Comenius University in Bratislava

Faculty of Mathematics, Physics and Informatics

Geometry of the (generalized) Kaluza-Klein theory with scalar field

Diploma Thesis

Bratislava 2020

Bc. Dominik Novotný

Comenius University in Bratislava

Faculty of Mathematics, Physics and Informatics

Geometry of the (generalized) Kaluza-Klein theory with scalar field

Diploma Thesis

Study program : Theoretical Physics

Field of study : 4.1.1 Physics

Department : Theoretical Physics

Supervisor : doc. RNDr. Marián Fecko, PhD.

Bratislava 2020

Bc. Dominik Novotný

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. Dominik Novotný
Študijný program:	teoretická fyzika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor:	fyzika
Typ záverečnej práce:	diplomová
Jazyk záverečnej práce:	anglický
Sekundárny jazyk:	slovenský

Názov: Geometry of the (generalized) Kaluza-Klein theory with scalar field
Geometria (zovšeobecnenej) Kaluzovej-Kleinovej teórie so skalárnym poľom

Anotácia: Theodor Kaluza publikoval v r.1921 článok, v ktorom (istým spôsobom) „zjednotil gravitáciu s elektromagnetizmom". Hlavnou myšlienkou bolo zapísať samotnú gravitáciu (podľa vzoru nedávno vzniknutej všeobecnej teórie relativity), ale v piatich rozmeroch (Einsteinova teória pracuje so 4 rozmermi) a „pozrieť sa na výsledok 4-rozmerne". Napodiv, zadarmo z toho vypadla obyčajná gravitácia plus elektromagnetizmus. Neskoršie zovšeobecnenia jednak pridali skalárne pole (to sa ukazuje byť nevyhnutnou súčasťou teórie) a jednak prešli od elektromagnetizmu k všeobecným kalibračným poliam. Ukazuje sa výhodné chápať nový „časopriestor" (analóg pôvodného 5-rozmerného priestoru) ako hlavný fibrovaný G-priestor s konexiou.

Cieľ: Zrátať skalárnu krivosť v (zovšeobecnenej) Kaluzovej-Kleinovej geometrii so skalárnym poľom, chápanej ako geometrii na hlavnej G-fibrácii s konexiou.

Literatúra: C.N.Pope: Kaluza–Klein Theory, Lecture notes (available online)
M.Fecko: Differential geometry and Lie groups for physicists, Cambridge University Press 2006

Kľúčové slová: Kaluzova-Kleinova teória, hlavný fibrovaný priestor s grupou G a konexiou, skalárna krivosť

Vedúci:	doc. RNDr. Marián Fecko, PhD.
Katedra:	FMFI.KTF - Katedra teoretickej fyziky
Vedúci katedry:	doc. RNDr. Tomáš Blažek, PhD.
Dátum zadania:	16.11.2018

Dátum schválenia: 16.11.2018

prof. Ing. Roman Martoňák, DrSc.
garant študijného programu

....

študent vedúci práce

36222819

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname:	Bc. Dominik Novotný
Study programme:	Theoretical Physics (Single degree study, master II. deg., full time form)
Field of Study:	Physics
Type of Thesis:	Diploma Thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Geometry of the (generalized) Kaluza-Klein theory with scalar field

Annotation: In 1921, Theodor Kaluza published a paper in which he (in a way) "unified gravity with electromagnetism". The main idea behind was to study the pure gravity (treated according to just discovered general theory of relativity), however in 5 dimensions (the Einstein theory works in 4 dimensions) and then „look at the result from the 4-dimensional perspective". Surprisingly enough, what one gets is the usual Einstein gravity plus, free of charge, the electro-magnetic field. Later generalizations added the scalar field (it turned out to be an inevitable part of the theory) and replaced electro-magnetic theory by a general gauge-field theory. It turns out to be useful to regard the new „space-time" (the counter-part of the original 5-dimensional space) as the principle G-bundle with a connection.

Aim: Computation of the scalar curvature within the (generalized) Kaluza-Klein geometry with the scalar field, regarded as the geometry on a principal G-bundle with a connection.

Literature: C.N.Pope: Kaluza–Klein Theory, Lecture notes (available online)
M.Fecko: Differential geometry and Lie groups for physicists, Cambridge University Press 2006

Keywords: Kaluza-Klein theory, principal G-bundle with connection, scalar curvature

Supervisor:	doc. RNDr. Marián Fecko, PhD.
Department:	FMFI.KTF - Department of Theoretical Physics
Head of department:	doc. RNDr. Tomáš Blažek, PhD.
Assigned:	16.11.2018
Approved:	16.11.2018

prof. Ing. Roman Martoňák, DrSc.
Guarantor of Study Programme

....
Student

....
Supervisor

Acknowledgement

I would like to thank my supervisor Marián Fecko for the opportunity to work on a interesting thesis and also for his patience when discussing problems which occurred while writing this thesis.

Abstrakt

Autor : Bc. Dominik Novotný

Názov : Geometria (zovšeobecnenej) Kaluzovej-Kleinovej teórie so skalárnym poľom

Typ záverečnej práce : diplomová práca

Univerzita : Univerzita Komenského v Bratislave

Fakulta : Fakulta matematiky, fyziky a informatiky

Katedra : Katedra teoretickej fyziky

Školiteľ: doc. RNDr. Marián Fecko, PhD.

Bratislava : FMFI UK, 2020

62 strán

Mnoho fyzikov je posadnutých myšlienkou nájsť teóriu všetkého. Jeden z prvých pokusov pochádza od Theodora Kaluzu, ktorý sa pokúsil spojiť gravitáciu a elektromagnetizmus. My sa pozrieme na zovšeobecnenie tejto teórie, najskôr bez skalárneho poľa, berúc najprv do úvahy všeobecnú kalibračnú grupu. Ďalej sa sústredíme už len na grupu $U(1)$, ale pridáme skalárne pole. Hlavným technickým nástrojom bude v oboch prípadoch využitie pojmu vertikálny stupeň diferencálnych foriem.

Abstract

Author : Bc. Dominik Novotný

Title : Geometry of the (generalized) Kaluza-Klein theory with scalar field

Type of thesis : diploma thesis

University : Comenius University in Bratislava

Faculty : Faculty of Mathematics, Physics and Informatics

Department : Department of Theoretical Physics

Supervisor : doc. RNDr. Marián Fecko, PhD.

Bratislava : FMFI UK, 2020

62 pages

Many physicist are obsessed by finding a theory of everything. On of the first attempts come from Theodor Kaluza merging gravity and electromagnetism. First we are going to take a look on a generalization of this theory, yet without a scalar field, by assuming a general gauge group. In the next part we focus just on the $U(1)$ group, but we add the scalar field. The main technical apparatus will in both cases rely on utilizing the concept of vertical degree of differential forms.

Foreword

Ones my lecturer in the elementary course on mechanics said that physics is a Babylonian science. What he meant, is that when a part of physics gets lost or forgotten it can be knitted back from the parts that are left. In contrast, mathematics is based on axioms which can not be restored once lost. The example that provided Theodor Kaluza shows that also previously unknown parts can be knitted out of the already familiar parts. Kaluza took the then modern "pattern" developed by Einstein, and knitted something out of it in five dimensions and discover a little miracle. What previously in four dimensions gave "just" gravity, in five dimensions, when done correctly, gave in addition also electromagnetism. In this thesis we take just a little step further and see if it is possible to unravel this theory and knit it back together in a slightly other, arguably more elegant way and may be also knit a bit further. And since the "patterns" of local coordinates used by Kaluza are out of fashion and we like to go with the newer fashion we are going to use frame fields and bundles instead. But since scientist discovered a species who call themselves hipsters and fancy everything "old school", in order not to discriminate them, we are also going to mention local coordinates. What's left to say? I hope you enjoy reeding this thesis if only half as much as I enjoyed writing it.

Contents

Introduction

The Klauza-Klein theory, is one of the first unified field theories, unifying gravitation and electromagnetism in a classical way. The first attempt was by Gunnar Nordström in 1914 [9]. However this was pre general relativity. He used his own theory of gravitation, where the gravitational field was given as an four-vector and the electromagnetic field by a six-vector. He assumed the theory to take place in a five dimensional space where neither the gravitational four-vector, nor the electromagnetic six-vector were dependent on the fifth coordinate.

A similar idea (independently from Nordström) had Theodor Kaluza in 1921[6]. He considered our space-time as a four dimensional part of $\mathbb{R}^5$. It showed up, that the Einstein's general relativity equations in the 5 dimensional space consists of Einstein's equations, as we know it from the usual 3+1 dimensional case, and Maxwell's equations. Since we are aware only of space and time, he argued, that the derivatives with respect to the fifth dimension are either zero or negligible in terms of higher order, thereby suggesting the so called "cylinder condition". This could be seen as just hand waving.

An improvement came by Oskar Kein, whose idea was the compactification of the extra dimension and, in context of then recent discoveries (1926) by Heisenberg, Schrödinger and de Broglie, he gave also a quantum interpretation of the theory. As seen in [7], he assumed that the space is closed in the direction of the coordinate of the extra dimension with a period l. He managed to calculate this period $l = 0.8 \cdot 10^{-32} m$ which together with the periodicity supported the theory of Kaluza. This also gave hint why the fifth dimension might not contribute to the physics. A more detailed tratmenrt is given in [8].

Another approach came in the thirties by Oswald Veblen and Banesh Hoffmann [15]. Instead of using affine geometry they used projective geometry. That meant, that we do not need to worry about the extra dimension, since it gets "absorbed into" the ordinary space-time.

The approach we are going to take is a more recent one. We are going to formulate the theory in terms of a G-principal bundle $\pi : P \to M$ and a connection on P [1], [13]. What may be new is the use of vertical degree of forms when computing connection forms.

Chapter 1

Data of the theory

Let us consider a principal G-bundle $\pi : P \rightarrow (M, \hat{g})$ with RLC connection [1] on M. M is a space-time (in physics 1+3 dimensional), G is a general compact Lie group - the symmetry group of our theory. On M we have a orthonormal frame field denoted $\hat{e}_a$, hence the coframe field is $\hat{e}^a$. The metric tensor on M is

$$\hat{g} = \eta_{ab}\hat{e}^a \otimes \hat{e}^b, \tag{1.1}$$

and the covariant derivative:

$$\hat{\nabla}_v \hat{e}^a = -\hat{\omega}^a_b(v)\hat{e}^b, \tag{1.2}$$

where $\hat{\omega}^a_b$ are the connection forms. In this text we are going to denote everything which comes from M with a hat $\hat{\_}$.

Further, we have a G-connection on P, characterized by its connection forms ω^i. Note that this connection has nothing to do with RLC. Since ω^i are just one forms (and also linear independent) they can be used as part

[1] The standard name for this connection is either Riemann connection or Levi-Civita connection, since both were great mathematicians, not to discriminate we will therefore use the abbreviation RLC connection.(see [3])

of a coframe field on P. So our (co)frame field on P is

$$e_i := \xi_{E_i} \qquad e_a := \hat{e}_a^h \tag{1.3}$$

$$e^i := \omega^i \qquad e^a := \pi^* \hat{e}^a. \tag{1.4}$$

Because of the connection ω^i, every vector in P can be decomposed in its horizontal and vertical parts. The vertical part is by definition the one which projects to 0 by π_*. For the horizontal part we first need to define a connection in P.

Definition 1. *A connection on a principal G-bundle $\pi : P \to M$ is a(n arbitrary) horizontal G-invariant distribution $\mathcal{D}^h$ on the total space P (or anything which is equivalent to this object).*[2] *[3]*

As a result, each tangent space T_pP of the manifold P can be uniquely decomposed into the sum of its vertical and horizontal parts:

$$T_pP = \mathrm{Ver}_pP \oplus \mathrm{Hor}_pP \tag{1.5}$$

For convenience in the following text we use indices a, b, c for the horizontal part and i, j, k for the vertical part of vectors on P.

1.1 Metric on P

By default we have no metric on P so our first task is to construct one. The crucial point in Kaluza's construction was that the components of the metric are independent from the fifth coordinate. Only then were gravity and electromagnetism merged. When we translate this into our modern language, it means that we demand its invariance with respect to the action

[2] A connection in this sense may also be encoded into the connection form.

of G on P. The arguments of a metric tensor $g(U, V)$ can be decomposed to its vertical and horizontal parts. Therefore we can define the metric tensor g as follows:

- g is such that horizontal vectors are perpendicular to vertical vectors.
- For the horizontal part $g(\mathrm{hor}U, \mathrm{hor}V) := g(\pi_* U, \pi_* V)$.
- For the vertical part we take $g(\xi_X, \xi_Y) := K(X, Y)$, where K is an Ad-invariant metric in the Lie algebra $\mathcal{G}$.

An Ad invariant metric in $\mathcal{G}$ can be obtained e.g. by defining the Killing-Cartan form in the Lie algebra $\mathcal{G}$

$$K(X, Y) := Tr(\mathrm{ad}_X \mathrm{ad}_Y) \equiv \left\langle E^i, \mathrm{ad}_X \mathrm{ad}_Y Ei \right\rangle . \tag{1.6}$$

This form is symmetric, bilinear and Ad-invariant. The symmetry follows from a property of trace - operators do commute within trace. Bilinearity follows from linearity of ad_X:

$$\begin{aligned} K(X + \lambda Z, Y) &= Tr(\mathrm{ad}_{X+\lambda Z} \mathrm{ad}_Y) = Tr((\mathrm{ad}_X + \lambda \mathrm{ad}_Z) \mathrm{ad}_Y) \\ &= Tr((\mathrm{ad}_X \mathrm{ad}_Y) + \lambda Tr(\mathrm{ad}_Z \mathrm{ad}_Y) & (1.7) \\ K(X, Y + \lambda Z) &= Tr(\mathrm{ad}_X \mathrm{ad}_{Y+\lambda Z}) = Tr(\mathrm{ad}_X (\mathrm{ad}_Y + \lambda \mathrm{ad}_Z)) \\ &= Tr(\mathrm{ad}_X \mathrm{ad}_Y +) + \lambda Tr(\mathrm{ad}_X \mathrm{ad}_Z) & (1.8) \end{aligned}$$

Ad-invariance[3]:

$$\begin{aligned} K(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) &= Tr((\mathrm{Ad}_g \mathrm{ad}_X \mathrm{Ad}_{g^{-1}})(\mathrm{Ad}_g \mathrm{ad}_Y \mathrm{Ad}_{g^{-1}}) & (1.9) \\ &= Tr(\mathrm{ad}_X \mathrm{ad}_Y) = K(X, Y) & (1.10) \end{aligned}$$

[3] (1.9) holds, since we put the represented values of the inputs of the Killing-Cartan form and ad is the derived representation of Ad

Its matrix of components with respect to the basis $E_i \in \mathcal{G}$ is

$$
\begin{aligned}
k_{ij} &\equiv K(E_i, E_j) = \left\langle E^k, \mathrm{ad}_i \mathrm{ad}_j E_k \right\rangle &(1.11)\\
&= \left\langle E^k, [E_i, [E_j, E_k]] \right\rangle = \left\langle E^k, c^l_{jk} [E_i, E_l] \right\rangle &(1.12)\\
&= \left\langle E^k, c^l_{jk} c^m_{il} E_m \right\rangle = c^l_{jk} c^m_{il} \left\langle E^k, E_m \right\rangle = c^l_{jk} c^m_{il} \delta^k_m &(1.13)\\
&= c^k_{il} c^l_{jk} &(1.14)
\end{aligned}
$$

So the metric tensor on P reds:

$$g = \pi^* \hat{g} + \mathcal{K} = \eta_{ab} e^a \otimes e^b + \eta_{ij} e^i \otimes e^j, \ ^4 \tag{1.15}$$

where $e_i \equiv \xi_{E_i}$ and $e_a \equiv \hat{e}^h_a$ [5] and e^a, e^i are the corresponding dual bases, see (1.3)(1.4). Note that such metric tensor is not only G-invariant as a whole, but it vertical and horizontal parts are G-invariant separately.

1.2 Vertical degree of forms

Let us abandon our G-principal bundle for a while to introduce the concept of vertical degree of forms in a more general setting. We are going to proceed as in [4].

Let us have a space L which is composed of two subspaces $L = L_1 \oplus L_2$ and call them [6]

$L \supset L_1 \equiv$ vertical

$L \supset L_2 \equiv$ horizontal

[4] In general we have k_{ij}, but since these are just constants we can digonalize the matric by choseing a suitable basis, hance get η_{ij}

[5] $()^h$ indicates the horizontal lift, such that $\pi_* \hat{e}^h_a = \hat{e}_a$ and $\hat{e}^h_a \in \mathrm{Hor}_p P$.

[6] There is no deeper reason two call L_1 vertical and L_2 horizontal, we could chose it the other way round and it would not make any difference. It is a bit like with dual spaces, you effectively can not distinguish which space is the reference (L) and which is the dual to it (L^*), since $(L^*)^* \cong L$ canonically.

We choose a basis in both of this spaces, $e_i \in L_1$ and $e_a \in L_2$. So the adapted basis on L is $e_\alpha \equiv (e_i, e_a)$ and the adapted dual basis in L^* is $e^\alpha \equiv (e^i, e^a)$. Then the duality condition

$$\langle e^\alpha, e_\beta \rangle = \delta^\alpha_\beta,$$

decomposes into four cases:

$$\langle e^a, e_b \rangle = \delta^a_b \qquad \left\langle e^i, e_j \right\rangle = \delta^i_j \qquad \langle e^a, e_i \rangle = 0 \qquad \left\langle e^i, e_a \right\rangle = 0 \tag{1.16}$$

A general p-form α in L can be written as:

$$\alpha = \frac{1}{p!}\alpha_{\alpha\ldots\beta} e^\alpha \wedge \ldots \wedge e^\beta. \tag{1.17}$$

Since e^α is either e^i or e^a, α can be alternatively expressed as:

$$\begin{aligned} \alpha &= \hat{\alpha} + e^i \wedge \hat{\alpha}_i + e^i \wedge e^j \wedge \hat{\alpha}_{ij} + \ldots & (1.18) \\ &= \alpha_{(0)} + \alpha_{(1)} + \alpha_{(2)} + \ldots & (1.19) \end{aligned}$$

where $\hat{\alpha}$ in (1.18) only contains e^a. Note that $\alpha_{(q)}$ in (1.19) is a well defied object, since it is independent with respect to a change of frame, so together with (1.18), we can use this as a definition of the terms on the right hand side of the equation. Hence we see that a p-form $\alpha \in L = L_1 \oplus L_2$ is characterized by another integer (in addition to p), namely q, which we can call the vertical degree.

Definition 2. *The vertical degree of $\alpha_{(q)}$ is $q \in \mathbb{Z}$.*

Now we can introduce the concept of a horizontal form.

Definition 3. *A p-form in $L = L_1 \oplus L_2$ is called horizontal, if its vertical degree is 0 – $\alpha = \alpha_{(0)}$*

1.2.1 Some useful observations

One can first of all notice, that if α is horizontal, any vertical argument makes it 0.

$$i_{e_j}\alpha_{(0)} = 0 \tag{1.20}$$

This can be further generalized as:

$$i_{e_j}\alpha = 0 \quad \Leftrightarrow \quad \alpha = \alpha_{(0)} \tag{1.21}$$

$$i_{e_k}i_{e_j}\alpha = 0 \quad \Leftrightarrow \quad \alpha = \alpha_{(0)} + \alpha_{(1)} \tag{1.22}$$

$$i_{e_l}i_{e_k}i_{e_j}\alpha = 0 \quad \Leftrightarrow \quad \alpha = \alpha_{(0)} + \alpha_{(1)} + \alpha_{(2)} \tag{1.23}$$

$$\text{ect.} \tag{1.24}$$

As the last observation, I would like to introduce a operator which gives back the vertical degree of the form on which it is acting:

$$Q := j^k i_k \qquad i_k \equiv i_{e_k}, \qquad j^k\alpha := e^k \wedge \alpha \tag{1.25}$$

$$Q\alpha_{(q)} = q\alpha_{(q)} \tag{1.26}$$

For the proof see appendix I.

1.2.2 Example: Calculation of the curvature forms on a G-principal bundle

Back to our G-principal bundle. One of our desires is to calculate the curvature 2-forms Ω^i, since it will be helpful when calculating the connection forms of the RLC connection.

We have in P (see (1.3),(1.4)):

$$e_\alpha \equiv (e_i, e_a), \qquad e^\alpha \equiv (e^i, e^a), \qquad \langle e^\alpha, e_\beta \rangle = \delta^\alpha_\beta \tag{1.27}$$

We decompose each tangent space $L \equiv T_pP$ into its vertical and horizontal part in the sense of *connection theory*. Now, we see that this decomposition matches the decomposition studied in this section, included the meaning of vertical and horizontal.

$$L_1 := \mathrm{Ver}T_pP \qquad L_2 := \mathrm{Hor}T_pP \tag{1.28}$$

By definition, the curvature 2-form Ω is the exterior covariant derivative $D := \mathrm{hor} \circ d$ of the connection form ω.

$$\Omega^i := D\omega^i := \mathrm{hor}(d\omega^i) \equiv (de^i)_{(0)} \tag{1.29}$$

The defining properties of a connection form ω are:

$$R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega \tag{1.30}$$

$$i_{\xi_X}\omega = X \tag{1.31}$$

The infinitesimal version of the first one is:

$$\mathcal{L}_{\xi_X}\omega = -\mathrm{ad}_X\omega = -[X, \omega] \tag{1.32}$$

And since the Lie derivative on forms is $\mathcal{L}_v = di_v + i_v d$, this reduces to:

$$\begin{aligned} \mathcal{L}_{\xi_X}\omega &= di_{\xi_X}\omega + i_{\xi_X}d\omega && (1.33)\\ &= dX + i_{\xi_X}d\omega = i_{\xi_X}d\omega && (1.34)\\ i_{\xi_X}d\omega &= -[X, \omega] && (1.35) \end{aligned}$$

Finally for $X = E_j$, $\omega = \omega^i E_i$, $e^i = \omega^i$ we get:

$$\begin{aligned} i_{e_j}(de^i)E_i &= -[E_j, \omega^i E_i] = -c^k_{ji}\omega^i E_k && (1.36)\\ i_{e_j}(de^i) &= -c^i_{jk}e^k && (1.37) \end{aligned}$$

Now, when we act by an interior product (with a vertical argument) on (1.37) we get:

$$\begin{aligned} i_{e_k} i_{e_j}(de^i) &= -c^i_{jk} && (1.38) \\ i_{e_l} i_{e_k} i_{e_j}(de^i) &= 0 && (1.39) \end{aligned}$$

When we recall the useful observations (1.22)-(1.24) from previous section we conclude:

$$de^i = (de^i)_{(0)} + (de^i)_{(1)} + (de^i)_{(2)} \tag{1.40}$$

Then from (1.38) we get

$$(de^i)_{(2)} = -\frac{1}{2} c^i_{jk} e^j \wedge e^k, \tag{1.41}$$

and from (1.37) we get

$$(de^i)_{(1)} = 0. \tag{1.42}$$

So together we get

$$de^i = (de^i)_{(0)} - \frac{1}{2} c^i_{jk} e^j \wedge e^k. \tag{1.43}$$

And by abusing the notation:

$$\begin{aligned} (d\omega^i)_{(0)} &= d\omega^i + \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k && (1.44) \\ &=: \mathrm{hor}(d\omega^i) && (1.45) \end{aligned}$$

So we derived the celebrated *Cartan structure equations*:

$$\Omega^i = d\omega^i + \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k \tag{1.46}$$

Chapter 2

KK-theory on a principle G-bundle without a scalar field

This chapter is meant only as a mathematical exercise to hone our skills, since in general we are not allowed to put an ansatz (like setting $\Phi = 0$) in to the action (or the Lagrangian). The equations obtained by such an action often leaner than when we do it the right way - computing all the equations from the action and only then apply the ansatz. It shows up that in this case it is even worse, as this ansatz (setting $\Phi = 0$) violates a field equation for Φ. For further informations why see [12]. Nonetheless this exercise proves itself handy when we proceed to the more elaborate case with the scalar field.

2.1 Connection form on P

In this section we are going to calculate the RLC connection forms on P. This can be done using an ansatz [1]. Rather than guessing an appropriate ansatz and hoping that we have not missed some important part of it, we are going to use the concept of vertical degree of forms. By this method we can preform a rather elegant direct calculation, we need only in to know what properties are expected from the connection forms.

This means, for the RLC connection, that we want our connection forms to be symmetric (zero torsion)

$$T^a \equiv de^\alpha + \omega^\alpha_\beta \wedge e^\beta = 0, \tag{2.1}$$

which rewritten in terms of our horizontal and vertical indexes is

$$\begin{aligned} 0 &= de^a + \omega^a_b \wedge e^b + \omega^a_i \wedge e^i & (2.2)\\ 0 &= de^i + \omega^i_a \wedge e^a + \omega^i_j \wedge e^j, & (2.3)\end{aligned}$$

and metric

$$\omega_{ab} = -\omega_{ba} \qquad \omega_{ai} = -\omega_{ia} \qquad \omega_{ij} = -\omega_{ji} \tag{2.4}$$

with respect to the orthonormal basis (1.3) on P. This gives

$$\begin{aligned} \eta_{ab}{\omega^b}_i &= -\eta_{ab}{\omega_i}^b, \\ &= -\eta_{ab}\eta^{bc}\omega_{ic}, \\ &= -\eta_{ab}\eta^{bc}\eta_{ij}{\omega^i}_c, \\ {\omega^a}_i &= -\eta^{ab}\eta_{ij}{\omega^i}_b. & (2.5)\end{aligned}$$

Our goal is to find such ω^α_β so that this holds. The forms de^a, de^i are already known from (1.46), and (2.1) on M

$$\begin{aligned} de^a &= -\pi^*\left(\hat{\omega}^a_b \wedge \hat{e}^b\right) & (2.6)\\ de^i &= \Omega^i - \frac{1}{2}c^i_{jk}e^j \wedge e^k & (2.7)\end{aligned}$$

We can rewrite this in terms of vertical degree [1]

$$\begin{aligned} de^a &= (de^a)_0, & (2.8)\\ de^i &= (de^i)_0 + (de^i)_2. & (2.9)\end{aligned}$$

[1] Since by the mapping π^* we get only horizontal forms, Ω^i is horizontal by definition and $c^i_{jk}e^j \wedge e^k$ has clearly of *vertical degree* 2.

The connection forms are 1-forms so the decomposition in terms of vertical degree is

$$\omega^a_b = (\omega^a_b)_0 + (\omega^a_b)_1, \tag{2.10}$$

$$\omega^i_a = (\omega^i_a)_0 + (\omega^i_a)_1, \tag{2.11}$$

$$\omega^i_j = (\omega^i_j)_0 + (\omega^i_j)_1. \tag{2.12}$$

Now when we combine everything and add it to equations (2.2) and (2.3) we get:

$$0 = (de^a)_0 + (\omega^a_b)_0 \wedge (e^b)_0 + (\omega^a_b)_1 \wedge (e^b)_0 - \eta^{ab}\eta_{ij}(\omega^j_b)_0 \wedge (e^i)_1 - \eta^{ab}\eta_{ij}(\omega^j_b)_1 \wedge (e^i)_1 \tag{2.13}$$

$$0 = (de^i)_0 + (de^i)_2 + (\omega^i_a)_0 \wedge (e^a)_0 + (\omega^i_a)_1 \wedge (e^a)_0 + (\omega^i_j)_0 \wedge (e^j)_1 + (\omega^i_j)_1 \wedge (e^j)_1 \tag{2.14}$$

Since on the LHS of both equations is zero, the form in each vertical degree separately must be zero. For vertical degree 0:

$$0 = (de^a)_0 + (\omega^a_b)_0 \wedge (e^b)_0 \tag{2.15}$$

$$0 = (de^i)_0 + (\omega^i_a)_0 \wedge (e^a)_0 \tag{2.16}$$

For vertical degree 1:

$$0 = (\omega^a_b)_1 \wedge (e^b)_0 - \eta^{ab}\eta_{ij}(\omega^j_b)_0 \wedge (e^i)_1 \tag{2.17}$$

$$0 = (\omega^i_a)_1 \wedge (e^a)_0 + (\omega^i_j)_0 \wedge (e^j)_1 \tag{2.18}$$

For vertical degree 2:

$$0 = \eta^{ab}\eta_{ij}(\omega^j_b)_1 \wedge (e^i)_1 \tag{2.19}$$

$$0 = (de^i)_2 + (\omega^i_j)_1 \wedge (e^j)_1 \tag{2.20}$$

When we add concrete familiar objects from (2.6) and (2.7,we get from vertical degree 0:

$$
\begin{aligned}
0 &= -\pi^*\left(\hat{\omega}^a_b \wedge \hat{e}^b\right) + (\omega^a_b)_0 \wedge e^b && (2.21)\\
&\Rightarrow (\omega^a_b)_0 = \pi^*\left(\hat{\omega}^a_b\right) && (2.22)\\
0 &= -\frac{1}{2}\Omega^i_{ab} e^b \wedge e^a + (\omega^i_a)_0 \wedge e^a && (2.23)\\
&\Rightarrow (\omega^i_a)_0 = \frac{1}{2}\Omega^i_{ab}\pi^*\hat{e}^b && (2.24)
\end{aligned}
$$

from vertical degree 1:

$$
\begin{aligned}
0 &= (\omega^a_b)_1 \wedge e^b - \frac{1}{2}\eta^{ab}\eta_{ij}\Omega^j_{bc}\pi^*\hat{e}^c \wedge e^i && (2.25)\\
0 &= (\omega^a_b)_1 \wedge \pi^*\hat{e}^b - \frac{1}{2}\Omega_i{}^a{}_b\pi^*\hat{e}^b \wedge e^i && (2.26)\\
&\Rightarrow (\omega^a_b)_1 = -\frac{1}{2}\Omega_i{}^a{}_b\omega^i && (2.27)\\
0 &= (\omega^i_a)_1 \wedge \pi^*\hat{e}^a + (\omega^i_j)_0 \wedge \omega^j && (2.28)\\
0 &= (\omega^i_a)_j\omega^j \wedge \pi^*\hat{e}^a + (\omega^i_j)_a\pi^*\hat{e}^a \wedge \omega^j && (2.29)\\
&\Rightarrow (\omega^i_a)_j = (\omega^i_j)_a && (2.30)
\end{aligned}
$$

from vertical degree 2:

$$
\begin{aligned}
0 &= \eta^{ab}\eta_{ij}(\omega^j_b)_1 \wedge (e^i)_1 && (2.31)\\
&\Rightarrow (\omega^j_b)_1 = 0 = (\omega^i_j)_0 && (2.32)\\
0 &= -\frac{1}{2}c^i_{jk}\omega^j \wedge \omega^k + (\omega^i_j)_1 \wedge \omega^j && (2.33)\\
&\Rightarrow (\omega^i_j)_1 = -\frac{1}{2}c^i_{jk}\omega^k && (2.34)
\end{aligned}
$$

When we combine all this together we get the connection forms on P:

$$\omega^a_b = \pi^*\hat{\omega}^a_b - \frac{1}{2}\Omega_i{}^a{}_b e^i \qquad (2.35)$$

$$\omega^i_a = \frac{1}{2}\Omega^i_{ab} e^b \qquad (2.36)$$

$$\omega^a_i = -\frac{1}{2}\Omega_i{}^a{}_b e^b \qquad (2.37)$$

$$\omega^i_j = -\frac{1}{2}c^i_{jk} e^k \qquad (2.38)$$

The same formulas can be find [1] (see equation (5.3)).

2.2 Scalar curvature R and Hilbert action

Scalar curvature is, in general, easily calculated from the curvature forms, see appendix II. When we do all the computing we get the final result:

$$R = \pi^*\hat{R} + \frac{1}{4}\Omega^i_{ab}\Omega_i{}^{ba} - \frac{1}{4}c^i_{jk}c_i{}^{kj} \qquad (2.39)$$

The Hilbert action of a gravitational field is (up to a conventional factor $(-1/16\pi)$) the volume integral of the scalar curvature, regarded as a functional of the metric tensor

$$S_H \equiv S_H[g] := -\frac{1}{16\pi}\int R_g \omega_g$$

where $R_g \equiv R$ is the scalar curvature of the RLC connection on $(M,\ g)$ and ω_g is the metric volume form [3]. In our case:

$$\boxed{S_H = -\frac{1}{16\pi}\int_P \left(\pi^*\hat{R} + \frac{1}{4}\Omega^i_{ab}\Omega_i{}^{ba} - \frac{1}{4}c^i_{jk}c_i^{kj}\right)\omega_\eta} \qquad (2.40)$$

So we see that the action consists of three parts. The last one is constant, so it is physically uninteresting. The first term on its own is the Hilbert action as we know it from the general relativity on M. The second term is the action of the gauge field, so after variation we get something like a generalized Maxwell equation in vacuum.(When we do this for $G = U(1)$ we in fact get the Maxwell equations in vacuum).

2.3 Equations of motion

The equations of motion are obtained from the geodesic equations. The acceleration is defined as

$$a = \nabla_v v. \tag{2.41}$$

A geodesics is a curve on which is the acceleration zero, so it has to satisfy the following equation:

$$\begin{aligned} 0 &= \nabla_v v^\alpha e_\alpha && (2.42) \\ 0 &= \dot{v}^\alpha e_\alpha + v^\alpha \omega_\alpha^\beta(v) e_\beta && (2.43) \\ 0 &= \dot{v}^\alpha + v^\beta \omega_\beta^\alpha(v) && (2.44) \end{aligned}$$

This decomposes in to two equations in indices $i,\ a$:

$$\begin{aligned} 0 &= \dot{v}^a + v^b \omega_b^a(v) + v^i \omega_i^a(v) && (2.45) \\ 0 &= \dot{v}^i + v^a \omega_a^i(v) + v^j \omega_j^i(v) && (2.46) \end{aligned}$$

We put in the connection forms:

$$\begin{aligned} \omega_b^a &= \pi^* \hat{\omega}_b^a - \frac{1}{2} \Omega_i{}^a{}_b e^i && (2.47) \\ \omega_a^i &= \frac{1}{2} \Omega_{ab}^i e^b && (2.48) \\ \omega_i^a &= -\frac{1}{2} \Omega_i{}^a{}_b e^b && (2.49) \\ \omega_j^i &= -\frac{1}{2} c_{jk}^i e^k && (2.50) \end{aligned}$$

So we get:

$$\begin{aligned}
\dot{v}^i &= \frac{1}{2}v^a v^b \Omega^i_{ab} + \frac{1}{2}v^j v^k c^i_{jk} = 0 &\quad (2.51)\\
&\Rightarrow v^i = const. &\quad (2.52)\\
0 &= -\frac{1}{2}v^b v^i \Omega_i{}^a{}_b - \frac{1}{2}v^i v^b \Omega_i{}^a{}_b &\quad (2.53)\\
\dot{v}^a + v^b v^c \pi^*(\hat{\omega}^a_b(\hat{e}_c)) &= v^i v^b \Omega_i{}^a{}_b &\quad (2.54)
\end{aligned}$$

In the equation (2.54) we see on the left hand side is the covariant derivative of v^a which can be obtain by lifting from M. Since $v^i = const.$ we can denote this $qF^a_b \equiv v^i \Omega_i{}^a{}_b$. So we get the equations of motion

$$\boxed{\hat{\nabla}_v v^a = v^b q F^a_b.} \qquad (2.55)$$

But this are the well known equations for the motion of charged particle in an electromagnetic field due to the Lorentz force (at leas in case $G = U(1)$).

2.4 Field equations

When we perform a variation, with respect to the metric tensor g_{ab}, of the Hilbert action (2.40) we get Einstein's equations in vacuum:

$$G_{ab} = 0 \qquad (2.56)$$

It turns out that this is equivalent to

$$R_{ab} = 0 \qquad (2.57)$$

So to obtain the equations of motion, we need to calculate the Ricci tensor. Our data are the curvature forms Ω^α_β. For a detailed calculation see appendix III. The Ricci tensor is:

$$R_{ab} = \pi^* \hat{R}_{ab} - \frac{1}{2}\Omega_i{}^c{}_a \Omega^i_{cb} \qquad (2.58)$$

$$R_{ai} = -\frac{1}{2}\nabla_b \Omega_i{}^b{}_a \tag{2.59}$$

$$R_{ij} = -\frac{1}{4}\Omega_j{}^a{}_b \Omega_i{}^b{}_a + \frac{1}{4}c^k_{il}c^l_{kj} \tag{2.60}$$

So finally we get the equations of motion:

$$\pi^* \hat{R}_{ab} = \frac{1}{2}\Omega_i{}^c{}_a \Omega^i_{cb} \tag{2.61}$$

$$\nabla_b \Omega_i{}^b{}_a = 0 \tag{2.62}$$

$$\frac{1}{4}\Omega_j{}^a{}_b \Omega_i{}^b{}_a = \frac{1}{4}c^k_{il}c^l_{kj} \tag{2.63}$$

Term on the right hand side of the first equation (2.61) resembles the stress-energy tensor of the gauge field. So the first equation looks like the Einstein's gravitational equation on M with an external source from the gauge field

The second equation (2.62) is a generalized version of the Maxwell equation. It is similar to $\nabla_b F^b_a = 0$, which is the Maxwell equation in vacuum (in a curved space). The difference is in the index i from the Lie algebra, so there are multiple copies of the Maxwell equation for each i.

The third one(2.63) is doubtful, since this would be analogous (in the case of electromagnetic field) to $F^{\mu\nu}F_{\mu\nu} = const.$, which does not hold in general. But after all we can not be surprised to get a strange field equation (2.63) since we used the wrong procedure (namely ignored altogether the scalar field) to obtain it.

This proves how dangers is to put an ansatz into the action, since two out of three equations look fairly reasonable. In the case of the original Kaluza-Klein theory it took about twenty years to discover this mistake, and realizing that the original KK-theory in fact does not provide a satisfactory unification of gravity and electromagnetism.

Chapter 3

Kaluza-Klein theory with scalar field for $U(1)$

In order to remedy the fiasco from the last chapter we need to add more degrees of freedom. Since by adding the scalar field, we add only one additional degree of freedom, we will reduce our calculation just for $U(1)$. In the case of an higher dimensional field it would be necessary to add an tensor field to cover up all the necessary degrees of freedom. If we would just add one scalar field, in case of a general compact Lie group, it would be a similar mistake as in the case of no scalar field. Even if it would be a slight improvement, it would not be the whole story.

Before we introduce the scalar field, we write down, at first, the metric on P in terms of local coordinates (x, y), where

· $x = x^\mu$ are coordinates lifted from M

· $y = y^p$ are coordinates on the fibre $(p = 1, \ldots, \dim G)$

Recall our metric on P is

$$g = \pi^* \hat{g} + K(\omega, \omega) \tag{3.1}$$

The local coordinates relate to the frame field as follows:

$$e^a = e^a_\mu(x) dx^\mu \tag{3.2}$$

$$e^i = e^i_\mu(x,y) dx^\mu + e^i_p(x,y) dy^p \tag{3.3}$$

The first half of the metric we get by lifting it from M:

$$\hat{g} = \hat{g}_{\mu\nu}(x) dx^\mu \otimes dx^\nu \qquad \Rightarrow \qquad \pi^*\hat{g} = \hat{g}_{\mu\nu}(x) dx^\mu \otimes dx^\nu \tag{3.4}$$

Then, we have to take a look at the second half. First of all, it is worth mentioning that $e^i_\mu(x,y)$ and $e^i_p(x,y)$ also depend on y. On the other hand $e^a_\mu(x)$ depends only on x, since we obtain it by lifting from M. This has to be compatible with the defining properties of ω^i - (1.31) and (1.32). If we insert this in (3.1) we get:

$$\begin{aligned} g = & \left(\pi^*\hat{g}_{\mu\nu} + \mathcal{K}_{\mu\nu}\right) dx^\mu \otimes dx^\nu \\ & + \mathcal{K}_{\mu p} dx^\mu \otimes dy^p + \mathcal{K}_{p\mu} dy^p \otimes dx^\mu \\ & + \mathcal{K}_{pq} dy^p \otimes dy^q \end{aligned} \tag{3.5}$$

where

$$\mathcal{K}_{\mu\nu} = k_{ij} e^i_\mu e^j_\nu \tag{3.6}$$

$$\mathcal{K}_{\mu p} = \mathcal{K}_{p\mu} = k_{ij} e^i_\mu e^j_p \tag{3.7}$$

$$\mathcal{K}_{pq} = k_{ij} e^i_p e^j_q \tag{3.8}$$

3.1 Introducing the scalar field

There are several ways for introducing the scalar field into the theory[1]. In this text we are going to stick a similar approach as in [12].

[1] For example see [11] or [12].

However, as in most other paper also in [12] it is done only for the special case where $G = U(1)$. That means for us, in order to compare (3.5) to the metric use in [12], we have to rewrite (3.5) for $U(1)$. That means we have to calculate the Killing-Cartan form. But in case of $U(1)$, it is easy, since $U(1)$ is abelian group, so this form is identically zero. In order to get something interesting, i.e. something which is non zero, we have to ask, first, why we used the Killing-Cartan form in the first place. We did that in order to get an Ad invariant metric. And this is needed, since this was a crucial fact in Kaluza's construction. In the five dimensional case we need that the components of the metric are independent of the fifth coordinate. Only then were gravity and electromagnetism merged.

Since $U(1)$ is abelian, $Ad = id$, so any metric is Ad invariant. So we can simply choose $k_{11} = 1$. The Lie algebra $u(1) = i\mathbb{R}$ with a basis $E_1 = i$. We have just one $y^1 \equiv y$, so the ξ_1 is just ∂_y. From the conditions (1.31) and (1.32) we get:

$$\omega^1 = e^1 = iA_\mu(x)dx^\mu + idy \tag{3.9}$$

This gives

$$\begin{aligned} \mathcal{K}_{\mu\nu} &= A_\mu A_\nu & (3.10) \\ \mathcal{K}_{\mu p} = \mathcal{K}_{p\mu} &= A_\mu & (3.11) \\ \mathcal{K}_{pq} &= 1 & (3.12) \end{aligned}$$

$$\begin{aligned} g &= (\pi^* \hat{g}_{\mu\nu} + A_\mu A_\nu)\, dx^\mu \otimes dx^\nu \\ &\quad + A_\mu dx^\mu \otimes dy + A_\mu dy \otimes dx^\mu \\ &\quad + dy \otimes dy \end{aligned} \tag{3.13}$$

That means, the matrix of the metric tensor looks like:

$$g_{\alpha\beta} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\mu & 1 \end{pmatrix} \tag{3.14}$$

Well, the component g_{55} is surely not the most general one. To get something more general we can introduce the scalar field, e.g. as in [12]:

$$g_{\alpha\beta} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\mu & e^{2\beta\phi} \end{pmatrix} \tag{3.15}$$

That means for our orthonormal frame field:

$$e^5 := e^{\beta\Phi}\omega^1 \qquad e^a := e^{\alpha\Phi}\pi^*\hat{e}^a \tag{3.16}$$

$$e_5 := e^{-\beta\Phi}\xi_{E_1} \qquad e_a := e^{-\alpha\Phi}\hat{e}^h_a \tag{3.17}$$

Now we can see, that as an supplement to the original metric tensor, the scalar fields provides just one additional degree of freedom, which is not enough for a general G. On the other hand for $U(1)$ it is enough and gives actually the whole story.

3.2 Connection forms with Φ

To calculate the connection form on P with Φ we use as previously the vertical degree of forms. In this case this technique proves itself even more useful, than in the case without the scalar field, since to write down the most general ansatz with a scalar field is a quite tricky task. This happens mainly, because there are many places where we can add the scalar field in an ansatz. Luckily, with this technique, we do not need to worry about that.

We want a RLC connection on P, i.e. to be symmetric (zero torsion)

$$0 = \alpha(e_b\Phi)e^b \wedge e^a + \exp\{\alpha\Phi\}\pi^*(d\hat{e}^a) + \omega^a_b \wedge e^b + \omega^a_5 \wedge e^5 \quad (3.18)$$

$$0 = \beta(e_a\Phi)e^a \wedge e^5 + \exp\{\beta\Phi\}d\omega^5 + \omega^5_a \wedge e^a + \omega^5_5 \wedge e^5 \quad (3.19)$$

and metric

$$\omega_{ab} = -\omega_{ba} \qquad \omega_{a5} = -\omega_{5a} \qquad \omega_{55} = -\omega_{55} \overset{!}{=} 0 \quad (3.20)$$

with respect to the orthonormal basis (3.17) on P. As before

$$\omega^a{}_5 = -\eta^{ab}\omega^5{}_b. \quad (3.21)$$

Our goal is to find ω^α_β for which this holds. The forms de^a, $d\omega^i$ are already known from (1.46), and (2.1) on M

$$\pi^* d\hat{e}^a = -\pi^*\left(\hat{\omega}^a_b \wedge \hat{e}^b\right) \quad (3.22)$$

$$d\omega^1 = \Omega^1 \quad (3.23)$$

The equation (2.7) reduces in case of $U(1)$ to (3.23), since the structure constants are zero. We can rewrite this as

$$\pi^* d\hat{e}^a = (\pi^* d\hat{e}^a)_0 \quad (3.24)$$

$$d\omega^1 = (d\omega^1)_0 \quad (3.25)$$

The connection forms are 1-forms so

$$\omega^a_b = (\omega^a_b)_0 + (\omega^a_b)_1 \quad (3.26)$$

$$\omega^5_a = (\omega^5_a)_0 + (\omega^5_a)_1 \quad (3.27)$$

$$\omega^5_5 = 0 \quad (3.28)$$

Now when we combine everything and add it to equations (3.18) and (3.19) we get: (3.19) we get:

$$
\begin{aligned}
0 &= \alpha(e_b\Phi)(e^b)_0 \wedge (e^a)_0 + \exp\{\alpha\Phi\}(d\pi^*\hat{e}^a)_0 \\
&\quad +(\omega^a_b)_0 \wedge (e^b)_0 + (\omega^a_b)_1 \wedge (e^b)_0 \\
&\quad -\eta^{ab}\eta_{55}(\omega^5_b)_0 \wedge (e^5)_1 - \eta^{ab}\eta_{55}(\omega^5_b)_1 \wedge (e^5)_1 \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \beta(e_a\Phi)(e^a)_0 \wedge (e^5)_1 + \exp\{\beta\Phi\}(d\omega^5)_0 \\
&\quad +(\omega^5_a)_0 \wedge (e^a)_0 + (\omega^5_a)_1 \wedge (e^a)_0 \qquad (3.30)
\end{aligned}
$$

Since on the LHS of both equations is zero, so the form in each vertical degree separately must be zero. For vertical degree 0:

$$
\begin{aligned}
0 &= \alpha(e_b\Phi)(e^b)_0 \wedge (e^a)_0 + \exp\{\alpha\Phi\}(d\pi^*\hat{e}^a)_0 + (\omega^a_b)_0 \wedge (e^b)_0 \qquad (3.31) \\
0 &= \exp\{\beta\Phi\}(d\omega^5)_0 + (\omega^5_a)_0 \wedge (e^a)_0 \qquad (3.32)
\end{aligned}
$$

For vertical degree 1:

$$
\begin{aligned}
0 &= (\omega^a_b)_1 \wedge (e^b)_0 - \eta^{ab}\eta_{55}(\omega^5_b)_0 \wedge (e^i)_1 \qquad (3.33) \\
0 &= \beta(e_a\Phi)(e^a)_0 \wedge (e^5)_1 + (\omega^5_a)_1 \wedge (e^a)_0 \qquad (3.34)
\end{aligned}
$$

Since we have only one vertical form e^5 the vertical degree 2 does not exists.

When we add concrete familiar objects from (3.22) and (3.23,we get from vertical degree 0 [2] [3]

$$
\begin{aligned}
0 &= \alpha(e_b\Phi)e^b \wedge e^a - (\pi^*\hat{\omega}^a_b) \wedge e^b + (\omega^a_b)_0 \wedge e^b && (3.35)\\
(\omega^a_b)_0 \wedge e^b &= \alpha(e_b\Phi)e^a \wedge e^b + (\pi^*\hat{\omega}^a_b) \wedge e^b \\
(\omega^a_b)_0 &= \pi^*\hat{\omega}^a_b + \alpha j^c \left(\eta^{ad} i_d(e_b\Phi)e^c - \eta^{ae}\eta_{cf} i_b(e_e\Phi)e^f\right) \\
(\omega^a_b)_0 &= \pi^*\hat{\omega}^a_b + \alpha j^c \left(\eta^{ad}(e_b\Phi)\delta^c_d - \eta^{ae}\eta_{cf}(e_e\Phi)\delta^f_b\right) \\
&\Rightarrow (\omega^a_b)_0 = \pi^*\hat{\omega}^a_b + \alpha\left((e_b\Phi)e^a - \eta^{ac}\eta_{bd}(e_c\Phi)e^d\right) && (3.36)\\
0 &= -\frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}\Omega^1_{ab}e^b \wedge e^a + (\omega^5_a)_0 \wedge e^a && (3.37)\\
&\Rightarrow (\omega^5_a)_0 = \frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}\Omega^1_{ab}e^b && (3.38)
\end{aligned}
$$

from vertical degree 1:

$$
\begin{aligned}
0 &= (\omega^a_b)_1 \wedge e^b - \eta^{ab}\eta_{55}\frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}\Omega^1_{bc}e^c \wedge e^i \\
0 &= (\omega^a_b)_1 \wedge e^b + \frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}{\Omega_1}^a{}_b e^5 \wedge e^b && (3.39)\\
&\Rightarrow (\omega^a_b)_1 = -\frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}{\Omega_1}^a{}_b e^5 && (3.40)\\
0 &= \beta(e_a\Phi)e^a \wedge e^5 + (\omega^5_a)_1 \wedge e^a && (3.41)\\
&\Rightarrow (\omega^5_a)_1 = \beta(e_a\Phi)e^5 && (3.42)
\end{aligned}
$$

For the sake of simplicity we denote $\Omega^1_{ab} \equiv \mathcal{F}_{ab}$. When we combine this all together we get the connection forms on P:

$$
\begin{aligned}
\omega^a_b &= \pi^*\hat{\omega}^a_b + \alpha\left((e_b\Phi)e^a - \eta^{ac}\eta_{bd}(e_c\Phi)e^d\right) - \frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}\mathcal{F}^a{}_b e^5 \\
\omega^5_a &= \frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}\mathcal{F}_{ab}e^b + \beta(e_a\Phi)e^5 \\
\omega^a_5 &= -\frac{1}{2}\exp\{(\beta-2\alpha)\Phi\}\mathcal{F}^a{}_b e^b - \beta\eta^{ab}(e_b\Phi)e^5 \\
\omega^5_5 &= 0 && (3.43)
\end{aligned}
$$

[2] For solution to the first one see apendix in [2]

[3] $\Omega^5 = \frac{1}{2}\Omega^5_{ab}\pi^*(\hat{e}^a \wedge \hat{e}^b) = \frac{1}{2}\exp\{-2\alpha\Phi\}\Omega^5_{ab}e^a \wedge e^b$

Note that α and β are not completely arbitrary. For a deeper understanding why see [12]. Let us just use the result obtained there and state:

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\alpha, \tag{3.44}$$

where $D = \dim M$.

3.3 Hilbert action and field equations

Similarly as in the previous chapter we calculate the Scalar curvature from the connection forms. If you are interested, the whole calculation is listed in appendix IV. Here we list just the result:

$$\begin{aligned} R &= \exp\{-2\alpha\Phi\}\pi^*\hat{R} - 2\alpha\exp\{-2\alpha\Phi\}(\hat{\Box}\Phi) - \frac{1}{2}\eta^{ab}(e_a\Phi)(e_b\Phi) \\ &\quad -\frac{1}{4}\exp\{-2D\alpha\Phi\})\mathcal{F}_{ab}\mathcal{F}^{ab} \end{aligned} \tag{3.45}$$

No we can with fanfare announce the long awaited Hilbert action with scalar curvature:

$$\begin{aligned} S_H[g,\Phi] &= \frac{1}{16\pi}\int_P \Big(\exp\{-2\alpha\Phi\}\pi^*\hat{R} - 2\alpha\exp\{-2\alpha\Phi\}(\hat{\Box}\Phi) \\ &\quad -\frac{1}{2}\eta^{ab}(e_a\Phi)(e_b\Phi) - \frac{1}{4}\exp\{-2D\alpha\Phi\})\mathcal{F}_{ab}\mathcal{F}^{ab}\Big)\omega_g \end{aligned} \tag{3.46}$$

Now we can work out the field equations, similar as in the case without the scalar field they are $R_{\alpha\beta} = 0$. The calculation of the Ricci tensor can be found in appendix V. Our field equations are:

$$\begin{aligned} \pi^*\hat{R}_{ab} &= \alpha\eta_{ab}\hat{\Box}\Phi + \exp\{2\alpha\Phi\}\frac{1}{2}(e_a\Phi)(e_b\Phi) \\ &\quad -\frac{1}{2}\exp\{-2(D-1)\alpha\Phi\}\mathcal{F}^c_a\mathcal{F}_{bc} \end{aligned} \tag{3.47}$$

$$0 = \hat{\nabla}_b(\exp\{-2(D-1)\alpha\Phi\}\mathcal{F}^b_a) \tag{3.48}$$

$$\hat{\Box}\Phi = -\frac{1}{4(D-2)\alpha}\exp\{-2(D-1)\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_a \tag{3.49}$$

From the last equation it is clearly visible, that we can not set $\Phi = 0$.

Conclusion

We convinced ourselves that Kaluza's original idea was not complete since the lack of the scalar field led to a doubtful field equation (2.63). The remedy came when we included the scalar field. There we can see that the scalar field can not be simply set to zero, as tempting as it can be, since would be violating the field equation for Φ (3.49). But regardless the mistake that Kaluza and Klein made their discovery opened the door for discovering new physics by calculating things in other (higher-dimensional) spaces than is our usual Minkowski space. It also provided a cornerstone for developing the geometry of bundles which can be now used to interpret higher dimensional theories as string theories in our space-time.

An interesting future study could be to take a deeper look at the case of a general compact Lie group and corresponding additional fields (analogous of our scalar field for the five dimensional case) and work out how to add the missing degrees of freedom.

Appendix I

Proof of $Q\alpha_{(q)} = q\alpha_{(q)}$

Here we are going to prove $Q\alpha_{(q)} = q\alpha_{(q)}$ which was listed in (1.26) as a useful observation.

$$Q = j^k i_k \tag{I.1}$$

The definitions of i_v an j_v are:

$$(i_v\alpha)(u,\ldots,w) \;:=\; \alpha(v,u,\ldots,w) \qquad \alpha \in \Lambda^p L^*,\ p \geq 1 \tag{I.2}$$

$$j_v\alpha \;:=\; \tilde{v} \wedge \alpha \qquad \tilde{v} \equiv g(v,\cdot) \tag{I.3}$$

It is enough to prove this on a monomial $e^i \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b$, which is a p form with vertical degree q.

$$j^k\left(i_k e^i \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b\right) \;=\; j^k\left(\delta^i_k \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b + \ldots + (-1)^q \delta^j_k e^i \wedge \ldots \wedge e^a \wedge \ldots \wedge e^b\right)$$

After acting by the interior product they are just q terms left (see (1.20)).

$$\begin{aligned}
&= \left(\delta^i_k e^k \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b + \ldots + (-1)^q \delta^j_k e^k \wedge e^i \wedge \ldots \wedge e^a \wedge \ldots \wedge e^b\right)\\
&= \left(e^i \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b + \ldots + (-1)^q e^j \wedge e^i \wedge \ldots \wedge e^a \wedge \ldots \wedge e^b\right)\\
&= \left(e^i \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b + \ldots + e^i \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b\right)\\
&= q e^i \wedge \ldots \wedge e^j \wedge e^a \wedge \ldots \wedge e^b
\end{aligned} \tag{I.4}$$

Appendix II

Calculation of scalar curvature without scalar field

Before we start calculating the scalar curvature, we are going to prove some theorems which come in handy in the calculation.

Theorem 1. *Let R_{ab} be the Ricci tensor. R_{ab} may be encoded into Ricci forms R_a and that these forms are simply related to the curvature forms Ω^a_b and the scalar curvature R*

$$R_a := R_{ab}e^b \qquad R_a = i_b\Omega^b_a \qquad i^a R_a \equiv i^a i_b \Omega^b_a = R$$

Proof. The curvature forms are defined:

$$\Omega^a_b := \frac{1}{2}R^a_{bcd}(e^c \wedge e^d) \tag{II.1}$$

By direct computation we get:

$$i_b\Omega^b_a = \frac{1}{2}i_b R^b_{acd}(e^c \wedge e^d) = \frac{1}{2}R^b_{acd}\delta^c_b e^d - \frac{1}{2}R^b_{acd}e^c\delta^d_b = \frac{1}{2}R^b_{abd}e^d - \frac{1}{2}R^b_{acb}e^c \tag{II.2}$$

Let us use the antisymmetry in the last two indices of the Riemann tensor:

$$= \frac{1}{2}R^b_{abd}e^d + \frac{1}{2}R^b_{abc}e^c = \frac{1}{2}R_{ad}e^d + \frac{1}{2}R_{ad}e^d = R_{ad}e^d = R_a \tag{II.3}$$

□

Theorem 2. *for the RLC connection the Ricci (and then also Einstein) tensor is symmetric*

$$\text{RLC connection} \quad \Rightarrow \quad R_{ab} = R_{ba} \; G_{ab} = G_{ba}$$

Proof. The Einstein tensor is:

$$G_{ab} = R_a b - \frac{1}{2} R g_a b \tag{II.4}$$

The metric tensor is symmetric (and non-degenerate) by definition so it is enough to show that RLC $\Rightarrow$ symmetry of Ricci tensor. From Ricci identity we get:

$$\Omega \wedge e = 0 \tag{II.5}$$

So if we act on the LHS by i_b we get also 0:

$$i_b(\Omega^b_a \wedge e^a) = (i_b \Omega^b_a) \wedge e^a + \Omega^b_a \wedge (i_b e^a) = R_{ab} e^b \wedge e^a + \Omega^b_a \delta^a_b = R_{ab} e^b \wedge e^a + \Omega^a_a \tag{II.6}$$

However, in an orthonormal basis is $\Omega^a_a = 0$, thus necessarily:

$$R_{ab} e^b \wedge e^a \stackrel{!}{=} 0 \tag{II.7}$$

Since wedge is an antisymmetric product this is only (non-trivially) satisfied by a symmetric Ricci tensor. □

We calculate the scalar curvature R using the curvature forms.

$$\begin{aligned} R &= i^\alpha i_\beta \Omega^\beta_\alpha = \eta^{\alpha\gamma} i_\gamma i_\beta \Omega^\beta_\alpha && \text{(II.8)} \\ &= i_\alpha i_\beta \Omega^{\beta\alpha} = \Omega^{\beta\alpha}(e_\beta, e_\alpha) && \text{(II.9)} \\ &= \Omega^{ab}(e_a, e_b) + \Omega^{ai}(e_a, e_i) + \Omega^{ia}(e_i, e_a) + \Omega^{ij}(e_i, e_j) && \text{(II.10)} \\ &= \Omega^{ab}(e_a, e_b) + 2\Omega^{ai}(e_a, e_i) + \Omega^{ij}(e_i, e_j) && \text{(II.11)} \end{aligned}$$

Luckily we do not need know the full reading of the curvature forms. We just need to know what it looks like when adding the needed arguments. To express the curvature forms from already known connexion forms we are going to use the Cartan structure equation:

$$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta \tag{II.12}$$

This means for us:

$$\begin{aligned}
\Omega^{ab}(e_a, e_b) &= \eta^{bc}\Omega^a_c(e_a, e_b) &\text{(II.13)}\\
&= \eta^{bc}\left(d\omega^a_c + \omega^a_d \wedge \omega^d_c + \omega^a_i \wedge \omega^i_c\right)(e_a, e_b) &\text{(II.14)}\\
&= \left(d\omega^{ab} + \omega^a_d \wedge \omega^{db} + \omega^a_i \wedge \omega^{ib}\right)(e_a, e_b) &\text{(II.15)}
\end{aligned}$$

For the first term:

$$\begin{aligned}
d\omega^a_b &= \pi^* d\hat{\omega}^a_b - \frac{1}{2}(d\Omega_i{}^a{}_b) \wedge e^i - \frac{1}{2}(\Omega_i{}^a{}_b) de^i\\
&= \pi^* d\hat{\omega}^a_b - \frac{1}{2}(d\Omega_i{}^a{}_b) \wedge e^i\\
&\quad -\frac{1}{4}\Omega_i{}^a{}_b\Omega^i_{cd} e^c \wedge e^d + \frac{1}{4}\Omega_i{}^a{}_b c^i_{jk} e^j \wedge e^k\\
d\omega^{ab} &= \pi^* d\hat{\omega}^{ab} - \frac{1}{2}(d\Omega_i{}^{ab}) \wedge e^i\\
&\quad -\frac{1}{4}\Omega_i{}^{ab}\Omega^i_{cd} e^c \wedge e^d + \frac{1}{4}\Omega_i{}^{ab} c^i_{jk} e^j \wedge e^k\\
d\omega^{ab}(e_a, e_b) &= \pi^* d\hat{\omega}^{ab}(e_a, e_b) - \frac{1}{4}\Omega_i{}^{ab}\Omega^i_{cd}(\delta^c_a\delta^d_b - \delta^c_b\delta^d_a)\\
&= \pi^* d\hat{\omega}^{ab}(e_a, e_b) - \frac{1}{4}\Omega_i{}^{ab}\Omega^i_{ab} + \frac{1}{4}\Omega_i{}^{ab}\Omega^i_{ba}\\
&= \boxed{\pi^* d\hat{\omega}^{ab}(e_a, e_b) - \frac{1}{2}\Omega_i{}^{ab}\Omega^i_{ab}} &\text{(II.16)}
\end{aligned}$$

For the second term:

$$\begin{aligned}
\omega^a_d \wedge \omega^{db}(e_a, e_b) &= \left(\pi^*\hat{\omega}^a_d - \frac{1}{2}\Omega_i{}^a{}_d e^i\right) \wedge \left(\pi^*\hat{\omega}^{db} - \frac{1}{2}\Omega_i{}^{db} e^i\right)(e_a, e_b)\\
&= \boxed{\pi^*\left(\hat{\omega}^a_d \wedge \hat{\omega}^{db}\right)(e_a, e_b)} &\text{(II.17)}
\end{aligned}$$

For the third term:

$$\begin{aligned}
\omega^a_i \wedge \omega^{ib}(e_a, e_b) &= \left(-\frac{1}{2}\Omega_i{}^a{}_c e^c\right) \wedge \left(\frac{1}{2}\Omega^{ib}_d e^d\right)(e_a, e_b) \\
&= -\frac{1}{4}\Omega_i{}^a{}_c \Omega^{ib}_d \left(\delta^c_a \delta^d_b - \delta^c_b \delta^d_a\right) \\
&= \boxed{-\frac{1}{4}\Omega_i{}^a{}_a \Omega^{ib}_b + \frac{1}{4}\Omega_i{}^a{}_b \Omega^{ib}_a} \qquad \text{(II.18)}
\end{aligned}$$

But $\Omega^i_{ab} = -\Omega^i_{ba}$ so $\Omega_i{}^a{}_a = \Omega^{ib}_b = 0$. So together we get

$$\Omega^{ab}(e_a, e_b) = \pi^* \left(\hat{d\omega}^{ab} + \hat{\omega}^a_d \wedge \hat{\omega}^{db}\right)(e_a, e_b) - \frac{1}{2}\Omega_i{}^{ab}\Omega^i_{ab} + \frac{1}{4}\Omega_i{}^a{}_b \Omega^{ib}_a \quad \text{(II.19)}$$

In the first term we recognize the scalar curvature $\hat{R}$ on M:

$$\boxed{\Omega^{ab}(e_a, e_b) = \pi^* \hat{R} + \frac{3}{4}\Omega_i{}^{ab}\Omega^i_{ba}} \qquad \text{(II.20)}$$

For the other terms of R:

$$\Omega^{ai}(e_a, e_i) = \left(d\omega^{ai} + \omega^a_b \wedge \omega^{bi} + \omega^a_j \wedge \omega^{ji}\right)(e_a, e_i) \qquad \text{(II.21)}$$

$$\begin{aligned}
d\omega^{ai}(e_a, e_i) &= \left(-\frac{1}{2}(d\Omega^{ia}_b) \wedge e^b - \frac{1}{2}\Omega^{ia}_b de^b\right)(e_a, e_i) \\
&= \left(-\frac{1}{2}(d\Omega^{ia}_b) \wedge e^b + \frac{1}{2}\Omega^{ia}_b \pi^*(\hat{\omega}^b_c \wedge \hat{e}^c)\right)(e_a, e_i) \\
&= \frac{1}{2}(d\Omega^{ia}_b)\delta^b_a(e_i) = \frac{1}{2}(d\Omega^{ia}_a)(e_i) = 0 \qquad \text{(II.22)}
\end{aligned}$$

$$\begin{aligned}
\omega^a_b \wedge \omega^{bi}(e_a, e_i) &= \left(\pi^*\hat{\omega}^a_b - \frac{1}{2}\Omega_j{}^a{}_b e^j\right) \wedge \left(-\frac{1}{2}\Omega^{ib}_c e^c\right)(e_a, e_i) \\
&= \frac{1}{4}\Omega_j{}^a{}_b \Omega^{ib}_c e^j \wedge e^c(e_a, e_i) \\
&= -\frac{1}{4}\Omega_j{}^a{}_b \Omega^{ib}_c \delta^j_i \delta^c_a \\
&= -\frac{1}{4}\Omega_i{}^a{}_b \Omega^{ib}_a = \boxed{-\frac{1}{4}\Omega_i{}^{ab}\Omega^i_{ba}} \qquad \text{(II.23)}
\end{aligned}$$

$$
\begin{aligned}
\omega^a_j \wedge \omega^{ji}(e_a, e_i) &= \left(-\frac{1}{2}\Omega_j{}^a{}_b e^b\right) \wedge \left(\frac{1}{2}c^{ji}_k e^k\right)(e_a, e_i) \\
&= \frac{1}{4}\Omega_j{}^a{}_b c^{ji}_k e^b \wedge e^k(e_a, e_i) \\
&= \frac{1}{4}\Omega_j{}^a{}_b c^{ji}_k \delta^b_a \delta^k_i \\
&= \frac{1}{4}\Omega_j{}^a{}_a c^{ji}_i = 0 \qquad \text{(II.24)}
\end{aligned}
$$

$$
\boxed{\Omega^{ai}(e_a, e_i) = -\frac{1}{4}\Omega_i{}^{ab}\Omega^i_{ba}} \qquad \text{(II.25)}
$$

$$
\Omega^{ij}(e_i, e_j) = \left(d\omega^{ij} + \omega^i_a \wedge \omega^{aj} + \omega^i_k \wedge \omega^{kj}\right)(e_i, e_j) \qquad \text{(II.26)}
$$

$$
\begin{aligned}
d\omega^{ij}(e_i, e_j) &= -\frac{1}{2}c_k{}^{ij} de^k(e_i, e_j) && \text{(II.27)} \\
&= -\frac{1}{4}c_k{}^{ij}\left(\Omega^k_{ab}e^a \wedge e^b - c^k_{lm}e^l \wedge e^m\right)(e_i, e_j) && \text{(II.28)} \\
&= \frac{1}{4}c_k{}^{ij}c^k_{lm}(\delta^l_i\delta^m_j - \delta^l_j\delta^m_i) && \text{(II.29)} \\
&= \frac{1}{4}c_k{}^{ij}c^k_{ij} - \frac{1}{4}c_k{}^{ij}c^k_{ji} && \text{(II.30)} \\
&= \boxed{\frac{1}{2}c_k{}^{ij}c^k_{ij}} && \text{(II.31)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^i_a \wedge \omega^{aj}(e_i, e_j) &= \left(\frac{1}{2}\Omega^i_{ab}e^b\right) \wedge \left(-\frac{1}{2}\Omega^{ja}_c e^c\right)(e_i, e_j) && \text{(II.32)} \\
&= 0 && \text{(II.33)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^i_k \wedge \omega^{kj}(e_i, e_j) &= \left(-\frac{1}{2}c^i_{kl}e^l\right) \wedge \left(-\frac{1}{2}c_m{}^{kj}e^m\right)(e_i, e_j) && \text{(II.34)} \\
&= \frac{1}{4}c^i_{kl}c_m{}^{kj}e^l \wedge e^m(e_i, e_j) && \text{(II.35)} \\
&= \frac{1}{4}c^i_{kl}c_m{}^{kj}(\delta^l_i\delta^m_j - \delta^l_j\delta^m_i) && \text{(II.36)} \\
&= \boxed{\frac{1}{4}c^i_{ki}c_j{}^{kj} - \frac{1}{4}c^i_{kj}c_i{}^{kj}} && \text{(II.37)}
\end{aligned}
$$

Using Jcaobi identity we can show $c^i_{ki}{c_j}^{kj} = 0$, so we get

$$\boxed{\Omega^{ij}(e_i, e_j) = \frac{1}{4}c^i_{kj}{c_i}^{kj}} \ . \tag{II.38}$$

So putting equations (II.20), (II.25) and (II.38) together we get the scalar curvature on P:

$$R \ = \ \pi^*\hat{R} + \frac{1}{4}\Omega^i_{ab}{\Omega_i}^{ba} - \frac{1}{4}c^i_{jk}{c_i}^{kj} \tag{II.39}$$

Appendix III

Calculation of Ricci tensor without scalar field

The procedure to obtain the Ricci tensor out of the curvature forms will be as following:

$$\begin{aligned}
\Omega^{\alpha}_{\beta} &= \frac{1}{2}R^{\alpha}_{\beta\gamma\delta}e^{\gamma}\wedge e^{\delta} &\text{(III.1)}\\
i_{\rho}\Omega^{\alpha}_{\beta} &= i_{\rho}\frac{1}{2}R^{\alpha}_{\beta\gamma\delta}e^{\gamma}\wedge e^{\delta} &\text{(III.2)}\\
&= \frac{1}{2}R^{\alpha}_{\beta\gamma\delta}\left(\delta^{\gamma}_{\rho}e^{\delta}-e^{\gamma}\delta^{\delta}_{\rho}\right) &\text{(III.3)}\\
&= \frac{1}{2}\left(R^{\alpha}_{\beta\rho\delta}e^{\delta}-R^{\alpha}_{\beta\gamma\rho}e^{\gamma}\right) &\text{(III.4)}\\
&= \frac{1}{2}\left(R^{\alpha}_{\beta\rho\delta}e^{\delta}+R^{\alpha}_{\beta\rho\gamma}e^{\gamma}\right) &\text{(III.5)}\\
&= R^{\alpha}_{\beta\rho\gamma}e^{\gamma} &\text{(III.6)}\\
i_{\sigma}i_{\rho}\Omega^{\alpha}_{\beta} &= i_{\sigma}R^{\alpha}_{\beta\rho\gamma}e^{\gamma} &\text{(III.7)}\\
&= R^{\alpha}_{\beta\rho\gamma}\delta^{\gamma}_{\sigma} &\text{(III.8)}\\
R^{\alpha}_{\beta\gamma\delta} &= i_{\delta}i_{\gamma}\Omega^{\alpha}_{\beta} &\text{(III.9)}
\end{aligned}$$

So for the Ricci tensor we get:

$$R_{\alpha\beta} := R^{\gamma}_{\alpha\gamma\beta} = i_{\beta}i_{\gamma}\Omega^{\gamma}_{\alpha} = \Omega^{\gamma}_{\alpha}(e_{\gamma}, e_{\beta}) \tag{III.10}$$

That decouples into 3 equations[1].

$$R_{ab} = \Omega^c_a(e_c, e_b) + \Omega^i_a(e_i, e_b) \tag{III.11}$$

$$R_{ai} = \Omega^b_a(e_b, e_i) + \Omega^j_a(e_j, e_i) \tag{III.12}$$

$$R_{ij} = \Omega^a_i(e_a, e_j) + \Omega^k_i(e_k, e_j) \tag{III.13}$$

We start by computing R_{ab}

$$\Omega^c_a(e_c, e_b) = \left(d\omega^c_a + \omega^c_d \wedge \omega^d_a + \omega^c_i \wedge \omega^i_a\right)(e_c, e_b) \tag{III.14}$$

$$\begin{aligned}
d\omega^c_a(e_c, e_b) &= \left(\pi^* d\hat{\omega}^c_a - \frac{1}{2}(d\Omega_i{}^c{}_a) \wedge e^i - \frac{1}{2}(\Omega_i{}^c{}_a) de^i\right)(e_c, e_b) && \text{(III.15)}\\
&= \left(\pi^* d\hat{\omega}^c_a - \frac{1}{4}\Omega_i{}^c{}_a \Omega^i_{de} e^d \wedge e^e + \frac{1}{4}\Omega_i{}^c{}_a c^i_{jk} e^j \wedge e^k\right)(e_c, e_b) && \text{(III.16)}\\
&= \pi^* d\hat{\omega}^c_a(e_c, e_b) - \frac{1}{4}\Omega_i{}^c{}_a \Omega^i_{de}\delta^d_c \delta^e_b + \frac{1}{4}\Omega_i{}^c{}_a \Omega^i_{de}\delta^d_b \delta^e_c && \text{(III.17)}\\
&= \pi^* d\hat{\omega}^c_a(e_c, e_b) - \frac{1}{4}\Omega_i{}^c{}_a \Omega^i_{cb} + \frac{1}{4}\Omega_i{}^c{}_a \Omega^i_{bc} && \text{(III.18)}\\
&= \pi^* d\hat{\omega}^c_a(e_c, e_b) - \frac{1}{2}\Omega_i{}^c{}_a \Omega^i_{cb} && \text{(III.19)}
\end{aligned}$$

$$\begin{aligned}
\omega^c_d \wedge \omega^d_a(e_c, e_b) &= \left(\pi^*\hat{\omega}^c_d - \frac{1}{2}\Omega_i{}^c{}_d e^i\right) \wedge \left(\pi^*\hat{\omega}^d_a - \frac{1}{2}\Omega_j{}^d{}_a e^j\right)(e_c, e_b) && \text{(III.20)}\\
&= \pi^*(\hat{\omega}^c_d \wedge \hat{\omega}^d_a)(e_c, e_b) && \text{(III.21)}
\end{aligned}$$

$$\begin{aligned}
\omega^c_i \wedge \omega^i_a(e_c, e_b) &= \left(-\frac{1}{2}\Omega_i{}^c{}_d e^d\right) \wedge \left(\frac{1}{2}\Omega^i_{ae} e^e\right)(e_c, e_b) && \text{(III.22)}\\
&= -\frac{1}{4}\Omega_i{}^c{}_d \Omega^i_{ae}\left(\delta^d_c \delta^e_b - \delta^d_b \delta^e_c\right) && \text{(III.23)}\\
&= -\frac{1}{4}\Omega_i{}^c{}_c \Omega^i_{ab} + \frac{1}{4}\Omega_i{}^c{}_b \Omega^i_{ac} && \text{(III.24)}\\
&= \frac{1}{4}\Omega_i{}^c{}_b \Omega^i_{ac} && \text{(III.25)}
\end{aligned}$$

[1] Just 3, no 4 since the Ricci tensor is symmetric.

$$\Omega^i_a(e_i, e_b) = \left(d\omega^i_a + \omega^i_d \wedge \omega^d_a + \omega^i_j \wedge \omega^j_a\right)(e_i, e_b) \tag{III.26}$$

$$d\omega^i_a(e_i, e_b) = \frac{1}{2}(d\Omega^i_{ac})e^c(e_i, e_b) + \frac{1}{2}\Omega^i_{ac}(de^c)(e_i, e_b) \tag{III.27}$$

$$= \frac{1}{2}(d\Omega^i_{ac})(e_i)\delta^c_b - \frac{1}{2}\Omega^i_{ac}\pi^*(\hat{\omega}^c_d \wedge \hat{e}^d)(e_i, e_b) \tag{III.28}$$

$$= \frac{1}{2}(d\Omega^i_{ab})(e_i)\stackrel{!}{=}0 \tag{III.29}$$

This is zero, since the Ricci tensor is symmetric in indexes a, b, but Ω^i_{ab} is antisymmetric.

$$\omega^i_d \wedge \omega^d_a(e_i, e_b) = \left(\frac{1}{2}\Omega^i_{dc}e^c\right) \wedge \left(\pi^*\hat{\omega}^d_a - \frac{1}{2}\Omega_j{}^d{}_a e^j\right)(e_i, e_b) \tag{III.30}$$

$$= \frac{1}{4}\Omega^i_{dc}\delta^c_b\Omega_j{}^d{}_a\delta^j_i \tag{III.31}$$

$$= \frac{1}{4}\Omega^i_{cb}\Omega_i{}^c{}_a \tag{III.32}$$

$$\omega^i_j \wedge \omega^j_a(e_i, e_b) = \left(-\frac{1}{2}c^i_{jk}e^k\right) \wedge \left(\frac{1}{2}\Omega^j_{ac}e^c\right)(e_i, e_b) \tag{III.33}$$

$$= -\frac{1}{4}c^i_{jk}\delta^k_i\frac{1}{2}\Omega^j_{ac}\delta^c_b \tag{III.34}$$

$$= -\frac{1}{4}c^i_{ji}\frac{1}{2}\Omega^j_{ab}\stackrel{!}{=}0 \tag{III.35}$$

So the R_{ab} component of the Ricci tensor reads

$$R_{ab} = \pi^*d\hat{\omega}^c_a(e_c, e_b) + \pi^*(\hat{\omega}^c_d \wedge \hat{\omega}^d_a)(e_c, e_b) \tag{III.36}$$

$$-\frac{1}{2}\Omega_i{}^c{}_a\Omega^i_{cb} + \frac{1}{4}\Omega_i{}^c{}_b\Omega^i_{ac} + \frac{1}{4}\Omega^i_{cb}\Omega_i{}^c{}_a \tag{III.37}$$

$$= \pi^*\hat{R}_{ab} - \frac{1}{2}\Omega_i{}^c{}_a\Omega^i_{cb} - \frac{1}{4}\Omega_i{}^c{}_a\Omega^i_{cb} + \frac{1}{4}\Omega_i{}^c{}_a\Omega^i_{cb} \tag{III.38}$$

$$= \boxed{\pi^*\hat{R}_{ab} - \frac{1}{2}\Omega_i{}^c{}_a\Omega^i_{cb}} \tag{III.39}$$

We proceed to R_{ai}:

$$\Omega^b_a(e_b, e_i) = \left(d\omega^b_a + \omega^b_c \wedge \omega^c_a + \omega^b_j \wedge \omega^j_a\right)(e_b, e_i) \tag{III.40}$$

$$
\begin{aligned}
d\omega^b_a(e_b, e_i) &= d\left(\pi^*\hat{\omega}^b_a - \frac{1}{2}\Omega_j{}^b{}_a e^j\right)(e_b, e_i) && \text{(III.41)}\\
&= -\frac{1}{2}(d\Omega_j{}^b{}_a)(e_b)\delta^j_i && \text{(III.42)}\\
&= -\frac{1}{2}(d\Omega_i{}^b{}_a)(e_b) && \text{(III.43)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^b_c \wedge \omega^c_a(e_b, e_i) &= \left(\pi^*\hat{\omega}^b_c - \frac{1}{2}\Omega_j{}^b{}_c e^j\right) \wedge \left(\pi^*\hat{\omega}^c_a - \frac{1}{2}\Omega_k{}^c{}_a e^k\right)(e_b, e_i) && \text{(III.44)}\\
&= -\frac{1}{2}\pi^*\hat{\omega}^b_c(e_b)\Omega_k{}^c{}_a\delta^k_i + \frac{1}{2}\Omega_j{}^b{}_c\delta^j_i\pi^*\hat{\omega}^c_a(e_b) && \text{(III.45)}\\
&= -\frac{1}{2}\Omega_i{}^c{}_a\pi^*\hat{\omega}^b_c(e_b) + \frac{1}{2}\Omega_i{}^b{}_c\pi^*\hat{\omega}^c_a(e_b) && \text{(III.46)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^b_j \wedge \omega^j_a(e_b, e_i) &= \left(-\frac{1}{2}\Omega_j{}^b{}_c e^c\right) \wedge \left(\frac{1}{2}\Omega^j_{ad} e^d\right)(e_b, e_i) && \text{(III.47)}\\
&= 0 && \text{(III.48)}
\end{aligned}
$$

$$
\Omega^j_a(e_j, e_i) = \left(d\omega^j_a + \omega^j_c \wedge \omega^c_a + \omega^j_k \wedge \omega^k_a\right)(e_j, e_i) \qquad \text{(III.49)}
$$

$$
\begin{aligned}
d\omega^j_a(e_j, e_i) &= d\left(\frac{1}{2}\Omega^j_{ab} e^b\right)(e_j, e_i) && \text{(III.50)}\\
&= 0 && \text{(III.51)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^j_c \wedge \omega^c_a(e_j, e_i) &= \left(\frac{1}{2}\Omega^j_{cb} e^b\right) \wedge \left(\pi^*\hat{\omega}^c_a - \frac{1}{2}\Omega_k{}^c{}_a e^k\right)(e_j, e_i) && \text{(III.52)}\\
&= 0 && \text{(III.53)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^j_k \wedge \omega^k_a(e_j, e_i) &= \left(-\frac{1}{2}c^j_{kl} e^l\right) \wedge \left(\frac{1}{2}\Omega^k_{ab} e^b\right)(e_j, e_i) && \text{(III.54)}\\
&= 0 && \text{(III.55)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ai} &= -\frac{1}{2}(d\Omega_i{}^b{}_a)(e_b) - \frac{1}{2}\Omega_i{}^c{}_a\pi^*\hat{\omega}^b_c(e_b) + \frac{1}{2}\Omega_i{}^b{}_c\pi^*\hat{\omega}^c_a(e_b) && \text{(III.56)}\\
&= -\frac{1}{2}\left(d\Omega_i{}^b{}_a + \Omega_i{}^c{}_a\pi^*\hat{\omega}^b_c - \Omega_i{}^b{}_c\pi^*\hat{\omega}^c_a\right)(e_b) && \text{(III.57)}\\
&= -\frac{1}{2}\left(D\Omega_i{}^b{}_a\right)(e_b) && \text{(III.58)}\\
&= \boxed{-\frac{1}{2}\nabla_b\Omega_i{}^b{}_a} && \text{(III.59)}
\end{aligned}
$$

When, we put R_{ai}, together, we spot something familiar - the three terms give together just one, with covariant exterior derivative[2]. which can be further simplified using the definition of the covariant derivative along a vector field[3].

And finally R_{ij}:

$$\Omega^a_i(e_a, e_j) = \left(d\omega^a_i + \omega^a_b \wedge \omega^b_i + \omega^a_k \wedge \omega^k_i\right)(e_a, e_j) \tag{III.60}$$

$$\begin{aligned} d\omega^a_i(e_a, e_j) &= d\left(-\frac{1}{2}\Omega_i{}^a{}_b e^b\right)(e_a, e_j) && \text{(III.61)} \\ &= \frac{1}{2}(d\Omega_i{}^a{}_b)(e_j)\delta^b_a = \frac{1}{2}(d\Omega_i{}^a{}_a)(e_j) = 0 && \text{(III.62)} \end{aligned}$$

$$\begin{aligned} \omega^a_b \wedge \omega^b_i(e_a, e_j) &= \left(\pi^*\hat{\omega}^a_b - \frac{1}{2}\Omega_k{}^a{}_b e^k\right) \wedge \left(-\frac{1}{2}\Omega_i{}^b{}_c e^c\right)(e_a, e_j) && \text{(III.63)} \\ &= -\frac{1}{4}\Omega_k{}^a{}_b \delta^k_j \Omega_i{}^b{}_c \delta^c_a && \text{(III.64)} \\ &= -\frac{1}{4}\Omega_j{}^a{}_b \Omega_i{}^b{}_a && \text{(III.65)} \end{aligned}$$

$$\begin{aligned} \omega^a_k \wedge \omega^k_i(e_a, e_j) &= \left(-\frac{1}{2}\Omega_k{}^a{}_b e^b\right) \wedge \left(-\frac{1}{2}c^k_{il} e^l\right)(e_a, e_j) && \text{(III.66)} \\ &= \frac{1}{4}\Omega_k{}^a{}_b \delta^b_a c^k_{il} \delta^l_j && \text{(III.67)} \\ &= \frac{1}{4}\Omega_k{}^a{}_a c^k_{ij} = 0 && \text{(III.68)} \end{aligned}$$

$$\Omega^k_i(e_k, e_j) = \left(d\omega^k_i + \omega^k_a \wedge \omega^a_i + \omega^k_l \wedge \omega^l_i\right)(e_k, e_j) \tag{III.69}$$

---

[2]See exercise 21.7.6 in [3].

[3]See exercise 21.3.6 in [3]. $\nabla_b \equiv \nabla_{e_b}$

$$
\begin{aligned}
d\omega_i^k(e_k, e_j) &= d\left(-\frac{1}{2}c_{il}^k e^l\right)(e_k, e_j) && \text{(III.70)}\\
&= -\frac{1}{2}c_{il}^k\left(\Omega^i - \frac{1}{2}c_{mn}^l e^m \wedge e^n\right)(e_k, e_j) && \text{(III.71)}\\
&= \frac{1}{4}c_{il}^k c_{mn}^l\left(\delta_k^m\delta_j^n - \delta_j^m\delta_k^n\right) && \text{(III.72)}\\
&= \frac{1}{4}c_{il}^k c_{kj}^l - \frac{1}{4}c_{il}^k c_{jk}^l && \text{(III.73)}\\
&= \frac{1}{2}c_{il}^k c_{kj}^l && \text{(III.74)}
\end{aligned}
$$

$$
\begin{aligned}
\omega_a^k \wedge \omega_i^a(e_k, e_j) &= \left(\frac{1}{2}\Omega_{ab}^k e^b\right) \wedge \left(-\frac{1}{2}\Omega_i{}^a{}_c e^c\right)(e_k, e_j) && \text{(III.75)}\\
&= 0 && \text{(III.76)}
\end{aligned}
$$

$$
\begin{aligned}
\omega_l^k \wedge \omega_i^l(e_k, e_j) &= \left(-\frac{1}{2}c_{lm}^k e^m\right) \wedge \left(-\frac{1}{2}c_{in}^l e^n\right)(e_k, e_j) && \text{(III.77)}\\
\text{‘} \quad &= \frac{1}{4}c_{lm}^k c_{in}^l\left(\delta_k^m\delta_j^n - \delta_j^m\delta_k^n\right) && \text{(III.78)}\\
&= \frac{1}{4}c_{lk}^k c_{ij}^l - \frac{1}{4}c_{lj}^k c_{ik}^l && \text{(III.79)}\\
&= -\frac{1}{4}c_{lj}^k c_{ik}^l && \text{(III.80)}
\end{aligned}
$$

So the R_{ij} component of the Ricci tensor reads

$$
\begin{aligned}
R_{ij} &= -\frac{1}{4}\Omega_j{}^a{}_b\Omega_i{}^b{}_a + \frac{1}{2}c_{il}^k c_{kj}^l - \frac{1}{4}c_{lj}^k c_{ik}^l && \text{(III.81)}\\
&= \boxed{-\frac{1}{4}\Omega_j{}^a{}_b\Omega_i{}^b{}_a + \frac{1}{4}c_{il}^k c_{kj}^l} && \text{(III.82)}
\end{aligned}
$$

Appendix IV

Calculation of scalar curvature with scalar field for $U(1)$

To compute the scalar curvature with the scalar field we are going to proceed as in appendix II.

$$
\begin{aligned}
R &= i_\alpha i_\beta \Omega^{\beta\alpha} = \Omega^{\beta\alpha}(e_\beta, e_\alpha) && \text{(IV.1)} \\
&= \Omega^{ab}(e_a, e_b) + \Omega^{a5}(e_a, e_5) + \Omega^{5a}(e_5, e_a) + \Omega^{55}(e_5, e_5) && \text{(IV.2)} \\
&= \Omega^{ab}(e_a, e_b) + 2\Omega^{5a}(e_a, e_5) && \text{(IV.3)}
\end{aligned}
$$

$\Omega^{55}(e_5, e_5) = 0$ because, in general $\Omega^{ij} = -\Omega^{ji}$.

We proceed with the firs term of R - $\Omega^{ab}(e_a, e_b)$. As before we are going to use the Cartan structure equation:

$$
\Omega^{ab}(e_a, e_b) = \left(d\omega^{ab} + \omega^a_d \wedge \omega^{db} + \omega^a_5 \wedge \omega^{5b}\right)(e_a, e_b) \qquad \text{(IV.4)}
$$

$$
\begin{aligned}
d\omega^{ab}(e_a, e_b) &= \pi^* d(\hat{\omega}^{ab})(e_a, e_b) \\
&\quad + \left(\alpha\eta^{bc}(e_d e_c \Phi) e^d \wedge e^a - \alpha\eta^{bc}(e_c\Phi)(\pi^*\omega^a_d) \wedge e^d\right)(e_a, e_b) \\
&\quad - \left(\alpha\eta^{ac}(e_d e_c \Phi) e^d \wedge e^b - \alpha\eta^{ac}(e_c\Phi)(\pi^*\omega^b_d) \wedge e^d\right)(e_a, e_b) \\
&\quad -\frac{1}{2}\exp\{(2\beta - 2\alpha)\Phi\}\mathcal{F}^{ab}\Omega^1(e_a, e_b) \\
&= \pi^* d(\hat{\omega}^{ab})(e_a, e_b) \\
&\quad +\alpha\eta^{bc}(e_d e_c\Phi)\left(\delta^d_a\delta^a_b - \delta^d_b\delta^a_a\right) \\
&\quad -\alpha\eta^{bc}(e_c\Phi)\left((\pi^*\omega^a_d)(e_a)\delta^d_b - (\pi^*\omega^a_d)(e_b)\delta^d_a\right) \\
&\quad -\alpha\eta^{ac}(e_d e_c\Phi)\left(\delta^d_a\delta^b_b - \delta^d_b\delta^b_a\right) \\
&\quad +\alpha\eta^{ac}(e_c\Phi)\left((\pi^*\omega^b_d)(e_a)\delta^d_b - (\pi^*\omega^b_d)(e_b)\delta^d_a\right) \\
&\quad -\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^{ab}\mathcal{F}_{cd}e^c \wedge e^d(e_a, e_b) \\
&= \pi^* d(\hat{\omega}^{ab})(e_a, e_b) \\
&\quad -\alpha(D-1)\eta^{bc}(e_b e_c\Phi) - \alpha\eta^{bc}(e_c\Phi)(\pi^*\omega^a_b)(e_a) \\
&\quad -\alpha(D-1)\eta^{ac}(e_a e_c\Phi) - \alpha\eta^{ac}(e_c\Phi)(\pi^*\omega^b_a)(e_b) \\
&\quad -\frac{1}{2}\exp\{-2D\alpha\Phi\}\mathcal{F}^{ab}\mathcal{F}_{ab} \\
&= \pi^* d(\hat{\omega}^{ab})(e_a, e_b) - 2(D-1)\alpha\eta^{ab}(e_a e_b\Phi) \qquad \text{(IV.5)} \\
&\quad -2\alpha(e_b\Phi)(\pi^*\omega^{ab})(e_a) - \frac{1}{2}\exp\{-2D\alpha\Phi\}\mathcal{F}^{ab}\mathcal{F}_{ab}
\end{aligned}
$$

$$
\begin{aligned}
\omega^a_d \wedge \omega^{db}(e_a, e_b) &= \pi^*\left(\hat{\omega}^a_d \wedge \hat{\omega}^{db}\right)(e_a, e_b) \\
&\quad +\alpha\eta^{bf}(e_f\Phi)(\pi^*\hat{\omega}^a_d) \wedge e^d(e_a, e_b) \\
&\quad -\alpha\eta^{dg}(e_g\Phi)(\pi^*\hat{\omega}^a_d) \wedge e^b(e_a, e_b) \\
&\quad +\alpha(e_d\Phi)e^a \wedge (\pi^*\hat{\omega}^{db})(e_a, e_b) \\
&\quad +\alpha^2\eta^{bf}(e_d\Phi)(e_f\Phi)e^a \wedge e^d(e_a, e_b) \\
&\quad -\alpha^2\eta^{dg}(e_d\Phi)(e_g\Phi)e^a \wedge e^b(e_a, e_b) \\
&\quad -\alpha\eta^{ac}\eta_{de}(e_c\Phi)e^e \wedge (\pi^*\hat{\omega}^{db})(e_a, e_b) \\
&\quad -\alpha^2\eta^{ac}\eta_{de}\eta^{bf}(e_c\Phi)(e_f\Phi)e^e \wedge e^d(e_a, e_b) \\
&\quad +\alpha^2\eta^{ac}\eta_{de}\eta^{dg}(e_c\Phi)(e_g\Phi)e^e \wedge e^b(e_a, e_b) \\
&= \pi^*\left(\hat{\omega}^a_d \wedge \hat{\omega}^{db}\right)(e_a, e_b) \\
&\quad +\alpha(e_b\Phi)(\pi^*\hat{\omega}^{ab})(e_a) - (D-1)\alpha(e_b\Phi)(\pi^*\hat{\omega}^{ab})(e_a) \\
&\quad +(D-1)\alpha(e_b\Phi)(\pi^*\hat{\omega}^{ba})(e_a) + (D-1)\alpha^2\eta^{ab}(e_a\Phi)(e_b\Phi) \\
&\quad -(D^2-D)\alpha^2\eta^{ab}(e_a\Phi)(e_b\Phi) - \alpha(e_b\Phi)(\pi^*\hat{\omega}^{ba})(e_a) \\
&\quad +(D-1)\alpha^2\eta^{ab}(e_a\Phi)(e_b\Phi) \\
&= \pi^*\left(\hat{\omega}^a_d \wedge \hat{\omega}^{db}\right)(e_a, e_b) \\
&\quad -2(D-2)\alpha(e_b\Phi)(\pi^*\hat{\omega}^{ab})(e_a) \\
&\quad -(D-2)(D-1)\alpha^2\eta^{ab}(e_a\Phi)(e_b\Phi) \qquad \text{(IV.6)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^a_5 \wedge \omega^{5b}(e_a, e_b) &= -\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_c\mathcal{F}^b_d e^c \wedge e^d(e_a, e_b) \\
&= \frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_a \qquad \text{(IV.7)}
\end{aligned}
$$

$$
\Omega^{5a}(e_5, e_a) = \left(d\omega^{5a} + \omega^5_b \wedge \omega^{ba}\right)(e_5, e_a) \qquad \text{(IV.8)}
$$

$$
\begin{aligned}
d\omega^{5a}(e_5, e_a) &= d\left(\frac{1}{2}\exp\{-D\alpha\Phi\})\mathcal{F}^a_b e^b + \beta\eta^{ab}(e_b\Phi)e^5\right)(e_5, e_a) \\
&= \frac{1}{2}\exp\{-D\alpha\Phi\})(d\mathcal{F}^a_b) \wedge e^b(e_5, e_a) \\
&\quad +\beta\eta^{ab}(e_c e_b\Phi)e^c \wedge e^5(e_5, e_a) \\
&\quad +\beta(\beta-\alpha)\eta^{ab}(e_b\Phi)(e_c\Phi)e^c \wedge e^5(e_5, e_a) \\
&= -\beta\eta^{ab}(e_a e_b\Phi) - \beta(\beta-\alpha)\eta^{ab}(e_a\Phi)(e_b\Phi) \qquad \text{(IV.9)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^5_b \wedge \omega^{ba}(e_5, e_a) &= -\frac{1}{4}\exp\{-2D\alpha\Phi\})\mathcal{F}_{bc}\mathcal{F}^{ba}e^c \wedge e^5(e_5, e_a) \\
&\quad +\beta(e_b\Phi)e^5 \wedge (\pi^*\hat{\omega}^{ba})(e_5, e_a) \\
&\quad +\alpha\beta\eta^{ad}(e_b\Phi)(e_d\Phi)e^5 \wedge e^b(e_5, e_a) \\
&\quad -\alpha\beta\eta^{bd}(e_b\Phi)(e_d\Phi)e^5 \wedge e^a(e_5, e_a) \\
&= \frac{1}{4}\exp\{-2D\alpha\Phi\})\mathcal{F}_{ab}\mathcal{F}^{ab} \qquad \text{(IV.10)} \\
&\quad +\beta(e_b\Phi)(\pi^*\hat{\omega}^{ba})(e_a) - (D-1)\alpha\beta\eta^{ab}(e_a\Phi)(e_b\Phi)
\end{aligned}
$$

Combining it altogether gives:

$$
\begin{aligned}
R &= \exp\{-2\alpha\Phi\}\pi^*\hat{R} \\
&\quad -[2(D-1)\alpha + 2\beta]\eta^{ab}(e_a e_b\Phi) - [2(D-1)\alpha + 2\beta](e_b\Phi)(\pi^*\hat{\omega}^{ab})(e_a) \\
&\quad -[(D-2)(D-1)\alpha^2 + 2\beta(\beta-\alpha) + 2(D-1)\alpha\beta]\eta^{ab}(e_a\Phi)(e_b\Phi) \\
&\quad -\frac{1}{4}\exp\{-2D\alpha\Phi\})\mathcal{F}_{ab}\mathcal{F}^{ab} \\
&= \exp\{-2\alpha\Phi\}\pi^*\hat{R} \\
&\quad -2\alpha\exp\{-2\alpha\Phi\}(\hat{\Box}\Phi) - \frac{1}{2}\eta^{ab}(e_a\Phi)(e_b\Phi) \\
&\quad -\frac{1}{4}\exp\{-2D\alpha\Phi\})\mathcal{F}_{ab}\mathcal{F}^{ab}
\end{aligned}
\qquad \text{(IV.11)}
$$

The terms in the second line give together a covariant d'Alembert operator (see exercise 15.2.9 in [3]). Since we want to take a look at the physics on

M we are going to express this in terms of the d'Alembert operator on M, hence the hat and extra $\exp\{-2\alpha\Phi\}$.

Appendix V

Calculation of Ricci tensor with scalar field for $U(1)$

For calculating the Ricci tensor we proceed as in appendix III.

We start by calculating the R_{ab} component

$$R_{ab} = \Omega_a^c(e_c, e_b) + \Omega_a^5(e_5, e_b) \quad \text{(V.1)}$$

$$= (d\omega_a^c + \omega_d^c \wedge \omega_a^d + \omega_5^c \wedge \omega_a^5 + d\omega_a^5 + \omega_d^5 \wedge \omega_a^d)(e_c, e_b) \quad \text{(V.2)}$$

$$
\begin{aligned}
d\omega_a^c(e_c,e_b) &= \pi^*(d\hat{\omega}_a^c)(e_c,e_b) \\
&\quad +\alpha(e_d e_a\Phi)e^d\wedge e^c(e_c,e_b) \\
&\quad -\alpha(e_a\Phi)(\hat{\omega}_d^c)\wedge e^d(e_c,e_b) \\
&\quad -\alpha\eta^{cd}\eta_{af}(e_e e_d\Phi)e^e\wedge e^f(e_c,e_b) \\
&\quad +\alpha\eta^{cd}\eta_{af}(e_d\Phi)(\hat{\omega}_g^f)\wedge e^g(e_c,e_b) \\
&\quad -\frac{1}{2}\exp\{(2\beta-2\alpha)\Phi\}\mathcal{F}_a^c\Omega^1(e_c,e_b) \\
&= \pi^*(d\hat{\omega}_a^c)(e_c,e_b) \\
&\quad -(D-1)\alpha(e_a e_b\Phi) \\
&\quad -\alpha(e_a\Phi)(\hat{\omega}_b^c)(e_c) \\
&\quad -\alpha\eta_{ab}\eta^{cd}(e_c e_d\Phi)+\alpha(e_a e_b\Phi) \\
&\quad +\alpha\eta^{cd}(e_d\Phi)(\hat{\omega}_{ab})(e_c)+\alpha(e_d\Phi)(\hat{\omega}_a^d)(e_b) \\
&\quad -\frac{1}{4}\exp\{(2\beta-2\alpha)\Phi\}\mathcal{F}_a^c\mathcal{F}_{de}e^d\wedge e^e(e_c,e_b) \\
&= \pi^*(d\hat{\omega}_a^c)(e_c,e_b)-\alpha(e_a\Phi)(\hat{\omega}_b^c)(e_c) \\
&\quad +\alpha\eta^{cd}(e_d\Phi)(\hat{\omega}_{ab})(e_c)+\alpha(e_d\Phi)(\hat{\omega}_a^d)(e_b) \\
&\quad -(D-2)\alpha(e_a e_b\Phi)-\alpha\eta_{ab}\eta^{cd}(e_c e_d\Phi) \\
&\quad +\frac{1}{2}\exp\{(2\beta-2\alpha)\Phi\}\mathcal{F}_a^c\mathcal{F}_{bc} \qquad \text{(V.3)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^c_d \wedge \omega^d_a(e_c, e_b) &= \pi^*\left(\hat{\omega}^c_d \wedge \hat{\omega}^{da}\right)(e_c, e_b) \\
&\quad +\alpha(e_a\Phi)(\pi^*\hat{\omega}^c_d) \wedge e^d(e_c, e_b) \\
&\quad -\alpha\eta^{de}\eta_{af}(e_e\Phi)(\pi^*\hat{\omega}^c_d) \wedge e^f(e_c, e_b) \\
&\quad +\alpha(e_d\Phi)e^c \wedge (\pi^*\hat{\omega}^d_a)(e_c, e_b) \\
&\quad +\alpha^2(e_a\Phi)(e_d\Phi)e^c \wedge e^d(e_c, e_b) \\
&\quad -\alpha^2\eta^{de}\eta_{af}(e_d\Phi)(e_e\Phi)e^c \wedge e^f(e_c, e_b) \\
&\quad -\alpha\eta^{ce}\eta_{df}(e_e\Phi)e^f \wedge (\pi^*\hat{\omega}^d_a)(e_c, e_b) \\
&\quad -\alpha^2\eta^{ce}\eta_{df}(e_e\Phi)(e_a\Phi)e^f \wedge e^d(e_c, e_b) \\
&\quad +\alpha^2\eta^{ce}\eta_{df}\eta^{dg}\eta_{ah}(e_e\Phi)(e_g\Phi)e^f \wedge e^h(e_c, e_b) \\
&= \pi^*\left(\hat{\omega}^c_d \wedge \hat{\omega}^{da}\right)(e_c, e_b) \\
&\quad +\alpha(e_a\Phi)(\pi^*\hat{\omega}^c_b)(e_c) \\
&\quad -\alpha\eta_{ab}(e_d\Phi)(\pi^*\hat{\omega}^{cd})(e_c) + \alpha(e_c\Phi)(\pi^*\hat{\omega}^c_a)(e_b) \\
&\quad +D\alpha(e_d\Phi)(\pi^*\hat{\omega}^d_a)(e_b) - \alpha(e_d\Phi)(\pi^*\hat{\omega}^d_a)(e_b) \\
&\quad +D\alpha^2(e_a\Phi)(e_b\Phi) - \alpha^2(e_a\Phi)(e_b\Phi) \\
&\quad -D\alpha^2\eta_{ab}\eta^{cd}(e_c\Phi)(e_d\Phi) + \alpha^2\eta_{ab}\eta^{cd}(e_c\Phi)(e_d\Phi) \\
&\quad -\alpha(e_d\Phi)(\pi^*\hat{\omega}^d_a)(e_b) + \alpha\eta^{cd}(e_d\Phi)(\pi^*\hat{\omega}_{ba})(e_c) \\
&\quad -\alpha^2(e_a\Phi)(e_b\Phi) + \alpha^2(e_a\Phi)(e_b\Phi) \\
&\quad +\alpha^2\eta_{ab}\eta^{cd}(e_c\Phi)(e_d\Phi) - \alpha^2(e_a\Phi)(e_b\Phi) \\
&= \pi^*\left(\hat{\omega}^c_d \wedge \hat{\omega}^{da}\right)(e_c, e_b) \\
&\quad +(D-2)\alpha^2(e_a\Phi)(e_b\Phi) \\
&\quad -(D-2)\alpha^2\eta_{ab}\eta^{cd}(e_c\Phi)(e_d\Phi) \\
&\quad +\alpha(e_a\Phi)(\pi^*\hat{\omega}^c_b)(e_c) \\
&\quad -\alpha\eta_{ab}(e_d\Phi)(\pi^*\hat{\omega}^{cd})(e_c) + \alpha\eta^{cd}(e_d\Phi)(\pi^*\hat{\omega}_{ba})(e_c) \\
&\quad +(D-3)\alpha(e_d\Phi)(\pi^*\hat{\omega}^d_a)(e_b) \qquad \text{(V.4)}
\end{aligned}
$$

$$\begin{aligned}
\omega^c_5 \wedge \omega^5_a(e_c, e_b) &= -\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^c_d\mathcal{F}_{ae}e^d \wedge e^e(e_c, e_b) \\
&= \frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^c_b\mathcal{F}_{ac}
\end{aligned} \tag{V.5}$$

$$\begin{aligned}
d\omega^5_a(e_5, e_b) &= \frac{1}{2}\exp\{-D\alpha\Phi\}(dF_{ac}) \wedge e^c(e_5, e_b) \\
&\quad +\beta(\beta-\alpha)(e_a\Phi)(e_c\Phi)e^c \wedge e^5(e_5, e_b) \\
&\quad +\beta(e_ae_c\Phi)e^c \wedge e^5(e_5, e_b) \\
&= \frac{1}{2}\exp\{-D\alpha\Phi\}(dF_{ab})(e_5) \\
&\quad -\beta(\beta-\alpha)(e_a\Phi)(e_b\Phi) - \beta(e_ae_b\Phi)
\end{aligned} \tag{V.6}$$

$$\begin{aligned}
\omega^5_c \wedge \omega^c_a(e_5, e_b) &= -\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}_{cd}\mathcal{F}^c_ae^d \wedge e^5(e_5, e_b) \\
&\quad +\beta(e_c\Phi)e^5 \wedge (\pi^*\hat{\omega}^c_a)(e_5, e_b) \\
&\quad +\alpha\beta(e_c\Phi)(e_a\Phi)e^5 \wedge e^c(e_5, e_b) \\
&\quad -\alpha\beta\eta^{ce}\eta_{af}(e_c\Phi)(e_e\Phi)e^5 \wedge e^f(e_5, e_b) \\
&= +\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}_{cb}\mathcal{F}^c_a \\
&\quad +\beta(e_c\Phi)(\pi^*\hat{\omega}^c_a)(e_b) \\
&\quad +\alpha\beta(e_a\Phi)(e_b\Phi) \\
&\quad -\alpha\beta\eta_{ab}\eta^{cd}(e_c\Phi)(e_d\Phi)
\end{aligned} \tag{V.7}$$

When we put everything together we get

$$\begin{aligned}
R_{ab} &= \exp\{-2\alpha\Phi\}\pi^*(\hat{R}) \\
&\quad +\frac{1}{2}\exp\{-2D\alpha\Phi\}\mathcal{F}_a^c\mathcal{F}_{bc} \\
&\quad -\frac{1}{2}(e_a\Phi)(e_b\Phi) \\
&\quad -\alpha\eta_{ab}\eta^{cd}(e_ce_d\Phi) - \alpha\eta_{ab}(e_d\Phi)(\pi^*\hat{\omega}^{cd})(e_c) \\
&= \boxed{\exp\{-2\alpha\Phi\}\left(\pi^*\hat{R}_{ab} - \alpha\eta_{ab}\hat{\Box}\Phi\right) - \frac{1}{2}(e_a\Phi)(e_b\Phi) + \frac{1}{2}\exp\{-2D\alpha\Phi\}\mathcal{F}_a^c\mathcal{F}_{bc}}
\end{aligned} \tag{V.8}$$

The terms in the fourth line give together a covariant d'Alembert operator (see exercise 15.2.9 in [3]). Since we want to take a look at the physics on M we are going to express this in terms of the d'Alembert operator on M, hence the hat and extra $\exp\{-2\alpha\Phi\}$.

We continue the calculation for R_{a5}:

$$R_{a5} = \Omega_a^b(e_b, e_5) \tag{V.9}$$

$$= (d\omega_a^b + \omega_d^b \wedge \omega_a^d + \omega_5^b \wedge \omega_a^5)(e_b, e_5) \tag{V.10}$$

$$\begin{aligned}
d\omega_a^b(e_b, e_5) &= -\frac{1}{2}2(\beta-\alpha)(e_c\Phi)\exp\{-D\alpha\Phi\}\mathcal{F}_a^b e^c \wedge e^5(e_b, e_5) \\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}(d\mathcal{F}_a^b) \wedge e^5(e_b, e_5) \\
&= -(\beta-\alpha)(e_b\Phi)\exp\{-D\alpha\Phi\}\mathcal{F}_a^b \\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}(d\mathcal{F}_a^b)(e_b)
\end{aligned} \tag{V.11}$$

$$
\begin{aligned}
\omega^b_d\wedge\omega^d_a(e_b,e_5) &= -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^d_a(\pi^*\hat\omega^b_d)\wedge e^5(e_b,e_5)\\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^d_a\alpha(e_d\Phi)e^b\wedge e^5(e_b,e_5)\\
&\quad +\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^d_a\alpha\eta^{bc}\eta_{de}(e_c\Phi)e^e\wedge e^5(e_b,e_5)\\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^b_d e^5\wedge(\pi^*\hat\omega^d_a)(e_b,e_5)\\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^b_d\alpha(e_a\Phi)e^5\wedge e^d(e_b,e_5)\\
&\quad +\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^b_d\alpha\eta^{dc}\eta_{ae}(e_c\Phi)e^5\wedge e^e(e_b,e_5)\\
&= -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^d_a(\pi^*\hat\omega^b_d)(e_b)\\
&\quad +\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^b_d(\pi^*\hat\omega^d_a)(e_b)\\
&\quad -\frac{1}{2}(D-2)\exp\{-D\alpha\Phi\}\mathcal{F}^b_a\alpha(e_b\Phi) \qquad \text{(V.12)}
\end{aligned}
$$

$$
\begin{aligned}
\omega^b_5\wedge\omega^5_a(e_b,e_5) &= -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^b_c\beta(e_a\Phi)e^c\wedge e^5(e_b,e_5)\\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}_{ad}\beta\eta^{bc}(e_c\Phi)e^5\wedge e^d(e_b,e_5)\\
&= -\frac{1}{2}\exp\{-D\alpha\Phi\}\beta\mathcal{F}^b_a(e_b\Phi) \qquad \text{(V.13)}
\end{aligned}
$$

$$
\begin{aligned}
R_{a5} &= -\frac{1}{2}\exp\{-D\alpha\Phi\}(d\mathcal{F}^b_a)(e_b)\\
&\quad -\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^d_a(\pi^*\hat\omega^b_d)(e_b)\\
&\quad +\frac{1}{2}\exp\{-D\alpha\Phi\}\mathcal{F}^b_d(\pi^*\hat\omega^d_a)(e_b)\\
&\quad +(D-1)\alpha\exp\{-D\alpha\Phi\}\mathcal{F}^b_a(e_b\Phi)\\
&= -\frac{1}{2}\exp\{-(D-1)\alpha\Phi\}(\hat\nabla_b\mathcal{F}^b_a)\\
&\quad +(D-1)\alpha\exp\{-D\alpha\Phi\}\mathcal{F}^b_a(e_b\Phi)\\
&= \boxed{-\frac{1}{2}\exp\{(D-3)\alpha\Phi\}\hat\nabla_b(\exp\{-2(D-1)\alpha\Phi\}\mathcal{F}^b_a)} \qquad \text{(V.14)}
\end{aligned}
$$

The terms in the first three lines give together the covariant derivative of $\mathcal{F}^b_d$ (this is almost the same as in (III.59)). Then we just add the remaining two terms together to obtain the result.

At last for the component R_{55}:

$$\begin{aligned} R_{55} &= \Omega^a_5(e_a, e_5) &\text{(V.15)}\\ &= (d\omega^a_5 + \omega^a_b \wedge \omega^b_5)(e_a, e_5) &\text{(V.16)}\end{aligned}$$

$$\begin{aligned} d\omega^a_5(e_a, e_5) &= -\frac{1}{2}\exp\{-D\alpha\Phi\}(d\mathcal{F}^a_b) \wedge e^b \\ &\quad -\beta\eta^{ab}(e_c e_b \Phi) e^c \wedge e^5 (e_a, e_5) \\ &\quad -\beta(\beta-\alpha)(e_b\Phi)(e_c\Phi) e^c \wedge e^5(e_a, e_5) \\ &= -\beta\eta^{ab}(e_a e_b\Phi) - \beta(\beta-\alpha)(e_a\Phi)(e_b\Phi) &\text{(V.17)}\end{aligned}$$

$$\begin{aligned} \omega^a_b \wedge \omega^b_5(e_a, e_5) &= -\beta\eta^{be}(e_e\Phi)(\pi^*\hat{\omega}^a_b) \wedge e^5(e_a, e_5) \\ &\quad -\alpha\beta\eta^{be}(e_b\Phi)(e_e\Phi) e^a \wedge e^5(e_a, e_5) \\ &\quad +\alpha\beta\eta^{ac}\eta_{bd}\eta^{be}(e_c\Phi)(e_e\Phi) e^d \wedge e^5(e_a, e_5) \\ &\quad +\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_e e^5 \wedge e^e(e_a, e_5) \\ &= -\beta\eta^{be}(e_e\Phi)(\pi^*\hat{\omega}^a_b)(e_a) \\ &\quad -(D-1)\alpha\beta\eta^{ab}(e_a\Phi)(e_b\Phi) \\ &\quad -\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_a &\text{(V.18)}\end{aligned}$$

$$\begin{aligned} R_{55} &= -\beta\eta^{ab}(e_a e_b\Phi) - ((D-1)\alpha\beta + \beta(\beta-\alpha))(e_a\Phi)(e_b\Phi) \\ &\quad -\beta\eta^{be}(e_e\Phi)(\pi^*\hat{\omega}^a_b)(e_a) - \frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_a \\ &= (D-2)\alpha\eta^{ab}(e_a e_b\Phi) + (D-2)\alpha(e_b\Phi)(\pi^*\hat{\omega}^{ab})(e_a) \\ &\quad -\frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_a \\ &= \boxed{(D-2)\alpha\exp\{-2\alpha\Phi\}\hat{\Box}\Phi - \frac{1}{4}\exp\{-2D\alpha\Phi\}\mathcal{F}^a_b\mathcal{F}^b_a} &\text{(V.19)}\end{aligned}$$

Here we get by the same manner as in (V.8) the covariant d'Alembert operator.

Bibliography

[1] M. Dubois-Violette and J. Madore. Conservation laws and integrability conditions for gravitational and yang-mills field equations. *Comm. Math. Phys.*, 108(2):213–223, 1987.

[2] M. Fecko. Kaluza-Klein theory, lecture notes. unpublished.

[3] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

[4] M. Fecko. Vertical degree of forms and some applications, 2018. http://davinci.fmph.uniba.sk/ fecko1/referaty/stara_lesna_2018.pdf.

[5] G. R. Jensen. Einstein metrics on principal fibre bundles. *J. Differential Geom.*, 8(4):599–614, 1973.

[6] T. Kaluza. Zum Unitätsproblem der Physik. (German) [on the unification problem in physics], 1921.

[7] O. Klein. The atomicity of electricity as a quantum theory law. *Nature*, 188:516–516, 1926.

[8] O. Klein. Quantentheorie und fünfdimensionale Relativitätstheorie. (German) [quantum theory and five-dimensional theory of relativity]. *Zeits. Phys.*, 37:895–906, 1926.

[9] G. Nordström. Über die möglichkeit, das elektromagnetische feld und das gravitationsfeld zu vereinigen. *Physikalische Zeitschrift*, 15:504 – 506, 1914.

[10] B. O'Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13(4):459–469, 12 1966.

[11] J. Overduin and P. Wesson. Kaluza-klein gravity. *Physics Reports*, 283(5-6):303–378, Apr 1997.

[12] C. Pope. Kaluza-Klein theory, lecture notes. http://people.physics.tamu.edu/pope/ihplec.pdf.

[13] A. Trautman. Fibre bundles associated with space-time. *Reports on Mathematical Physics*, 1(1):29 – 62, 1970.

[14] A. Trautman. Einstein - cartan theory. *Encyclopedia of Mathematical Physics*, pages 189–195, 2006.

[15] O. Veblen and B. Hoffmann. Projective relativity. *Phys. Rev.*, 36:810–822, Sep 1930.