangle - \langle n \mid \hat{s_y}|\, n-1\rangle\langle n-1 \mid \hat{s_x}|\, n\rangle) = \\
&= \frac{1}{2B^2}\left(C_{n+1}^- C_n^+ - C_{n-1}^+ C_n^-\right) = \\
&= \frac{1}{2B^2}((s+n+1)(s-n) - (s+n)(s-n+1)) = \\
&= \frac{n}{B^2}
\end{aligned} \tag{2.28}$$

Coming back to our original basis, we simply obtain

$$\vec{V}_n(\vec{B}) = \frac{n\vec{B}}{B^3} \tag{2.29}$$

and we can finally compute the Berry's phase for a particle with spin s in the state n in magnetic field $\vec{B}$ according to (2.19):

$$\gamma_n(C) = -\iint\limits_{int\,C} \vec{V}_n(\vec{R}) \cdot d\vec{S}_{\vec{R}} = -n \iint\limits_{int\,C} \frac{\vec{B} \cdot d\vec{S}_{\vec{B}}}{B^3} = -n \iint\limits_{int\,C} d\Omega \tag{2.30}$$

that is flux of a magnetic monopole living at the origin of the parametric space (multiplied by $-n$). The integral can be done and its result is

$$\gamma_n(C) = -n\,\Omega_C \tag{2.31}$$

where Ω_C stands for the solid angle, that is cut off by the circuit C at the origin of the coordinate system. Note, that the phase depends only on the projection of spin along the $\vec{B}$- direction. Moreover, we can produce any phase change by travelling along a suitable loop in the B- space.

To close this discourse about the Berry's original approach to his phase, let us give some elementary examples. For instance, taking fermions (that have half-integer spin) and slowly making whole turn with the magnetic field the state gains a phase $e^{i\gamma_n(C)} = e^{-i\frac{n}{2}2\pi} = -1$. That is a proof, that spinors of an electron become an extra phase of -1 when being rotated by 2π. In contrast, bosons (integer spin) does not get any phase by rotating about any axis. If we vary $\vec{B}$ round a cone of semiangle θ, its solid angle is $\Omega = 2\pi\,(1 - \cos\theta)$, so taking $\theta = 60°$ results in a change of the sign of the bosonic state from + to −.

3 Caution: Differential geometry detected!

A short chapter is insterted between the original Berry's approach to his phase and our differential geometric one. We will (imaginary) read the previous section again and try to uncover as many differential terms as possible and then rewrite the most important formulae in the language of differential geometry, that turns up to be very benefitial.

This section, however, does *not* stand as a replacement of a proper course of differential geometry. It is only a brief reminder of those concepts from the course, that we will use later[11].

3.1 Basic concepts of differential geometry

First of all, let us define some basic concepts of differential geometry, we will use in the following section. We will try to give a precise mathematical definition of the term and then a short explanation, "what does it actually do". Crucial terms are *p-form*, *wedge product*, *differential form* and *exterior derivative*.

We will start with so-called p-form. A tensor $\alpha \in T^0_p(L)$ is a p-form if it is skew-symmetric in each pair of indices, e.g. for any two vectors $v, w \in L$ one has

$$\alpha(\ldots, v, \ldots, w, \ldots) = -\alpha(\ldots, w, \ldots, v, \ldots) \tag{3.1}$$

Here $T^0_p(L)$ stands for the space of tensors of the type $\binom{0}{p}$ over a linear space L. Definition (3.1) makes no sence for $p < 2$. Therefore, let us define $\Lambda^0 L^* = T^0_0(L) \equiv \mathbb{R}$ and $\Lambda^1 L^* = T^0_1(L) \equiv L^*$, where L^* has the meaning of a dual space to L (in other words, 0-forms are real numbers and 1-forms are covectors). Additionally, set of such p-forms will be denoted as $\Lambda^p L^*$. Note, that if $dim\, L = n$, then $dim\, \Lambda^p L^* = \binom{n}{p}$.

Next step would be definition of so-called wedge product. The idea of it is to make a $(p+q)$-form from p- and q-forms. Naive guess would be tensor product, that makes a tensor of rank $(p+q)$ from two tensors of rank p- and q respectivelly. However, the result does not need to be skew-symmetric. To fix this, let us introduce an antisymetric projection operator:

$$\pi^A \colon T^0_p(L) \to \Lambda^p L^* \subset T^0_p(L) \tag{3.2}$$

Thanks to this projector ($\pi^A \circ \pi^A = \pi^A$) we can define a new form as an exterior (wedge) product of two forms ($\alpha \in \Lambda^p L^*$, $\beta \in \Lambda^q L^*$):

$$\begin{aligned} \wedge &\colon \Lambda^p L^* \times \Lambda^q L^* \to \Lambda^{p+q} L^* \\ \alpha \wedge \beta &:= \frac{(p+q)!}{p!\, q!}\, \pi^A(\alpha \otimes \beta) \end{aligned} \tag{3.3}$$

For practical use of the wedge product, one only has to know, that it is bilinear, asociative, $\mathbb{Z}$-graded commutative (e.g. $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$) and skew-symmetric (that means $\alpha \wedge \alpha = 0$ for any α). The last piece of information about p-form is its decomposition into wedge product of base 1-forms (e^a etc.)[12]

$$\alpha = \frac{1}{p!}\, \alpha_{a\ldots b}\, e^a \wedge \cdots \wedge e^b \tag{3.4}$$

[11] We will follow an excellent book by M. Fecko, [8] - chapters 5 - 8

[12] Einstein's summation convention is used.

We can now define a special p-form in n-dimensional L - the metric volume form. Recall, that $dim\ \Lambda^p L^* = \binom{n}{p}$. For $p = n$ we get an 1-dimensional space of n-forms. When we choose a base, then the most general n-form ω is $\omega = \lambda\, e^1 \wedge \cdots \wedge e^n$. It can be shown, that if we fix this only parameter (λ) by defining the value of the form for some basis, then it is defined for any other basis automatically. The most natural choise is the orthonormalised euclidian basis and then

$$\omega = e^1 \wedge \cdots \wedge e^n \tag{3.5}$$

Note, that its skew-symmetry (and the fact, that it is a tensor of type $\binom{0}{n}$) reminds us of a definition for volume of parallelotope stretched on vectors e^i. For the euclidian basis the volume of such a parallelotope is 1. That is why we name this n-form the volume form.

In a general basis f^i we get (g being the metric tensor and o being orientation, $o(f) = \pm 1$ for right- and left-handed basis respectivelly)

$$\omega_g = o(f)\sqrt{|det\ g(f_a, f_b)|}\ f^1 \wedge \cdots \wedge f^n \tag{3.6}$$

Some geometry has been introduced, but it has not been differential yet. Let us retrieve it. If we transport all previous concepts to a manifold M (by constructing a tangent space in every point $P \in M$, where live vectors $v \in L$ and covectors $\alpha \in L^*$, what imediatelly makes tensors live there too[13]), we get differential forms. Namely, in region $\mathcal{O}$ equipped by local coordinates x^i one has for vectors a coordinate basis ∂_i, then covector basis is a dual conjugate dx^i and a p-form is of the form

$$\alpha = \frac{1}{p!}\,\alpha_{i\ldots j}(x)\ \underbrace{dx^i \wedge \cdots \wedge dx^j}_{p-\ terms} \tag{3.7}$$

Note, that space of p-forms is denoted as $\Omega^p(M)$.

Additionally, if we have metric g on M, then the metric volume form on M is (o stands again for orientation)

$$\omega_g = o(x)\sqrt{|det(g_{ij}(x))|}\ dx^1 \wedge \cdots \wedge dx^n \tag{3.8}$$

For example, the volume form on sphere (of radius R) $S^2 \subset E^3$ is $\omega_g = R^2 \sin\vartheta\, d\vartheta \wedge d\varphi$, that will be crucial for us later on. The most important thing about the volume form is, that its integral over a region $D \subset M$ gives exactly the volume of such a region:

$$vol(D) = \int_D \omega_g \tag{3.9}$$

Everything is ready to introduce a special differential operator acting on differential p-forms- the exterior derivative. From the name one would naively await, that partial

[13] For more details, I recommend [8], sections 2.5 and 6.1.

derivative is sufficient, but it is not- it makes from a p-form an object with $p+1$ indices, but it is not a tensor. But if we (again) make the result skew-symmetric in all indices, we are done - the redundant terms in the result of partial derivative are symmetric in every pair of indices and thus cancelled by antisymmetrisation:

$$\begin{aligned} d\colon \Omega^p(M) &\to \Omega^{p+1}(M) \\ (d\alpha)_{i\ldots jk} &:= (-1)^p(p+1)\,\alpha_{[i\ldots j,k]} \end{aligned} \tag{3.10}$$

where the symbol $t_{[a\ldots b]} = (\pi^A t)_{a\ldots b} := \frac{1}{p!}\sum_\sigma (sgn\,\sigma)\, t_{\sigma(a,\ldots,b)}$, for σ being a permutation of indices[14] and abbreviated derivative notation is used, e.g. $h_{i,j} = \partial_j h_i$.

Again, it is not necessary to master this definition. We only need to know, that it is linear, nilpotent ($dd = 0$), it acts on a wedge product like graded Leibnitz rule ($d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p\, \alpha \wedge (d\beta)$ for $\alpha \in \Omega^p(M)$) and for $f \in \Omega^0(M)$ (a function on M) we have $df = grad\, f = f_{,i}\, dx^i$.

The last thing we need is the Stokes theorem - a central theorem of integral calculus of differential forms on manifolds. It is a generalisation of the famous statement, that any circuit integral over closed loop can be transcribed as a surface integral.

The most important observation is, that if we treat subintegrands as p-forms, we get correct results when transforming to another coordinates.[15], that can be written in an elegant way. Let $\Phi\colon (r, \varphi) \to (x, y)$ be the map from Cartesian to polar coordinates. Then for any region of integration C

$$\int_{\Phi(C)} \alpha = \int_C \Phi^* \alpha \tag{3.11}$$

that holds not only in two dimensions with this particular map, but for n dimensions and any (sufficiently smooth) map Φ. That is how differential forms enter the integral calculus.

Finally, we will introduce the Stokes theorem (without proof or motivation - both can be found in [8], section 7.5 for instance). For any n-dimensional region D on an orientable n-dimensional manifold M and for any $\alpha \in \Omega^{n-1}(M)$ it is true, that

$$\int_D d\alpha = \int_{\partial D} \alpha \tag{3.12}$$

where ∂D is the exterior boundary of D with its orientation in the direction of the external normal.

3.2 Berry could make it this way!

After an exhaustive and quite difficult introduction to some basic topics of differential geometry[16] we can finally uncover differential forms in Berry's computations and rewrite his statements[17].

[14] For example, $t_{[ab]} = \frac{1}{2!}\,(t_{ab} - t_{ba})$.

[15] For example, if $dx\,dy \longleftrightarrow dx \wedge dy$, then change to polar coordinates gives $rdr\,d\varphi \longleftrightarrow rdr \wedge d\varphi$.

[16] I really feel sorry for that, but it was necessary.

[17] Sketch of this manipulation can be found in [1].

From now on, the manifold M will be the parametric space of $\vec{R}$. Let us now remind of one property of the exterior derivative - it acts as a gradient on functions (being 0-forms n M). When we now look at the subintegrand in (2.8), we find, that

$$\partial_{\vec{R}} |n(\vec{R}(t))\rangle \cdot d\vec{R} = d\,|n(\vec{R}(t))\rangle \tag{3.13}$$

so there is directly a 1-form in the definition of the Berry's phase. Moreover, when we write

$$\gamma_n(C) = i \oint_{\partial D} \langle n \mid d \mid n \rangle \tag{3.14}$$

where $\partial D = C$ is the boundary of region $D = int\, C$, we are ready to use the Stokes theorem:

$$i \int_{\partial D} \langle n \mid d \mid n \rangle = i \int_D d(\langle n \mid d \mid n \rangle) \tag{3.15}$$

Using the fact, that $d\,d = 0$, we can compute the derivative under the integral (according to remarks on how the derivative acts) and write down

$$\gamma_n(C) = i \int_D \langle d\, n(\vec{R}(t))| \wedge |d\, n(\vec{R}(t))\rangle \tag{3.16}$$

That is how 1-form (Berry's connection $\mathcal{A}_n$) and 2-form (Berry's curvature Ω_n) come into the game of the Berry's phase:

$$\mathcal{A}_n(\vec{R}) = \; i\, \langle n(\vec{R}) \mid d\, |n(\vec{R})\rangle = i\, \langle n(\vec{R})|\, \partial_i\, |n(\vec{R})\rangle\, dR^i \tag{3.17}$$

$$\Omega_n(\vec{R}) = \; i\, \langle d\, n(\vec{R})| \wedge |d\, n(\vec{R})\rangle = i\, \langle \partial_i\, n(\vec{R}) \mid \partial_j\, n(\vec{R})\rangle\, dR^i \wedge dR^j \tag{3.18}$$

and then for $C = \partial D$ we have

$$\gamma_n(C) = \int_{\partial D} \mathcal{A}_n(\vec{R}) = \int_D \Omega_n(\vec{R}) \tag{3.19}$$

The remaining steps to the result for both general expression and for the particle with spin in magnetic field follow the same ideas as mentioned above. We will not do it, because we only wanted to show, how the differential geometry enters this topic. In next sections, we will use it widely to get particular results for particular situations.

4 Demo version: Particle with spin 1/2, 0 and 1

We are already full of general expressions and we would like to try, how it everything works in particular situations. The first would be the most frequent example - particle with spin $s = \frac{1}{2}$ (e.g. electron) in external magnetic field. Once we obtain this result, we can compute Berry's phase for other spins either directly by computing the Berry's curvature or by combination of spins $\frac{1}{2}$.

4.1 Spin 1/2

Let us start with the Hamiltonian of this system according to (2.20), namely

$$\hat{H} = g\,\frac{q\hbar}{2m}\,\hat{\vec{s}}\cdot\hat{\vec{B}} = \hat{H} = g\,\frac{q\hbar}{2m}\,\frac{\hat{\vec{\sigma}}}{2}\cdot\hat{\vec{B}} \tag{4.1}$$

where σ_i are Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.2}$$

all having eigenvalues ± 1. Let us set the basis so that $\vec{B} = (0, 0, B)$. Then eigenstates of such a Hamiltonian are simply $|+1/2\rangle = \binom{1}{0}$, $|-1/2\rangle = \binom{0}{1}$ with eigenvalues (energies) $\pm g\,\frac{q\hbar B}{4m}$ respectively corresponding to the two spin states of such a particle. When we now rotate the basis back to a general position given by a unit vector $\vec{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, we get the spin operator in the form

$$\vec{n}\cdot\vec{\sigma} = \begin{pmatrix} \cos\vartheta & \sin\vartheta\, e^{-i\varphi} \\ \sin\vartheta\, e^{i\varphi} & -\cos\vartheta \end{pmatrix} \tag{4.3}$$

with eigenstates corresponding to the eigenvalues ± 1 as follows

$$\left|+\frac{1}{2}\right\rangle = \begin{pmatrix} \cos\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix},\quad \left|-\frac{1}{2}\right\rangle = \begin{pmatrix} -\sin\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix} \tag{4.4}$$

that are the two states we were searching for (for the case $n = \pm 1/2$ in notation from the previous chapter).

To move further, we need to compute the exterior derivative of the state - we simply differentiate components themselves (Note, that we perform the exterior derivative on a parametric manifold with coordinates (= parameters) ϑ, φ, so the state itself is a $\mathbb{C}^2$-valued 0-form on the manifold S^2, giving birth to 1-form on the same manifold)[18] :

$$\begin{aligned} d\,|+\rangle &= \begin{pmatrix} d\,\left(\cos\frac{\vartheta}{2}\, e^{-i\varphi/2}\right) \\ d\,\left(\sin\frac{\vartheta}{2}\, e^{i\varphi/2}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\, e^{-i\varphi/2}\,\sin\frac{\vartheta}{2} \\ \frac{1}{2}\, e^{i\varphi/2}\,\cos\frac{\vartheta}{2} \end{pmatrix} d\vartheta + \begin{pmatrix} -\frac{i}{2}\, e^{-i\varphi/2}\,\cos\frac{\vartheta}{2} \\ \frac{i}{2}\, e^{i\varphi/2}\,\sin\frac{\vartheta}{2} \end{pmatrix} d\varphi = \\ &= \frac{1}{2}|-\rangle\, d\vartheta + \frac{i}{2}\begin{pmatrix} -\cos\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix} d\varphi \end{aligned} \tag{4.5a}$$

$$\begin{aligned} d\,|-\rangle &= \begin{pmatrix} d\,\left(-\sin\frac{\vartheta}{2}\, e^{-i\varphi/2}\right) \\ d\,\left(\cos\frac{\vartheta}{2}\, e^{i\varphi/2}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\, e^{-i\varphi/2}\,\cos\frac{\vartheta}{2} \\ -\frac{1}{2}\, e^{i\varphi/2}\,\sin\frac{\vartheta}{2} \end{pmatrix} d\vartheta + \begin{pmatrix} \frac{i}{2}\, e^{-i\varphi/2}\,\sin\frac{\vartheta}{2} \\ \frac{i}{2}\, e^{i\varphi/2}\,\cos\frac{\vartheta}{2} \end{pmatrix} d\varphi \\ &= -\frac{1}{2}|+\rangle\, d\vartheta + \frac{i}{2}\begin{pmatrix} \sin\frac{\vartheta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\vartheta}{2}\, e^{i\varphi/2} \end{pmatrix} d\varphi \end{aligned} \tag{4.5b}$$

[18] We will sometimes use the expression *spinor* instead of *state*, because in some way we can treat a n-component vector as a spinor. However, we will *not* use it in the proper way when concerning spinor fields, etc. We hope it causes no confusion.

Hence the Berry's connection is

$$\mathcal{A}_+ = i\,\langle +|\,d\,|+\rangle = \frac{i}{2}\langle +\,|-\rangle\,d\vartheta - \frac{1}{2}\left(e^{i\varphi/2}\,\cos\frac{\vartheta}{2}, e^{-i\varphi/2}\,\sin\frac{\vartheta}{2}\right)\cdot\begin{pmatrix}-\cos\frac{\vartheta}{2}\,e^{-i\varphi/2}\\ \sin\frac{\vartheta}{2}\,e^{i\varphi/2}\end{pmatrix}d\varphi$$
$$= -\frac{1}{2}\left(-\cos^2\frac{\vartheta}{2}+\sin^2\frac{\vartheta}{2}\right)d\varphi = \frac{1}{2}\,\cos\,\vartheta\,d\varphi \tag{4.6a}$$

$$\mathcal{A}_- = i\,\langle -|\,d\,|-\rangle = -\frac{i}{2}\langle -\,|+\rangle\,d\vartheta - \frac{1}{2}\left(-e^{i\varphi/2}\,\sin\frac{\vartheta}{2}, e^{-i\varphi/2}\,\cos\frac{\vartheta}{2}\right)\cdot\begin{pmatrix}\sin\frac{\vartheta}{2}\,e^{-i\varphi/2}\\ \cos\frac{\vartheta}{2}\,e^{i\varphi/2}\end{pmatrix}d\varphi$$
$$= -\frac{1}{2}\left(-\sin^2\frac{\vartheta}{2}+\cos^2\frac{\vartheta}{2}\right)d\varphi = -\frac{1}{2}\,\cos\,\vartheta\,d\varphi \tag{4.6b}$$

and finally the Berry's curvature

$$\Omega_+ = d\mathcal{A}_+ = d\left(\frac{1}{2}\,\cos\,\vartheta d\varphi\right) = -\frac{1}{2}\,\sin\vartheta\,d\vartheta\wedge d\varphi = -\frac{1}{2}\omega_g \tag{4.7a}$$

$$\Omega_- = d\mathcal{A}_- = d\left(-\frac{1}{2}\,\cos\,\vartheta d\varphi\right) = \frac{1}{2}\,\sin\vartheta\,d\vartheta\wedge d\varphi = +\frac{1}{2}\omega_g \tag{4.7b}$$

where according to the 3-rd chapter

$$\omega_g = \sin\vartheta\,d\varphi\wedge d\vartheta \tag{4.8}$$

is the metric volume form for sphere S^2 with radius $R=1$, e.g. unit sphere (note, that spinors of an electron live on such a sphere[19] - we used unit vector parametrised with spherical coordinates when obtaining spin operator in general direction). Then the Berry's phase is really what we awaited:

$$\gamma_{+1/2}(C) = -\frac{1}{2}\int_{int\,C}\omega_g = -\frac{1}{2}\,\Omega_C \tag{4.9a}$$

$$\gamma_{-1/2}(C) = \frac{1}{2}\int_{int\,C}\omega_g = \frac{1}{2}\,\Omega_C \tag{4.9b}$$

where Ω_C stands for the solid angle subtended by the C at origin of the unit sphere S^2. Note, that the solid angle hereby defined is *exactly* the area at the unit sphere (again substended by the C at its origin). Therefore it is easy to understand, why $\int_{int\,C}\omega_g \equiv \Omega_C$, what acts as a definition, though.

4.2 Spin 0

We will use the same procedure as sketched above. However, there is actually nothing worth of computing. If we represent a state of particle with spin s by a spinor (n-tuple of complex numbers, where the k-th one corresponds to the amplitude of probability, that the particle is in the state defined by the k-th element of basis of space of such spinors - eigenstate of the spin operator, that gives the k-th eigenvalue), then it is clear, that $n = 2s+1$. In addition, the spin operator is then represented by a regular

[19] To be more precise, they live in the whole 3-dimensional parametric space (B,ϑ,φ). However, the dependence on $\vec{B}$ is not present in the connection 1-form (due to computations, that were done - we can see the B only in formula for energy of these eigenstates), so *effectively* they really live on the sphere with unit radius.

$(2s+1) \times (2s+1)$ hermitian matrix (for every matrix element i, j: $a_{ij} = \overline{a_{ji}}$, or equivalently for the matrix $A = \overline{A^T} \equiv A^\dagger$).

Let us now write a condition for our desired state[20]

$$\hat{s}|m\rangle = s_m|m\rangle \tag{4.10}$$

In case $s = 0$, the spin operator is represented by 1×1 hermitian matrix - a real number. Note, that every regular matrix gets its eigenvalues onto the diagonal when diagonalised. That is why our operator is simply $\hat{s} = \hat{0} \equiv 0$ with any unit complex number as its eigenstate (do not forget, that it should still be the amplitude of probability). We choose the form

$$|0\rangle = 1 \tag{4.11}$$

that is correct ($1 \cdot 0 = 0$ and it is normalised too). Then the Berry's connetion is

$$\mathcal{A}_0 = i \langle 0| \, d \, |0\rangle = 0 \tag{4.12}$$

and immediately the Berry's curvature

$$\Omega_0 = d\mathcal{A}_0 = 0 = -0\,\omega_g \tag{4.13}$$

You can object, that we cheated in this section: what if we chose $|0\rangle = e^{i\,\alpha(\vartheta,\varphi)}$ instead of $|0\rangle = 1$? The answer is simple and holds for any eigenstate $|m\rangle$ of any spin operator $\hat{s}$. It is clear, that such a change of phase is mediated by an element of $U(1)$ group (any element of the group can be written as $e^{i\alpha}$ for arbitrary $\alpha \in \mathbb{R}$) and $U(1)$-gauge transformation changes only the Berry's potential, but the curvature is invariant under this transformation. Let us assume, that the new state is $|m'\rangle = e^{i\,\alpha}\,|m\rangle$. Then

$$\begin{aligned}
\mathcal{A}' &= i\langle m'|d|m'\rangle = i\,\langle m|e^{-i\,\alpha}\,d\left(e^{i\,\alpha}|m\rangle\right) = \\
&= i\,\langle m|e^{-i\,\alpha}\,\left(i\,e^{i\,\alpha}|m\rangle\,d\alpha + e^{i\,\alpha}\,d|m\rangle\right) = \\
&= i\,\langle m|d|m\rangle - d\alpha = \mathcal{A} - d\alpha
\end{aligned} \tag{4.14a}$$

that yields

$$\Omega' = d\mathcal{A}' = d\mathcal{A} - d\,d\alpha = d\mathcal{A} = \Omega \tag{4.14b}$$

because the exterior derivative is nilpotent. This result allows us to forget the extra phase factor in every single computation, that is to be done in this thesis.

What we do extra in comparison to the previous section, we will combine 1/2-spins so that they give a state with both total spin s and its z-projection m equal to zero.

Let us now remind of how the mechanism of spin combination works. Denote V_1 a $(2s_1+1)$-dimensional vector space with the basis $|s_1, m_1\rangle$, where $m_1 \in \{-s_1, \ldots, s_1\}$ and similarly V_2 let be a $(2s_2+1)$-dimensional vector space spanned by states $|s_2, m_2\rangle$ for $m_2 \in \{-s_2, \ldots, s_2\}$ (with integer gaps between values of both m_1, m_2 and, of course, s_1, s_2 being integer or half-integer). Then tensor product of these spaces is a $(2s_1+1)(2s_2+1)$-dimensional vector space with uncoupled basis $|s_1, m_1\rangle\,|s_2, m_2\rangle =$

[20] As promised, we returned to the notation, where n stands mostly for the dimension of a vector space and m is the z-projection of the total spin s.

$|s_1, m_1\rangle \otimes |s_2, m_2\rangle$. Operators act on these states like $(\hat{A} \otimes \hat{1})|s_1, m_1\rangle\,|s_2, m_2\rangle = \hat{A}|s_1, m_1\rangle \otimes |s_2, m_2\rangle$ and $(\hat{1} \otimes \hat{B})|s_1, m_1\rangle\,|s_2, m_2\rangle = |s_1, m_1\rangle \otimes ((\hat{B}|s_2, m_2\rangle)$. In such a vector space we can write any spin state as a linear combination of basis states with coefficients being the famous Clebsch- Gordan coefficients:

$$|s, m\rangle = \sum_{m_1+m_2=m} C^{s_1\,s_2\,s}_{m_1\,m_2}\,|s_1, m_1\rangle\,|s_2, m_2\rangle \tag{4.15}$$

We will now use the 2-dimensional basis $|1/2, m_1\rangle\,,\,|1/2, m_2\rangle$ for coupling of two spins ($m_1, m_2 \in \{-1/2, 1/2\}$). This can give as representation of spins $s = 0$ and $s = 1$, when we find the appropriate normalised linear combination.

When checking the result for spin $s = 0$ (which implies $m = 0$), we get this only combination:

$$|0,0\rangle = \frac{1}{\sqrt{2}}\,(|1/2,1/2\rangle|1/2,-1/2\rangle - |1/2,-1/2\rangle|1/2,1/2\rangle) = \frac{1}{\sqrt{2}}\,(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{4.16}$$

Note, that in the state representation we use the tensor product of two spins is tensor products of two (2-dimensional) vectors, what means, that we get a 4-dimensional vector representation of the spin $s = 0$. Precisely

$$\begin{aligned} |\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle &= \begin{pmatrix} \cos\frac{\vartheta}{2}\,e^{-i\varphi/2} \\ \sin\frac{\vartheta}{2}\,e^{i\varphi/2} \end{pmatrix} \otimes \begin{pmatrix} -\sin\frac{\vartheta}{2}\,e^{-i\varphi/2} \\ \cos\frac{\vartheta}{2}\,e^{i\varphi/2} \end{pmatrix} \\ &= \left(-\frac{1}{2}\,e^{-i\varphi}\,\sin\,\vartheta, \cos^2\frac{\vartheta}{2}, -\sin^2\frac{\vartheta}{2}, \frac{1}{2}\,e^{i\varphi}\,\sin\,\vartheta\right)^T \end{aligned} \tag{4.17a}$$

$$\begin{aligned} |\downarrow\uparrow\rangle = |\downarrow\rangle \otimes |\uparrow\rangle &= \begin{pmatrix} -\sin\frac{\vartheta}{2}\,e^{-i\varphi/2} \\ \cos\frac{\vartheta}{2}\,e^{i\varphi/2} \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\vartheta}{2}\,e^{-i\varphi/2} \\ \sin\frac{\vartheta}{2}\,e^{i\varphi/2} \end{pmatrix} \\ &= \left(-\frac{1}{2}\,e^{-i\varphi}\,\sin\,\vartheta, -\sin^2\frac{\vartheta}{2}, \cos^2\frac{\vartheta}{2}, \frac{1}{2}\,e^{i\varphi}\,sin\,\vartheta\right)^T \end{aligned} \tag{4.17b}$$

Then from (4.15) we obtain

$$|0,0\rangle = \frac{1}{\sqrt{2}}\,(0,1,-1,0)^T \tag{4.18}$$

Then for calculating $\mathcal{A} = i\,\langle 0,0|\,d\,|0,0\rangle$ we need to make the exterior derivative of this vector, that is immediatelly zero, what implies $\mathcal{A} = 0$, then $\Omega_0 = 0$, hence $\gamma_0 = 0$ and we are succesfully done.

4.3 Spin 1

Following the procedure once again, we will start from the basis in that $\vec{B} = (0, 0, B)$, which yields that eigenstates of Hamiltonian are simply $|1\rangle = (1,0,0)^T$, $|0\rangle = (0,1,0)^T$ and $|-1\rangle = (0,0,1)^T$, because this three are directly the eigenvectors of a 3×3 matrix s_z for spin $s = 1$, that is $s_z = diag(1, 0, -1)$. Rotating back to an arbitrary direction given by $\vec{n} = (\sin\,\vartheta\,\cos\,\varphi, \sin\,\vartheta\,\sin\,\varphi, \cos\,\vartheta)$, we will get the operator of spin in general position. Its eigenstates would be then our desired states needed for calculation of the Berry's phase. Recall therefore the two remaining operators in $x-$ and $y-$ direction:

$$s_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{4.19}$$

Then the spin operator in general direction $\hat{\vec{s}}_n = \vec{s}\cdot\vec{n}$ is

$$\hat{\vec{s}}_n = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\cos\vartheta & \sin\vartheta\, e^{-i\varphi} & 0 \\ \sin\vartheta\, e^{i\varphi} & 0 & \sin\vartheta\, e^{-i\varphi} \\ 0 & \sin\vartheta\, e^{i\varphi} & -\sqrt{2}\cos\vartheta \end{pmatrix} \tag{4.20}$$

Finding eigenstates of this matrix is slightly more difficult in comparison to the case $s = 1/2$, that indicates serious problem when travelling to the destination called general spin. However, this particular subtlety is not to be discussed here.[21] Hence the eigenstates are[22]

$$|1\rangle = \begin{pmatrix} \frac{1+\cos\vartheta}{2}\, e^{-i\varphi} \\ \frac{\sqrt{2}\sin\vartheta}{2} \\ \frac{1-\cos\vartheta}{2}\, e^{i\varphi} \end{pmatrix},\ |0\rangle = \begin{pmatrix} -\frac{\sqrt{2}\sin\vartheta}{2}\, e^{-i\varphi} \\ \cos\vartheta \\ \frac{\sqrt{2}\sin\vartheta}{2}\, e^{i\varphi} \end{pmatrix},\ |-1\rangle = \begin{pmatrix} \frac{1-\cos\vartheta}{2}\, e^{-i\varphi} \\ -\frac{\sqrt{2}\sin\vartheta}{2} \\ \frac{1+\cos\vartheta}{2}\, e^{i\varphi} \end{pmatrix} \tag{4.21}$$

To move further, let us $d-$te[23] them:

$$d\,|1\rangle = \begin{pmatrix} -i\,\frac{1+\cos\vartheta}{2}\, e^{-i\varphi} \\ 0 \\ i\,\frac{1-\cos\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\varphi + \begin{pmatrix} -\frac{\sin\vartheta}{2}\, e^{-i\varphi} \\ \frac{\sqrt{2}\cos\vartheta}{2} \\ \frac{\sin\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\vartheta \tag{4.22a}$$

$$d\,|0\rangle = \begin{pmatrix} i\,\frac{\sqrt{2}\sin\vartheta}{2}\, e^{-i\varphi} \\ 0 \\ i\,\frac{\sqrt{2}\sin\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\varphi + \begin{pmatrix} -\frac{\sqrt{2}\cos\vartheta}{2}\, e^{-i\varphi} \\ -\sin\vartheta \\ \frac{\sqrt{2}\cos\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\vartheta \tag{4.22b}$$

$$d\,|-1\rangle = \begin{pmatrix} -i\,\frac{1-\cos\vartheta}{2}\, e^{-i\varphi} \\ 0 \\ i\,\frac{1+\cos\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\varphi + \begin{pmatrix} \frac{\sin\vartheta}{2}\, e^{-i\varphi} \\ -\frac{\sqrt{2}\cos\vartheta}{2} \\ -\frac{\sin\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\vartheta \tag{4.22c}$$

Finally, for example the case $m = 1$ gives:

$$\begin{aligned} \mathcal{A}_1 &= i\,\langle 1|\, d\,|1\rangle = \\ &= i\left(\frac{1+\cos\vartheta}{2}\, e^{i\varphi},\ \frac{\sqrt{2}\sin\vartheta}{2},\ \frac{1-\cos\vartheta}{2}\, e^{-i\varphi}\right)\cdot\begin{pmatrix} -i\,\frac{1+\cos\vartheta}{2}\, e^{-i\varphi} \\ 0 \\ i\,\frac{1-\cos\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\varphi + \\ &+ i\left(\frac{1+\cos\vartheta}{2}\, e^{i\varphi},\ \frac{\sqrt{2}\sin\vartheta}{2},\ \frac{1-\cos\vartheta}{2}\, e^{-i\varphi}\right)\cdot\begin{pmatrix} -\frac{\sin\vartheta}{2}\, e^{-i\varphi} \\ \frac{\sqrt{2}\cos\vartheta}{2} \\ \frac{\sin\vartheta}{2}\, e^{i\varphi} \end{pmatrix} d\vartheta\ = \\ &= \left(\left(\frac{1+\cos\vartheta}{2}\right)^2 - \left(\frac{1-\cos\vartheta}{2}\right)^2\right) d\varphi + f(\vartheta)\, d\vartheta = \\ &= \cos\vartheta\, d\varphi + f(\vartheta)\, d\vartheta \end{aligned} \tag{4.23}$$

where the face of $f(\vartheta)$ is not important for us, because the last step is d-ting again to obtain the curvature and this second term of (4.23) then becomes identically zero: $d\,(f(\vartheta)d\vartheta) = f(\vartheta)_{,\vartheta}\, d\vartheta\wedge d\vartheta = 0$. Therefore $\Omega_1 = d\mathcal{A}_1 = -\sin\vartheta\, d\vartheta\wedge d\varphi = -\,\omega_g$. The two residual computations are not necessary, because one uncovers nothing new. To sum up, one really gets $\Omega_0 = 0 = 0\,\omega_g$ and $\Omega_{-1} = \sin\vartheta\, d\vartheta\wedge d\varphi = +\,\omega_g$.

[21] We will deal with it in the next chapter.

[22] Can be found in [2] too.

[23] ... has nothing to do with date, though.

In the second approach via spin composition, we need to know Clebsch- Gordan coefficients again. Following (4.15) and steps bellow, we can directly write down all 3 eigenstates as a appropriate composition of two $1/2-$ spins. Note, that 2 of 3 are trivial, since $m_1+m_2=1$ only if $m_1=m_2=1/2$ and $m_1+m_2=-1$ only if $m_1=m_2=-1/2$:

$$|1\rangle \equiv |1,1\rangle = |1/2,1/2\rangle\,|1/2,1/2\rangle = |\uparrow\uparrow\rangle \tag{4.24a}$$

$$|0\rangle \equiv |1,0\rangle = \frac{1}{\sqrt{2}}\,(|1/2,1/2\rangle|1/2,-1/2\rangle + |1/2,-1/2\rangle|1/2,1/2\rangle) = \\ = \frac{1}{\sqrt{2}}\,(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{4.24b}$$

$$|-1\rangle \equiv |1,-1\rangle = |1/2,-1/2\rangle\,|1/2,-1/2\rangle = |\downarrow\downarrow\rangle \tag{4.24c}$$

We need to calculate only 2 of them, because the case (4.24b) can be obtained from (4.17a)- (4.17b) by a sign- swap in (4.16):

$$|1,1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix}\cos\,\frac{\vartheta}{2}\,e^{-i\varphi/2}\\ \sin\,\frac{\vartheta}{2}\,e^{i\varphi/2}\end{pmatrix} \otimes \begin{pmatrix}\cos\,\frac{\vartheta}{2}\,e^{-i\varphi/2}\\ sin\,\frac{\vartheta}{2}\,e^{i\varphi/2}\end{pmatrix} = \\ = \left(\cos^2\,\frac{\vartheta}{2}\,e^{-i\varphi}, \frac{1}{2}\sin\,\vartheta, \frac{1}{2}\sin\,\vartheta, \sin^2\,\frac{\vartheta}{2}\,e^{i\varphi}\right)^T \tag{4.25a}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\,\left(-e^{-i\varphi}\,\sin\,\vartheta, \cos(2\vartheta), -\cos(2\vartheta), e^{i\varphi}\,\sin\,\vartheta\right)^T \tag{4.25b}$$

$$|1,-1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix}-\sin\,\frac{\vartheta}{2}\,e^{-i\varphi/2}\\ \cos\,\frac{\vartheta}{2}\,e^{i\varphi/2}\end{pmatrix} \otimes \begin{pmatrix}-\sin\,\frac{\vartheta}{2}\,e^{-i\varphi/2}\\ \cos\,\frac{\vartheta}{2}\,e^{i\varphi/2}\end{pmatrix} = \\ = \left(\sin^2\,\frac{\vartheta}{2}\,e^{-i\varphi}, -\frac{1}{2}\,\sin\,\vartheta, -\frac{1}{2}\,\sin\,\vartheta, \cos^2\,\frac{\vartheta}{2}\,e^{i\varphi}\right)^T \tag{4.25c}$$

To move further, we need $d-$te theese eigenstates. Let us demontrate it on the (4.25c) case[24].

So, let us do some maths[25]:

$$\mathcal{A}_{-1} = i\,\langle\downarrow\downarrow|\,d\,|\downarrow\downarrow\rangle = \\ = i\,\left(\sin^2\,\frac{\vartheta}{2}\,e^{i\varphi}, -\frac{1}{2}\,\sin\,\vartheta, -\frac{1}{2}\,\sin\,\vartheta, \cos^2\,\frac{\vartheta}{2}\,e^{-i\varphi}\right)\cdot\begin{pmatrix}-i\,\sin^2\,\frac{\vartheta}{2}\,e^{-i\varphi}\\ 0\\ 0\\ i\,\cos^2\,\frac{\vartheta}{2}\,e^{i\varphi}\end{pmatrix} d\varphi + \\ + i\,\left(\sin^2\,\frac{\vartheta}{2}\,e^{i\varphi}, -\frac{1}{2}\,\sin\,\vartheta, -\frac{1}{2}\,\sin\,\vartheta, \cos^2\,\frac{\vartheta}{2}\,e^{-i\varphi}\right)\cdot\begin{pmatrix}\sin\,\vartheta\,e^{-i\varphi}\\ -\frac{1}{2}\,\cos\vartheta\\ -\frac{1}{2}\,\cos\vartheta\\ -\sin\,\vartheta\,e^{i\varphi}\end{pmatrix} d\vartheta\; = \\ = \left(\sin^4\,\frac{\vartheta}{2} - \cos^4\,\frac{\vartheta}{2}\right) d\varphi + f(\vartheta)\,d\vartheta = -\cos\,\vartheta\,d\varphi + f(\vartheta)\,d\vartheta \tag{4.26}$$

and eventually $\Omega_{-1} = d\,\mathcal{A}_{-1} = \sin\,\vartheta\,d\vartheta\wedge d\varphi = \omega_g = -m\,\omega_g$ for $m=-1$.

[24] In contrast to the previous section, where the $|1\rangle$- state was done explicitly.

[25] Eventually... :)

To sum this section up, it seems, that both ways of finding the Berry's curvature[26] are straightforward and lead directly to a happy-end. However, in a general case (that we are on to) is the situation complicated: is it possible to find a closed form for eigenstates of a spin operator in a general direction? Can one explicitly compute the tensor product of a general number of $s = 1/2$ spins? Let us now deal with it properly.

[26] Or better: proving, that in particular situations it has the form it should have.

5 Berry would be proud!

5.1 Rotating spins

As we mentioned at the end of the previous section, we would like to deal with eigenstates of spin operator in general direction for arbitrary spin s. However, the method sketched above is rather instructive than useful.That is why we need something else. The idea we will use is simple: instead of "rotating operators" (evaluating eigenstates of operators in general direction) we will directly rotate eigenstates to get the result, we want. How does it work?

Let us start with a state of a particle with the total spin s and its z- projection m, denoted $|s,m\rangle$. If we now take a state oriented in z- direction with total spin s, z-projection is necessary of values $\{-s,\dots,-1,0,1,\dots s\}$. The corresponding vector representation is $\left\{\begin{pmatrix}1\\0\\\vdots\end{pmatrix},\dots,\begin{pmatrix}0\\\vdots\\1\end{pmatrix}\right\}$.

Next step to the state in general direction is to rotate this state. If we rotate $|s,m\rangle$ about the z axis by angle φ and then about new (rotated) y' axis by angle ϑ, we get a quantum state with total spin s, its z- projection m in the direction of a general unit vector $\vec{n}=(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)$.

Recall now generators of such rotations, that are well-known spin operators $\hat{s}_i$. In other words, if we want to rotate a ket- state $|\psi\rangle$ about an arbitrary axis $\vec{n}$ by angle α, we need to act on this state with an exponential operator $e^{-i\alpha\vec{n}\cdot\vec{s}}$:

$$|\psi'\rangle_{\vec{n}}=e^{-i\,\alpha\,\vec{n}\cdot\vec{s}}\,|\psi\rangle \tag{5.1}$$

As mentioned above, we deal with these 2 rotations:

$$\hat{R}_z(\varphi)=e^{-i\,\varphi\,\hat{s}_z} \tag{5.2a}$$

$$\hat{R}_{y'}(\vartheta)=e^{-i\,\vartheta\,\hat{s}_{y'}}=e^{-i\,\varphi\,\hat{s}_z}e^{-i\,\vartheta\,\hat{s}_y}e^{i\,\varphi\,\hat{s}_z} \tag{5.2b}$$

and finally the operator rotating from z-axis to axis in general position is

$$\hat{R}_{\vec{n}}(\varphi,\vartheta)=\hat{R}_{y'}(\vartheta)\hat{R}_z(\varphi)=e^{-i\,\varphi\,\hat{s}_z}e^{-i\,\vartheta\,\hat{s}_y} \tag{5.3}$$

It is important to remind us, that matrix of this operator is unitary, which means, that in our basis the operator has matrix, that satisfies $RR^\dagger=RR^*=1_n$, or in other words $R^\dagger=R^{-1}$. Formally, at this moment we could write our Berry's connection in terms of these matrices (note, that the state $|s,m\rangle$ itself does not depend on φ,ϑ):

$$\mathcal{A}=i\,\langle s,m|\,R^\dagger(\vartheta,\varphi)\,dR(\vartheta,\varphi)\,|s,m\rangle=i\,\langle s,m|\,R^{-1}(\vartheta,\varphi)\,dR(\vartheta,\varphi)\,|s,m\rangle \tag{5.4}$$

but as you can see, it did not help us too much.

To move further, let us deal with the eigenstate in general dierction calculated via (5.3):

$$|s,m\rangle_{\vec{n}}=e^{-i\,\varphi\,\hat{s}_z}e^{-i\,\vartheta\,\hat{s}_y}\,|s,m\rangle_{\vec{z}} \tag{5.5}$$

First of all, the matrix of the z- operator is diagonal, e.g. $s_z=diag(-s,\dots,s)$ having $(2s+1)$ values on its dianogal. Then the whole operator of the rotation about z axis

is diagonal too, having exponentials of the eigenvalues of the dianogal:

$$e^{-i\varphi s_z} = \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!} \begin{pmatrix} s & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -s \end{pmatrix}^k = \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!} \begin{pmatrix} s^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (-s)^k \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!} s^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!} (-s)^k \end{pmatrix} = \begin{pmatrix} e^{-i\varphi s} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-i\varphi(-s)} \end{pmatrix} \tag{5.6}$$

The last thing we need to know, is to evaluate (in a similar way) elements of the second rotation operator (about y- axis). This is, however, a serious problem. The matrix of this operator is not diagonal, which causes problem in evaluating its exponential.

Namely, recall the lowering and raising operators $s_\pm$, that act on eigenstates of s_z and s^2 ($|s,m\rangle$) in a well-known way:

$$s_\pm |s,m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\, |s, m\pm 1\rangle \tag{5.7}$$

where $s_\pm = s_x \pm i s_y$. This yields for an (k,l) element of the matrix of s_y following relation:[27]

$$(s_y)_{k\,l} = \frac{1}{2i}\, (\delta_{k,\,l+1} - \delta_{k+1,\,l})\, \sqrt{(s+1)(k+l-1) - kl} \tag{5.8}$$

We can see, that the matrix has only off-diagonal elements lying near to diagonal. To avoid computing exponential of such a matrix[28], let us introduce the *Wigner $\mathcal{D}$-matrices.*

It is true, that any arbitrary rotation can be made up from 3 particular rotations defined by Euler angles:
- about z axis by α
- about (new) y' axis by β
- about (new) z'' axis by γ

Using arguments from the beginning of this section one has the corresponding rotation operator:

$$R(\alpha,\beta,\gamma) = e^{-i\,\alpha\,\hat{s}_z}\, e^{-i\,\beta\,\hat{s}_y}\, e^{-i\,\gamma\,\hat{s}_y} \tag{5.9}$$

It is important, that this operator does not change s-value of the state it acts on. That is why we can express the result in terms of $|s,m'\rangle$ states "mixed up" by a particular matrix, that is the Wigner $\mathcal{D}$-matrix:

$$R(\alpha,\beta,\gamma)|s,m\rangle = \sum_{m'=-s}^{s} \mathcal{D}^s_{m',\,m}(\alpha,\beta,\gamma)\, |s,m'\rangle \tag{5.10}$$

[27] Compare with (4.18) for the case $s = 1$.

[28] Computing exponential of the spin matrix is generally a difficult, but frequently treated topic. For example, it can be shown, that for spin $s = 1/2$ the following equality holds: $exp\big(-\frac{i}{2}\,\alpha(\vec{\sigma}\cdot\vec{n})\big) = \mathbf{1}_2\,\cos(\alpha/2) - i\,(\vec{\sigma}\cdot\vec{n})\,\sin(\alpha/2)$, that can be used to get the eigenstate in general direction by using this formula 2-times: about z by φ and about y by ϑ. However, transition to higher spins is nontrivial. For example, in case $s = 1$ (that we actually secretly used to find eigenstates of our 3×3 matrix instead of direct computation of eigenvectors) one gets (so called Rodrigues formula) $exp\,(-i\,\alpha\,(\vec{s}\cdot\vec{n})) = \mathbf{1}_3 + i\,(\vec{s}\cdot\vec{n})\,\sin\,\alpha + (\vec{s}\cdot\vec{n})^2(\cos\,\alpha - 1)$. A very exhaustive calculation on this topic can be found in [6], where a closed formula for general spin is derived. However, it is of an awkward form, that is not very useful for us.

This equality acts as a definition of Wigner matrices. Thanks to (5.9) we see, that when dealing with rotation about z axis (α, γ), we have to acts with exponential of s_z on its eigenstates $|s, m'\rangle$, that returns (in similar way to (5.6)) the eigenvalue to the exponent. Therefore the only nontrivial part is (as awaited) associated with the y-rotation:

$$\mathcal{D}^s_{m',m}(\alpha, \beta, \gamma) = e^{-i\,m'\alpha}\, e^{-i\,m\gamma}\, d^s_{m',m}(\beta) \tag{5.11}$$

that is definition of so-called Wigner (small) $d-$matrix, giving

$$d^s_{m',m}(\beta) = \langle s, m'|\, e^{-i\,\beta\, s_y}\, |s, m\rangle \tag{5.12}$$

Thanks to this discourse, we can rewrite former results in terms of Wigner matrices.

Namely, $R_{\vec{n}}(\varphi, \vartheta) = \mathcal{D}(\varphi, \vartheta, 0) = e^{-i\,m'\varphi}\, d^s_{m',m}(\vartheta)$ and consequently

$$|s, m\rangle_{\vec{n}} = \sum_{m'=-s}^{s} e^{-i\,m'\varphi}\, d^s_{m',m}(\vartheta) |s, m'\rangle_{\vec{z}} \tag{5.13}$$

If we now use the vector representation of the eigenstates from the beginning of this section, every component of the final vector is one term from the sum:

$$|s, m\rangle_{\vec{n}(\varphi,\vartheta)} = \begin{pmatrix} d^s_{s,m}(\vartheta)\, e^{-i\,\varphi\, s} \\ d^s_{(s-1),m}(\vartheta)\, e^{-i\,\varphi\,(s-1)} \\ \vdots \\ d^s_{-s,m}(\vartheta)\, e^{i\,\varphi\, s} \end{pmatrix} \tag{5.14}$$

what means, that φ and ϑ are separated. That has some benefits in computation, as we will see. To move further, we would like to d-te this expression. On this purpose, we need an explicit formula for $d-$matrix. Thank God[29] Eugene Wigner left us this in his work about matrices from 1927[30]. Note, that we use the original convention, in which elements are all real:

$$d^j_{mm'}(\vartheta) = \sqrt{(j+m')!\,(j-m')!\,(j+m)!\,(j-m)!}\ \cdot$$
$$\cdot \sum_s \frac{(-1)^{m'-m+s}}{(j+m-s)!\,s!\,(m'-m+s)!\,(j-m'-s)!} \left(\cos\,\frac{\vartheta}{2}\right)^{2j+m-m'-2s} \left(\sin\,\frac{\vartheta}{2}\right)^{m'-m+2s} \tag{5.15}$$

Where the sumation runs though all such values s, that every factorial is well-defined.

This all yields:

$$\begin{aligned} \mathcal{A} &= i\,\langle s, m|\, d\, |sm\rangle = \\ &= i\,\left(d^s_{s,m}(\vartheta)\, e^{i\varphi s}, \ldots, d^s_{-s,m}(\vartheta)\, e^{-i\varphi s}\right) \cdot \begin{pmatrix} -i\,s\, d^s_{s,m}(\vartheta)\, e^{-i\varphi s}\, d\varphi + \left(d^s_{s,m}(\vartheta)\right)_{,\vartheta}\, e^{-i\varphi s}\, d\vartheta \\ \vdots \\ i\,s\, d^s_{-s,m}(\vartheta)\, e^{i\varphi s}\, d\varphi + \left(d^s_{-s,m}(\vartheta)\right)_{,\vartheta}\, e^{i\varphi s}\, d\vartheta \end{pmatrix} \\ &= \left(\sum_{k=-s}^{s} k\,\left(d^s_{k,m}(\vartheta)\right)^2\right) d\varphi + f(\vartheta)\, d\vartheta \end{aligned} \tag{5.16}$$

[29] ...or better unfortunatelly when seeing its form

[30] We deal in our thesis only with spin. When computing the total angular momentum being a sum of spin angular momentum and orbital angular momentum $j = s + l$, the matrix gets j-index instead of s without change of its function.

with second term in a form, that will not contribute to the value of Berry's curvature.

The problem, we are facing, is to deal with the first term. On this purpose, we need to make us familiar with a crucial identity, that holds for all Wigner matrices[31]:

$$\sum_{k=-j}^{j} d^j_{m\,k}(\beta_1)\, d^j_{k\,m'}(\beta_2)\, e^{-ik\varphi} = e^{-im\alpha}\, d^j_{m\,m'}(\beta)\, e^{-im'\gamma} \tag{5.17}$$

provided

$$\cot\alpha = \cos\beta_1\, \cot\varphi + \cot\beta_2\, \frac{\sin\beta_1}{\sin\varphi} \tag{5.18a}$$

$$\cos\beta = \cos\beta_1\, \cos\beta_2 - \sin\beta_1\, \sin\beta_2\, \cos\varphi \tag{5.18b}$$

$$\cot\gamma = \cos\beta_2\, \cot\varphi + \cot\beta_1\, \frac{\sin\beta_2}{\sin\varphi} \tag{5.18c}$$

To move on, let us differentiate equation (5.17) with respect to φ (the sum to the left is finite, what means, that the sum is differentiable). Note, that all Euler angles to the right are implicit functions of φ: $\alpha = \alpha(\varphi)$, etc. This yields:

$$\begin{aligned}\sum_{k=-j}^{j} k\, d^j_{m\,k}(\beta_1)\, d^j_{k\,m'}(\beta_2)\, e^{-ik\varphi} &= m\, e^{-im\alpha}\, d^j_{m\,m'}(\beta)\, e^{-im'\gamma}\cdot\frac{d\alpha}{d\varphi}\\ &\quad + i\, e^{-im\alpha-im'\gamma}\,\frac{d\, d^j_{m\,m'}(\beta)}{d\beta}\cdot\frac{d\beta}{d\varphi}\\ &\quad + m'\, e^{-im\alpha}\, d^j_{m\,m'}(\beta)\, e^{-im'\gamma}\cdot\frac{d\gamma}{d\varphi}\end{aligned} \tag{5.19}$$

Thanks to (5.18a-c) one has

$$\frac{d\alpha}{d\varphi} = \frac{sin^2\alpha}{sin^2\varphi}\cdot(\cos\beta_1 + \cot\beta_2\, \sin\beta_1\, \cos\varphi) \tag{5.20a}$$

$$\frac{d\beta}{d\varphi} = -\,\frac{\sin\beta_1\, \sin\beta_2\, \sin\varphi}{\sin\beta} \tag{5.20b}$$

$$\frac{d\gamma}{d\varphi} = \frac{sin^2\gamma}{sin^2\varphi}\cdot(\cos\beta_2 + \cot\beta_1\, \sin\beta_2\, \cos\varphi) \tag{5.20c}$$

We will not insert it into (5.19) for the sake of beauty. In order to get the left-hand side of (5.19) to the form in (5.16), we well set[32]

$$\beta_2 = -\beta_1 = \vartheta,\ \ \gamma = 0,\ \ m' = m \tag{5.21}$$

Using one of the properties of Wigner matrices ($d^j_{k\,l}(x) = d^j_{l\,k}(-x)$) we immediatelly get the desired expression. What does it do on the right-hand side? First of all, limits of derivatives give us the following (note again, that $\alpha = \alpha(\varphi)$ etc. from (5.18a-c), that has to be taken into account here):

$$\alpha_{,\varphi} = \cos\vartheta/2,\ \ \beta_{,\varphi} = \sin\vartheta,\ \ \gamma_{,\varphi} = \cos\vartheta/2 \tag{5.22}$$

[31] Can be found in [20], p. 77

[32] More preciselly, we will do $\gamma \to 0^+$. The reason is, that in (5.20b), there exists no two-sided limit as γ tends to 0, making change of sign when going from the other side ($\gamma \to 0^-$).

Moreover, another useful expression for the derivative od d-matrix is available (again in [20], p. 82)

$$\frac{d\, d^j_{m\,m'}(\beta)}{d\beta} = \frac{m' - m\cos\beta}{\sin\beta}\, d^j_{m\,m'}(\beta) - \sqrt{(j+m)(j-m+1)}\, d^j_{m-1,m'}(\beta) \tag{5.23}$$

and this all gives the right-hand side of (5.19) the following face:

$$\begin{aligned} \ldots &= m\, e^{-im(\alpha+\gamma)}\, d^j_{m\,m}(\beta)\, \frac{\cos\vartheta}{2} \\ &+ i\, e^{-im(\alpha+\gamma)} \left(m\, \frac{1-\cos\beta}{\sin\beta}\, d^j_{m\,m}(\beta) - \sqrt{(j+m)(j-m+1)}\, d^j_{m-1,m}(\beta) \right) \sin\vartheta \\ &+ m\, e^{-im(\alpha+\gamma)}\, d^j_{m\,m}(\beta)\, \frac{\cos\vartheta}{2} \end{aligned} \tag{5.24}$$

From (5.18a-c) using (5.21) one can see, that[33]

$$\alpha = \beta = \gamma = 0 \tag{5.25}$$

that extremly simplifies (5.24) - exponentials vanish and using the fact, that $d^j_{k\,l}(0) = \delta_{k\,l}$ we get

$$\ldots = m\, \frac{\cos\vartheta}{2} + 0 + m\, \frac{\cos\vartheta}{2} = m\, \cos\vartheta \tag{5.26}$$

and we are done[34]. The Berry's connection from (5.16) is now

$$\mathcal{A} = m\, \cos\vartheta\, d\varphi + f(\vartheta)\, d\vartheta \tag{5.27}$$

Finally d-ting conjures the Berry's curvature in the expected form:

$$\Omega_m = -m\, \sin\vartheta\, d\varphi \wedge d\vartheta = -m\, \omega_g \tag{5.28}$$

We have undergone a bit difficult, yet interesting way to obtain the result for the Berry's curvature using the whole machinery of Wigner matrices acting as crucial objects in rotating spin states.

5.2 Combining spins

Another approach is to be discussed here, that has been allready sketched in the chapter 4 when dealing with particular values of spin $s = 0,\ 1/2,\ 1$. It is based on the spin composition[35].

We are pretty familiar with eigenstates of the spin operator for spin $1/2$ and we would like to use it in order to get a general expression for the state $|s, m\rangle$. We will

[33] In [20], a more general proposition can be found: if $\varphi = 0$ and $\beta_1 + \beta_2 < \pi$, then $\alpha = \gamma = 0$ and $\beta = \beta_1 + \beta_2$.

[34] As a defense for the 30th footnote we can say, that it is unimportant, whether we take the limit $\gamma \to 0^+$ or $\to 0^-$, because terms in brackets of the second term in (5.28) cancell this whole term irrespective of the sign of $\sin\vartheta$ from the derivative $\beta_{,\varphi}$.

[35] We will follow section 4.10 from an excellent book from M. Chaichian and R. Hagedorn dedicated to angular momentum, [4].

deal with unrotated states (e.g. states oriented in the z-axial direction):

$$\hat{s}_+ |+\rangle = 0 \tag{5.29a}$$
$$\hat{s}_+ |-\rangle = |+\rangle \tag{5.29b}$$
$$\hat{s}_- |+\rangle = |-\rangle \tag{5.29c}$$
$$\hat{s}_- |-\rangle = 0 \tag{5.29d}$$
$$\hat{s}_z |+\rangle = \frac{1}{2} |+\rangle \tag{5.29e}$$
$$\hat{s}_z |-\rangle = -\frac{1}{2} |-\rangle \tag{5.29f}$$

where the meaning of symbols is taken from previous sections.

Let us now construct a system of $2p$ such single states of spin $1/2$. Each single state will be provided by a label ($i = 1, 2, \ldots, 2p$):

$$|(1, 2, \ldots, i)_+ (i+1, \ldots, 2p)_-\rangle = |+\rangle^1 \otimes \cdots \otimes |+\rangle^i \otimes |-\rangle^{i+1} \otimes \cdots \otimes |-\rangle^{2p} \tag{5.30}$$

what is a system consisting of i spins with z-projection $+1/2$ and $2p - i$ spins with z-prjection $-1/2$. It is clear, that any interchange of single states in the tensor product within the section of "up"- states or "down"- states will not change properties of the whole system. We have $i!$ possibilities to swap states within the "up"-states and then $(2p-i)!$ permutations in the other group. Moreover, we have $(2p)!$ permutations paying no attention in which set of states we are. Therefore, there are $\frac{(2p)!}{i!(2p-i)!} = \binom{2p}{i}$ different state- classes, that contain systems varying in permutations within the two sets of single states. If we take systems from the same class, their scalar product is equal to 1 (they are normalized), in case of different clases is equal to 0 (they are orthogonal to each other). To sum it up, a typical state consisting of i up-states and $(2p - i)$ down-states is of the form $|\,(\sigma_1, \ldots, \sigma_i)_+ (\sigma_{i+1}, \ldots, \sigma_{2p})_-\rangle$, where σ_i stands for permutation of i.

Let us now try to compute the z-projection of spin of the whole system or- in other words- let us construct the $\hat{s}_z$ operator for this system and then act with it on the system state. Remembering the procedure in the 4-th chapter (above (4.15)) we will construct an operator (capital letters will from now on stand for operators of the whole system):

$$\hat{S}_z = \hat{s}_z^1 + \cdots + \hat{s}_z^{2p} \tag{5.31}$$

such, that the i-th operator will act only on the i-th state of the tensor product:

$$\hat{s}_z^i = \hat{1}^1 \otimes \cdots \otimes \hat{1}^{i-1} \otimes \hat{s}_z \otimes \hat{1}^{i+1} \otimes \cdots \otimes \hat{1}^{2p} \tag{5.32}$$

Then we get

$$\begin{aligned}
&\hat{S}_z |\,(\sigma_1, \ldots, \sigma_i)_+ (\sigma_{i+1}, \ldots, \sigma_{2p})_-\rangle = \\
&= \sum_{i=1}^{2p} \left(\hat{1}^1 \otimes \cdots \otimes \hat{1}^{i-1} \otimes \hat{s}_z \otimes \hat{1}^{i+1} \otimes \cdots \otimes \hat{1}^{2p}\right) |\,(\sigma_1, \ldots, \sigma_i)_+ (\sigma_{i+1}, \ldots, \sigma_{2p})_-\rangle = \\
&= \left(\frac{1}{2} \cdot i + \left(-\frac{1}{2}\right) \cdot (2p - i)\right) |\,(\sigma_1, \ldots, \sigma_i)_+ (\sigma_{i+1}, \ldots, \sigma_{2p})_-\rangle = \\
&= (i - p) |\,(\sigma_1, \ldots, \sigma_i)_+ (\sigma_{i+1}, \ldots, \sigma_{2p})_-\rangle
\end{aligned} \tag{5.32}$$

And denoting $i - p = m$ yields

$$\hat{S}_z|\left(\sigma_1,\ldots,\sigma_{p+m}\right)_+\left(\sigma_{p+m+1},\ldots,\sigma_{2p}\right)_-\rangle = m|\left(\sigma_1,\ldots,\sigma_{p+m}\right)_+\left(\sigma_{p+m+1},\ldots,\sigma_{2p}\right)_-\rangle \tag{5.33}$$

so every such a state of the system defined by a permutation of single states with $p+m$ up-states and $p-m$ down states is an eigenstate of $\hat{S}_z$ with eigenvalue m.

To move further, we need to know the result of acting of $\hat{S}_+$ and $\hat{S}_-$ on this state. We can similarly build it together using $\hat{s}^i_+ = \hat{1}^1 \otimes \cdots \otimes \hat{1}^{i-1} \otimes \hat{s}_+ \otimes \hat{1}^{i+1} \otimes \cdots \otimes \hat{1}^{2p}$, but we need to specify, how they act on the system state first. It is clear (5.29a) that if $i \in (\sigma_1,\ldots,\sigma_{p+m})$, then the result is zero. Therefore i must be from $(\sigma_{p+m+1},\ldots,\sigma_{2p})$. If i really occurs in $(\sigma_{p+m+1},\ldots,\sigma_{2p})$, then the system changes in the i-th compoud due to (5.29a) like $\hat{s}^i_+|-\rangle^i = |+\rangle^i$ (otherwise the result is zero again). Consequently the "up" set of single states gets a new member, the mentioned i-th one:

$$\hat{s}^i_+|\left(\sigma_1,\ldots,i,\ldots,\sigma_{p+m}\right)_+\left(\sigma_{p+m+1},\ldots,\sigma_{2p}\right)_-\rangle = 0 \tag{5.34a}$$

$$\hat{s}^i_+|\left(\sigma_1,\ldots,\sigma_{p+m}\right)_+\left(\sigma_{p+m+1},\ldots,i,\ldots,\sigma_{2p}\right)_-\rangle =$$

$$=|\left(\sigma_1,\ldots,\sigma_{p+m},n\right)_+\left(\sigma_{p+m+1},\ldots,\sigma_{2p}\right)_-\rangle \tag{5.34b}$$

Note, that the final state has no more m as the eigenvalue of $\hat{S}_z$ - it becomes $m+1$, because we cutted one down-state and added one up-state.

This happens to all such operators $\hat{s}^i_+$: they produce either zero or one of states with $m+1$ as the eigenvalue of $\hat{S}_z$. Hence the desired $\hat{S}_+$ will generate a sum of $p-m$ terms, every one being a different state with of $m+1$ as the eigenvalue of $\hat{S}_z$. This puts us to the following idea: instead of one permutated system state let us take a sum of all such states, that can be made from an arbitrary one (for example the one from (5.30)) by a permutation σ:

$$|\widehat{s,m}\rangle = \frac{1}{(2s)!}\sum_\sigma |\left(\sigma_1,\ldots,\sigma_{s+m}\right)_+\left(\sigma_{s+m+1},\ldots,\sigma_{2s}\right)_-\rangle \tag{5.35}$$

summed over all possible permutations. However, this state is not normalised yet (labelled by the hat). The norm can be found very quickly.

As we mentioned in the beginning of this section, we deal with $\frac{(2p)!}{(2p-i)!i!} = \frac{(2s)!}{(s+m)!(s-m)!}$ different state classes with states orthogonal to each other (when taking two states from different classes). Hence, we can make summation (int the sum from (5.35)) within the classes and then summing only representatives of these classes:

$$|\widehat{s,m}\rangle = \frac{(s+m)!(s-m)!}{(2s)!}\sum_{class}|representative\rangle \tag{5.36}$$

The very last step is the realisation, that states from different classes are mutually orthogonal (as allready mentioned). Then we get

$$\langle\widehat{s,m}|\widehat{s,m}\rangle = \left(\frac{(s+m)!(s-m)!}{(2s)!}\right)^2 \cdot \frac{(2s)!}{(s+m)!(s-m)!} \tag{5.37}$$

where the second factor (making it being squared) comes from the bra-state and the inverse comes from the summation. Hence

$$|s,m\rangle = \sqrt{\frac{(2s)!}{(s+m)!(s-m)!}}\,\frac{1}{(2s)!}\sum_\sigma |\left(\sigma_1,\ldots,\sigma_{s+m}\right)_+\left(\sigma_{s+m+1},\ldots,\sigma_{2s}\right)_-\rangle \tag{5.38}$$

is the desired state given by a totally symmetrized direct (= tensor) product of $s = 1/2$ up- and down- states. To prove it, we will compute $\hat{S}_z|s,m\rangle$ and $\hat{S}_+|s,m\rangle$ (computation of $\hat{S}_-$ is analogous and does not need to be veryfied).

$$\begin{aligned}
\hat{S}_z|s,m\rangle &= \hat{S}_z\left(c_{s\,m}\frac{1}{(2s)!}\sum_\sigma |\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle\right) = \\
&= c_{s\,m}\,\frac{1}{(2s)!}\sum_\sigma \hat{S}_z|\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= c_{s\,m}\frac{1}{(2s)!}\sum_\sigma m|\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= m\left(c_{s\,m}\frac{1}{(2s)!}\sum_\sigma |\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle\right) = m\,|s,m\rangle
\end{aligned} \tag{5.39}$$

where $c_{s\,m}$ is the normalisation constant. Eventually

$$\begin{aligned}
\hat{S}_+|s,m\rangle &= c_{s\,m}\,\frac{1}{(2s)!}\sum_\sigma \hat{S}_+|\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= \frac{c_{s\,m}}{(2s)!}\sum_\sigma\sum_{i=1}^{2s}\hat{s}_+^i|\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= \frac{c_{s\,m}}{(2s)!}\sum_\sigma\sum_{i=s+m+1}^{2s}|\,(\sigma_1,\ldots,\sigma_{s+m},i)_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= \frac{c_{s\,m}}{(2s)!}\sum_{i=s+m+1}^{2s}\sum_\sigma|\,(\sigma_1,\ldots,\sigma_{s+m},i)_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= (s-m)\,\frac{c_{s\,m}}{(2s)!}\sum_\sigma|\,(\sigma_1,\ldots,\sigma_{s+m})_+\,(\sigma_{s+m+1},\ldots,\sigma_{2s})_-\rangle = \\
&= \sqrt{(s-m)(s+m+1)}\,|s,m+1\rangle
\end{aligned} \tag{5.40}$$

where we allready took into account the normalisation of the $|s,m+1\rangle$ state. The coefficient in front of it is familiar so far (see (5.7) when defining matrix operators $s_\pm$).

To sum this section up, we really constructed the $|s,m\rangle$ state by combining spin $s = 1/2$ states, that has required behaviour when letting operators act on it. Note, that we build the whole system-state from single states oriented in z-direction, that is only one special case of the general direction given by ϑ,φ. Now the step would be to rotate this state by particular angles about particular axes. For example, if we rotated the state by ϑ about y-axis, then the single states will transform in the following way:

$$\hat{R}_y(\vartheta)\,|+\rangle = \cos\vartheta\,|+\rangle + \sin\vartheta\,|-\rangle \tag{5.41a}$$

$$\hat{R}_y(\vartheta)\,|-\rangle = \cos\vartheta\,|-\rangle - \sin\vartheta\,|+\rangle \tag{5.41b}$$

Inserting these expressions in the permutated sum and expanding them using binomial theorem will immediatelly (after recollecting the terms with the same power of $\sin^a\vartheta\,\cos^b\vartheta$) give us the definition of Wigner d-matrix as displayed in (5.15). The aim is not to show it now, but only to underline the connection between these approaches[36].

[36]More detailed overview of this procedure can be found in Feynman lectures, [9].

Eventually, we have to underline, that the construction depicted in this section consists of using the same spin (single) states of spin $s = 1/2$ and then od deducting, what should the normalisation constant look like. Another way to build the general state $|s, m\rangle$ lies in (4.15), where the Clebsch- Gordan coefficients were introduced. It offers us the possibility to couple different states of even different spins. However, this is counter-productive: just imagine you want to compile spin state $|s, m\rangle$. Then you can couple states of various spins $s = 1/2, 1, 3/2, \ldots, s$ and even in various amounts. Then the concept using *only* $s = 1/2$ spins is just a small subset of many other available concepts. Therefore, we will not go into it.[37]

In order not to end this section in such a pessimistic way, we give a pro-argument for this idea, though. Recall once again the Wigner matrices. As we will see in a few moments, they represent an irreducible representation of rotations in the space of spin s (that is $(2s + 1)$ - dimensional). When we then form a reducible representation as a tensor product of two irreducible representations (given by matrices $\mathcal{D}^j$ and $\mathcal{D}^{j'}$), $\mathcal{D}^j \otimes \mathcal{D}^{j'}$, we can reduce it into a direct sum of another irreducible representations (all being invariant under rotations) using the following identity (for matrix elements):

$$\mathcal{D}^j_{k\,m}(\alpha, \beta, \gamma)\, \mathcal{D}^{j'}_{k'\,m'}(\alpha, \beta, \gamma) = \sum_{J=|j-j'|}^{j+j'} \sum_{M=-J}^{J} \sum_{K=-J}^{J} C^{j\,j'\,J}_{m\,m'}\, C^{j\,j'\,J}_{k\,k'}\, \mathcal{D}^J_{M\,K}(\alpha, \beta, \gamma) \tag{5.42}$$

what can be shortened into a very important formula from theory of group representations:

$$\mathcal{D}^j \otimes \mathcal{D}^{j'} = \mathcal{D}^{j+j'} \oplus \mathcal{D}^{j+j'-1} \oplus \cdots \oplus \mathcal{D}^{|j-j'|} \tag{5.43}$$

The only reason for displaying it here is to emphasize, that the way of construtcing $|s, m\rangle$ according to the main part of section (5.2) is not unique. We really *are* able to couple various spins, represent them as a direct sum of irreducible spin representations and then use them to build the general spin state. However, as mentioned, it is badly difficult.

[37] It is clear, that coupling of two spins is thanks to (4.15) a trivial problem - the only difficulty is to find the coefficient itself, but one has plenty of tables dedicated to this problem nowadays. However, when considering combination of more than 2 spins, it becomes a serious problem. One can proceed from 1 spin to another - just add a spin to a combination, that has allready been done. This is, however, only a sketch, that is really difficult, when performing exactly using proper mathematical concepts.

6 Advanced: Berry would be fascinated!

The last chapter of this thesis is dedicated to an advanced approach to the Berry's phase. It is based on using benefitial facts about Lie groups (being manifolds at the same time), their Lie algebras and representations. As this concept is a bit more demanding, we will explain basic ideas and terms from the group theory "along the way" giving definitions and descriptions of them[38]

We will start from the equation (5.4), that is the expression for the Berry's connection in terms of rotation matrices R[39]

$$\mathcal{A} = i \langle s, m | \, R^{-1}(\vartheta, \varphi) \, dR(\vartheta, \varphi) \, | s, m \rangle \tag{6.1}$$

We have to admit, that in this formula something is only partly true. The rotation matrices are generally matrices from $SU(2)$ group (that is subgroup of $GL(2, \mathbb{C})$, e.g. 2×2 matrices with complex values). That means the multiplication by them is only possible for 2-component columns (representing the state $|s, m\rangle$). There is only one value of spin, for which we represent its eigenstates as 2-compounded columns: $s = 1/2$ (see chapter 4). For other spins we need to represent these rotation matrices from $SU(2)$ by elements of the group $GL(2s + 1, \mathbb{C})$ in order rescue the validity of (6.1).

It is very important to say, that we did this (without mentioning it) in previous sections - we used representation of $SU(2)$ by $GL(2s+1, \mathbb{C})$ in all cases. It was the well-known irreducible representation of $SU(2)$ given by $\mathcal{D}$ matrices (to be more precise: the rows and columns of Wigner $\mathcal{D}$-matrices span the irreducible representations of the Lie algebras generated by operators of the total angular momentum j).

Therefore it is wise to correct the expression in the following way:

$$\mathcal{A} = i \langle s, m | \, \rho \left(R^{-1}(\vartheta, \varphi) \right) \, d\rho \left(R(\vartheta, \varphi) \right) \, | s, m \rangle \tag{6.2}$$

where $\rho \colon SU(2) \to GL(2s + 1, \mathbb{C})$ is the mentioned irreducible representation. We would now like to understand, what is hidden in the term $\rho \left(R^{-1}(\vartheta, \varphi) \right) \, d\rho \left(R(\vartheta, \varphi) \right)$.

First of all, representation is an injective homomorphism, what allows us to rewrite the term[40]

$$\rho(g^{-1}) \, d\rho(g) = \rho(g^{-1} dg) \tag{6.3}$$

where $g = R(\vartheta, \varphi)$ stands for the rotation matrix - an element of group $G = SU(2)$ (symbol dg stands for exterior derivative of each component of matrix $R(\vartheta, \varphi)$, that is element $g \in SU(2)$[41]). The expression in brackets is very similar to the one for so-called cannonical 1-form $\theta = x^{-1} \, dx$ and this object would be crucial for this whole section. However, in order to trully understand how it works, we need to introduce the theory first.

We will start with definition of a left-invariant p-form. A p-form α is left-invariant, if it satisfies

$$L_g^* \alpha = \alpha, \quad \forall g \in G \tag{6.4}$$

[38] To get more precise insight about Lie groups and its representations, I recommend chapters 10 - 12 from [8], that we follow in this part of our thesis.

[39] Tagging equation is not injective ...

[40] This is true for $s \geq 1/2$, because for $s = 0$ the representation is not injective. However, for $s = 0$ we have $\rho = \hat{1}$, then $d\rho(g) = 0$ and the whole connection and curvature is zero. Hence there is no need to take this case into account.

[41] We operate with matrix groups from the beginning of the chapter.

where the left translation

$$\begin{aligned} L_g &: G \to G \\ & h \mapsto L_g h := gh \end{aligned} \tag{6.5}$$

is a diffeomorphism and its pull-back is the well-defined for every tensor field of type $\binom{p}{q}$[42].

As we are interested in representations of $SU(2)$ by matrices from $GL(2s+1,\mathbb{C})$, we would like to investigate left-invariant 1-forms on $G = GL(2s+1,\mathbb{C})$. Moreover, we can take $GL(2s+1,\mathbb{C})$ as $GL(n,\mathbb{R})$ for a particular value of n (note, that complex matrices can be realised by two real matrices - one for each real and imaginary part respectively, $n = 2s+1$).

If we provide the (vector) space of matrices from $GL(n,\mathbb{R})$ by a standard Weyl basis E^i_j (such that $(E^i_j)^k_l = \delta^i_l \delta^k_j$), then $\forall x \in GL(n,\mathbb{R})$: $x = x^i_j E^j_i$. In this basis the left translation by $A \in GL(n,\mathbb{R})$ becomes $x^i_j \mapsto (L_A x)^i_j = A^i_k x^k_j$. Then for an arbitrary 1-form $\alpha = \alpha^j_i(x) dx^i_j$ the condition for left-invariance yields

$$L^*_A \alpha = \alpha^i_k (A^a_b x^b_a)\, A^k_j\, dx^j_i \stackrel{!}{=} \alpha^i_j(x)\, dx^j_i \tag{6.6}$$

that can be satisfied by taking

$$\alpha^i_j(x)_C = C^i_k (x^{-1})^k_j \tag{6.7}$$

for arbitrary constant matrix $C^i_j = \alpha^i_j(\mathbb{I})$. This is only one step from finding basis for the space of left-invariant 1-forms on $GL(n,\mathbb{R})$. We only need to take $C^i_j = E^i_j$ and we get

$$\alpha(x)_{E^i_j} \equiv \hat{\alpha}^i_j(x) = (x^{-1})^i_k\, dx^k_j \tag{6.8}$$

where the hat distinguishes this base left-invariant 1-form from the component α^i_j of a general 1-form.

To move further, we would like to get this basis to any subgroup $G \subset GL(n,\mathbb{R})$. For this purpose, consider homomorphism $f\colon G \mapsto GL(n,\mathbb{R})$ of Lie groups and consider the left-invariant 1-form $\alpha(x) = \alpha^i_j(x)\, dx^j_i$ on $GL(n,\mathbb{R})$. Then $f^*\alpha(x)$ is left-invariant 1-form on G:

$$L^*_A \circ f^*\alpha(x) = (f \circ L_A)^* \alpha(x) = \alpha^i_k(Ax(z^\mu))\, A^k_l\, dx^l_j(z^\mu) = \alpha^i_j(x(z^\mu))\, dx^j_i(z^\mu) \tag{6.9}$$

where the homomorphism maps $f\colon z^\mu \mapsto x^i_j(z^\mu)$ coordinates z^μ in G to coordinates x^i_j in $GL(n,\mathbb{R})$. If we now take injection $j\colon G \to GL(n,\mathbb{R}), \quad z^\mu \mapsto x^i_j(z^\mu)$ as the desired homomorphism f, we get $j(G) \subset GL(n,\mathbb{R})$ an isomorphic copy of G in $GL(n,\mathbb{R})$ and the whole information about left-invariant 1-forms on G will we available from matrix $\alpha = x^{-1}dx$ (on $GL(n,\mathbb{R})$) by taking its pull-back to G:

$$\alpha_G = x^{-1}(z)\, dx(z) = j^*(x^{-1}\, dx) = j^*(\alpha_{GL(n,\mathbb{R})}) \tag{6.10}$$

It may seem that we deviated from our goal to describe the canonical 1-form in order to conclude the discourse about Berry's phase, but that is not true. Namely, if

[42]Note, that without being diffeomorphism, one have problem with pull-back of vectors.

we now define the canonical 1-form and verify that the left-invariant 1-form is directly what we are searching for, everything becomes brighter.

We will start immediatelly with definition of the canonical 1-form (a.k.a. Mauer-Cartan 1-form): Let G be a Lie group, $\mathcal{G}$ being its Lie algebra ($\mathcal{G} \equiv T_eG$, e.g. the Lie algebra is a tangent vector space of the identity element $e \in G$). Then a $\mathcal{G}$-valued cannonical 1- form $\theta \in \Omega^1(G, \mathcal{G})$ on G is defined by

$$\langle \theta(g), v \rangle := L_{g^{-1}*} v, \quad v \in T_gG \tag{6.11}$$

One can read this definition as follows: a vector from the tangent vector space of $g \in G$ can be mapped from g to e by an appropriate (unique) left translation[43]. Note, that this definition holds for any Lie group G.

If we now take the computed left-invariant 1-form

$$\theta = x^{-1}\, dx = (x^{-1}\, dx)^i_j E^j_i = (x^{-1})^i_k\, dx^k_j\, E^j_i \tag{6.12}$$

we can easily check, that it has all properties it should have to be the canonical 1-form on $GL(n, \mathbb{R})$.

First of all, we see in (6.12), that it is decomposed using the Weyl basis as a basis for the Lie algebra $gl(n, \mathbb{R})$ (that are real matrices $n \times n$ without any restriction) That is why the 1-form *is* $gl(n, \mathbb{R})$- valued.

To verify, that for θ the condition (6.11) holds, let us transform this condition into equivalent (differential) form:

$$\langle \theta, V_C \rangle = C \tag{6.13}$$

for V_C being a general left-invariant vector field on $GL(n, \mathbb{R})$ parametrised by constant matrix C, namely $V_C = x^i_k\, C^k_j\, \partial^j_i$ (then $L^*_A V_C = V_C$ for any $A \in GL(n, \mathbb{R})$). Then one gets

$$\begin{aligned} \langle \theta, V_C \rangle &= \langle (x^{-1})^i_k\, dx^k_j\, E^j_i, x^a_c\, C^c_b\, \partial^b_a \rangle = (x^{-1})^i_k\, E^j_i\, C^c_a\, x^a_c\, \langle dx^k_j, \partial^b_a \rangle = \\ &= (x^{-1})^i_k\, E^j_i\, C^c_a\, \delta^k_a \delta^b_j = (x^{-1})^i_k\, x^k_c\, C^c_j\, E^j_i = \mathbb{I}^i_c\, C^c_j\, E^j_i = C^i_j\, E^j_i = C \end{aligned} \tag{6.14}$$

To sum it up, θ is a canonical 1-form on $GL(n, \mathbb{R})$ having values in $gl(n, \mathbb{R})$.

Now, following the procedure of getting left-invariant 1-form from $GL(n, \mathbb{R})$ to $G \subset GL(n, \mathbb{R})$, let us consider again the injective homomorphism $j\colon G \to GL(n, \mathbb{R})$ mapping $z^\mu \mapsto x^i_j(z^\mu)$. Then the canonical 1-form on G is given by

$$\theta_G = j^*(\theta_{GL(n,\mathbb{R})}) = j^*(x^{-1}\, dx) = j^*((x^{-1})^i_k\, dx^k_j\, E^j_i) = (\theta^a_\mu\, dz^\mu) E_a \tag{6.15}$$

where E_a are base matrices spanning only the part $\mathcal{G} \subset gl(n, \mathbb{R})$. We can now see a problem in (6.3), where $\rho(g^{-1}\, dg)$ occurs, because the term is being represented.

However, after a few moments of trying to fix this problem, one comes to the suggestion, that the following equality is true[44]

$$\rho(g^{-1}\, dg) = \rho'(\theta_G) \tag{6.16}$$

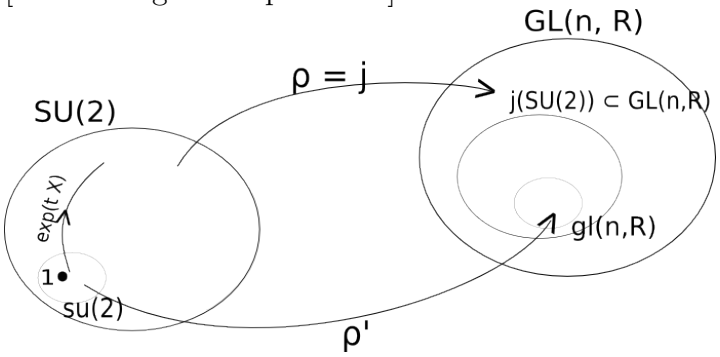
where ρ' stands for derived representation of Lie algebras[45]. To prove this, we only need to draw a picture and make us clear about where does the objects have their values.

[43] Recall, that $\langle \alpha, V \rangle \equiv \hat{1}(V; \alpha) = (dx^i \otimes \partial_i)(V; \alpha)$.

[44] As we mentioned, for $s = 0$ we have $\rho = \hat{1}$, then $\rho' = 0$ and $\rho'\theta = 0$, so the equality holds for $s = 0$. We will therefore assume again $s \geq 1/2$.

[45] If $\rho\colon G \to H$ is representation of Lie groups and the exponential map maps Lie algebras to Lie groups like $exp\colon \mathcal{G}, \mathcal{H} \to G, H$, then there exists a representation of their algebras $\rho'\colon \mathcal{G} \to \mathcal{H}$ such that $\rho(e^X) = e^{\rho'(X)}$.

Figure 2: Where is it? (Situation in our problem) [made using Inkscape editor]



As we can see, the pull-back $j^*(\theta)$ pulls the form from $GL(n,\mathbb{R})$ (valued in $gl(n,\mathbb{R})$) to $SU(2)$ (valued in $su(2)$), in terms of local coordinates z^μ of $su(2)$: $x^{-1}(z)dx(z) = \theta^a_\mu\, dz^\mu\, E_a$, where E_a is basis of $su(2)$ (subset of the basis E^i_j for the whole $gl(n,\mathbb{R})$). Then the derived representation ρ' represents this object back into $gl(n,\mathbb{R})$ (precisely $\rho'(su(2)) \subset gl(n,\mathbb{R})$), so it is of the face $\theta^a_\mu\, dz^\mu \hat{E}_a$, where the hat symbolizes the represented E_a base (again subset of E^i_j in the isomorphic copy of $SU(2)$ in $GL(n,\mathbb{R})$). As for the left-hand side of equation (6.16), we represent an object $g^{-1}\,dg$ directly to $GL(n,\mathbb{R})$, that is valued in $gl(n,\mathbb{R})$ too (see text bellow (6.14)), that is of the form $\theta^a_\mu\, dz^\mu\, E_a$. Again- only basis can be represented (leaving the rest untouched) and the represented one is, of course, the Weyl base (the one sufficient to span the isomorphic copy in $GL(n,\mathbb{R})$). The construction of the canonical 1-form is unique and here both "adepts" have desired properties of canonical 1-form, so it must be the same object. As a conclusion, (6.16) proves true.

The rest of our work is easy. If we know the canonical 1-form on $SU(2)$ (that would be the last untrivial part of this computation), we can continue from (6.16) in following way:

$$\rho'(\theta_{SU(2)}) = \rho'(e^a E_a) = e^a \rho'(E_a) \tag{6.17}$$

where e^a are left-invariant 1-forms on $SU(2)$ (see (6.12), where we directly took left-invariant 1-form as the canonical 1-form. The same situation is in this case). From the theory of irreducible representations of $SU(2)$ we know, that the represented base element is proportional to the matrix of operator of total angular momentum. In our case it yields

$$\rho'(E_a) = -i\,\hat{s}_a \tag{6.18}$$

with $\hat{s}_a$ being the spin operator in a-direction ($a \in (x,y,z)$). Hence the Berry's connection is

$$\mathcal{A} = i\,\langle s,m|\,\rho'(\theta)\,|s,m\rangle = e^a\,\langle s,m|\hat{s}_a|s,m\rangle = e^3\langle s,m|\hat{s}_z|s,m\rangle = m\,e^3 \tag{6.19}$$

because only the $\hat{s}_z$ operator leaves the $|s,m\rangle$ - state unchanged (contributing by factor m - its eigenvalue).

Let us therefore compute the left-invariant 1-form(s) on $SU(2)$. The most general matrix $A \in SU(2)$ is of the form

$$A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}, \quad |z|^2 + |w|^2 = 1, \quad z,w \in \mathbb{C} \tag{6.20}$$

that can be fulfilled by 3 Euler angles $\vartheta \in \langle 0,\pi\rangle$, $\varphi \in \langle 0,2\pi\rangle$ and $\psi \in \langle 0,4\pi\rangle$ as follows:

$$z = \cos\frac{\vartheta}{2}\, e^{-i/2(\psi+\varphi)}, \quad w = \sin\frac{\vartheta}{2}\, e^{-i/2(\psi-\varphi)} \tag{6.21}$$

Then (using property of $SU(2)$-matrices, that $A^{+} = A^{-1}$) the canonical 1-form is

$$\begin{aligned}\theta &= A^{-1}(\vartheta,\varphi,\psi)\, dA(\vartheta,\varphi,\psi) = A^{+}(\vartheta,\varphi,\psi)\, dA(\vartheta,\varphi,\psi) = \\ &= \begin{pmatrix} \cos\frac{\vartheta}{2}\, e^{i/2(\psi+\varphi)} & \sin\frac{\vartheta}{2}\, e^{i/2(\psi-\varphi)} \\ -\sin\frac{\vartheta}{2}\, e^{-i/2(\psi-\varphi)} & \cos\frac{\vartheta}{2}\, e^{-i/2(\psi+\varphi)} \end{pmatrix} \cdot d \begin{pmatrix} \cos\frac{\vartheta}{2}\, e^{-i/2(\psi+\varphi)} & -\sin\frac{\vartheta}{2}\, e^{i/2(\psi-\varphi)} \\ \sin\frac{\vartheta}{2}\, e^{-i/2(\psi-\varphi)} & \cos\frac{\vartheta}{2}\, e^{i/2(\psi+\varphi)} \end{pmatrix}\end{aligned} \tag{6.22}$$

The next few steps consist of d-ting the matrix and then matrix multiplication, that will not be displayed here. Important is the result[46]:

$$\theta = -\frac{i}{2} \begin{pmatrix} e^3 & e^1 - i\, e^2 \\ e^2 + i\, e^2 & -e^3 \end{pmatrix} \tag{6.23}$$

for

$$e^1 = \sin\psi\, d\vartheta - \sin\vartheta\, \cos\psi\, d\varphi \tag{6.24a}$$

$$e^2 = \cos\psi\, d\vartheta + \sin\vartheta\, \sin\psi\, d\varphi \tag{6.24b}$$

$$e^3 = d\psi + \cos\vartheta\, d\varphi \tag{6.24c}$$

Note, that the result in (6.23) can be reall written in terms of base elements of $su(2)$, the Pauli matrices (see (4.2)) multiplied by $-i/2$.

In our case, the rotation is given by ϑ, φ ($\psi = 0$)[47], what cancels few terms in (6.24). Mainly, $e^3 = \cos\vartheta\, d\varphi$ and eventually after inserting into (6.19) we get

$$\mathcal{A} = m\, \cos\vartheta\, d\varphi \tag{6.25}$$

and hence

$$\Omega_m = -m\, \sin\vartheta\, d\vartheta \wedge d\varphi = -m\, \omega_g \tag{6.26}$$

and we are done.[48]

[46] Can be found in [8], p. 264 too

[47] Note, that rotation given by the third Euler angle ψ would contribute to our state only by an extra phase factor, because it is a rotation about the (new) z-axis. As we mentioned before, the curvature 2-form is $U(1)$-invariant, so we do not need to deal with this rotation. Moreover, if we let the $d\psi$ stay in (6.24c), then consequental d-ting would cancel it immediately.

[48] As we mentioned, our parametric space is 3-dimensional space (B, ϑ, φ) (e.g. $3D$ without origin, because for $B = 0$ one has degenerated spectrum). When we then want to find eigenstates of our Hamiltonian, we can manage without any dependance on B - we get the same spinor for every value of B. This fact is then reflected in the formula for Berry's curvature- it does not depend on the radial value (B) and therefore states effectively live only on sphere. However, when we deal with rotations, the significant information about states is given by 2 rotations (ϑ, φ) and rotation given by ψ contributes only by a phase factor to the whole spinor - the essential information about spins is encoded in the 2 aout of 3 rotations. That means, spinors live on a manifold $SU(2)/U(1)$ (we get rid of the extra phase factor by factorisation of the whole rotation group by the group of redundant rotations), that is (as a manifold) S^2, so we are back to sphere, but we came to it in another way. In this approach, when we are on $SU(2)$ instead of S^2, we need to be careful in our computations too. For example, we should take $e^3 = d\psi + \cos\vartheta\, d\varphi$ from (6.24c), use Mauer- Cartan equation $de^a + \frac{1}{2} c^a_{b\,c}\, e^b \wedge e^c = 0$ ($c^a_{b\,c}$ are structure constants, $[E_b, E_c] = c^a_{b\,c} E_a$ for the basis E_i of the Lie algebra, in our case $SU(2)$ we get $c^a_{b\,c} = \epsilon_{a\,b\,c}$) and we get $de^3 = -1/2 c^3_{a\,b} e^a \wedge e^b = -c^3_{1\,2} e^1 \wedge e^2 = -\sin\vartheta\, d\vartheta \wedge d\varphi$, but this is on $SU(2)$ and now the factorisation is needed (it is only a formal step in our mind, because it does not change the formula, of course). Behind this all lies the fact, that the Berry's curvature 2-form lives on the principal $SU(2)$- bundle and needs to be pull-backed (pull-back of the section $\sigma = \psi(\vartheta, \varphi)$ - see text below (6.27)) to the basis of this bundle- $SU(2)/U(1)$.

This approach via the canonical 1-form seems absolutely distant from that one in previous sections, but it is not. The question is, what exactly is the object on the left-hand side of (6.3), namely $\rho(g^{-1})\, d\rho(g)$? It is a product of two represented group elements- rotations in the (Hilbert) space of spins. As we mentioned in the beginning of this section, such representations are mediated by Wigner $\mathcal{D}$ matrices. That means, that $\rho(g) = \mathcal{D}^s(\vartheta, \varphi, \psi)$ for an arbitrary spin s (e.g. the $(2s+1)$-dimensional representation of rotations). Then, however, the problematic term from (5.16) - sum of (by k multiplied) Wigner d-matrices squared arises exactly from hereby (by putting $\psi = 0$, etc.)

$$\rho(g^{-1})\, d\rho(g) \rightsquigarrow \sum_{k=-s}^{s} k\, \left(d^s_{k,m}(\vartheta)\right)^2 \tag{6.27}$$

That means, we proved (5.16) - (5.27) for the second time by using group methods, that is quite heart-warming[49][50].

[49] In fact, a lot of relations common to the one in (5.17) - (5.18) are often proved by means of group calculus instead of direct computation. The way is rather abstract, yet more straightforward.

[50] As a top of this whole section, I would like to sketch, how can one see this whole calculation from yet another point of view. It uses the theory of fiber bundles (as it is an advanced topic, we will not explain the theory here): The canonical 1-form consists officialy of 3 terms - $\theta = \theta^i E_i$ (E_i again as the basis of $su(2)$ Lie algebra). However, as we saw in (6.19), only the 3-rd term does not vanish and gives the birth of the Berry's onnection 1-form. That means, that we already constructed the connection form $\omega = e^3\, E_3$ living on $SU(2)$ in terms of fibre bundles. As it uses only 1 element of basis, it is $U(1)$-connection (note, that $U(1)$ is a subgroup of $SU(2)$, when we construct its element as follows: $e^{\alpha E_3} = diag(e^{-i/2\alpha}, e^{i/2\alpha})$ (see (5.6)), E_3 being generator of this group). Then the horizontal bundle is spanned by e_1, e_2 (because then $\langle e^3, e_1\rangle = \langle e^3, e_2\rangle = 0$, that acts as a definition of horizontal bundle - one has to characterize, which vector fields (to the right) or 1-forms (to the left) define the horizontal bundle). To sum it up, we have connection 1-form (depending on ψ on $SU(2)$ being the principal bundle) and hence curvature 2-form (without this dependence) on $SU(2)$ too. However, as mentioned, our states live on basis of this bundle and got to the bundle by a section $\sigma = \psi(\vartheta, \varphi)$ (note, that the basis is sphere (ϑ, φ) and the bundle has extra coordinate ψ - we have to define, how is ψ connetced with (ϑ, φ) - section $\psi(\vartheta, \varphi)$ does it excellently). The last step is to pull-back (towards section) our curvature to the basis $SU(2)/U(1) = S^2$ (formally, if we denote the former curvature on $SU(2)$ as $\hat{\omega}$, then our desired one on S^2 is $\omega = \sigma^* \hat{\omega}$). It is, again, only a formal step, because the face of our curvature 2-form does not change after this pull-back. Only then we have the Berry's curvature on the sphere inhabited by spinors (spin states).

Conclusion

When we glance back, we managed to prove the result $\Omega_C = -m\,\omega_g$ for Berry's curvature for spin s with its z-projection m in external magnetic fiel $\vec{B}$ in 3 different ways preciselly (and we sketched the fourth one). This fact is inspirative, because we conjoin different theoretical fields (e.g. group theory, theory of angular momentum, etc.) to prove the same thing. As we mentioned in the last footnote, this occurs in many situations, when we manage to see a problem (that seems to be insoluble) from another point of view, yielding benefits for computations.

From the third chapter on, we widely used differential geometry in order to get desired results. We computed the Berry's phase directly using eigenstates of spin operators for spin $s = 0,\ 1/2,\ 1$ in general direction given by ϑ, φ (living on a sphere S^2) and checked the results for $s = 0,\ 1$ using spin coupling, that is for 2 spins easy to compute. We did not emphasise, why is the use of differential geometry so practical. We hope, that it is clear, that the language of differential geometry is very elegant and effective, having impact on size of formulae too[51]. To prove this statement even more, one could investigate the holonomy itself in terms of Riemann curvature tensor and other objects conneted to parallel transport and linear connexion (note, that parallel transport of a vector corresponds to "walking straightforward"). This would, however, require even more theory introduced.

Eventually, Lie groups join the whole game and act symbiotic with differential geometry - its concept is arranged in such a way, that the continous structure of Lie group naturally cooperates with differential geometry on this group, being manifold for the geometry. In this chapter we described the idea of finding canonical 1-form on $SU(2)$ group. This can be trasferred to arbitrary group having a lot of possible applications (in concept of the geometrical Berry's phase too).

In our thesis, we omitted one other approach to the Phase, that could be understandable at this level too. One could simply take the central object $e^{-i\,m\int_C \omega_g}$ as a functional integral (sometimes called Feynman path integral too). This concept is

[51] Recall one example for all: Maxwell equations. In standard (Newton- Leibnitz) formalism, the whole system is:

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\,\partial_t \vec{B} \\ \nabla \times \vec{B} &= \mu_0 \vec{j} + 1/c^2\,\partial_t \vec{E} \end{aligned}$$

If we rewrite it using differential p-forms, we get:

$$\begin{aligned} d\,F &= 0 \\ d * F &= -\ J \end{aligned}$$

where $F = F_{\alpha\beta}\,dx^\alpha \wedge dx^\beta$ is a 2-form (the electromagnetic tensor), $A = A_\alpha\,dx^\alpha$ is the potential 1-form with A_α being vector potential ($A_\alpha = (\phi, -\vec{A})$ with well-known scalar and vector potentials respectivelly) and $J = *j$ is dual 3-form of electric current, if $j = j_\alpha dx^\alpha$ is the 1-form of current ($j_\alpha = (\rho, -\vec{j})$).

worth investigating, because it can have connections to the field theories (e.g. quantum field theory, etc.), that can be formulated via the path integral. However, the same argument holds - this would require an extensive introduction to functional analysis[52], that would enlarge our thesis and we would digress from our aim.

To end this whole thesis with a vision (that is crucial to the science itself), we can say that this topic has a great potential in further analysis - for instance, we assumed, that our eigenstates are non-degenerated for the whole time. We escaped from practical applications of the Berry's phase- for example measurable consequences (in optics)[53]. We assumed Abelian gauge in the whole thesis. In this spirit, we could give an exhaustive list of possible upgrades for our thesis. And - as a top- spin in magnetic field is only a drop in the whole wide ocean of geometrical phases born from adiabatic changes of external parameter(s)... :)

[52] About path integrals for (not only) spin systems, a great lecture [12] is available.
[53] Is widely analysed in [1], [15], for example.

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