COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Lie derivative of linear connection and its applications

Diploma thesis

2023 Frederik Ďalak

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Superisor: doc. RNDr. Marián Fecko, PhD.

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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. Frederik Ďalak
Študijný program:	teoretická fyzika (Jednoodborové štúdium, magisterský II. st., denná forma)
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Názov: Lie derivative of linear connection and its applications
Lieova derivácia lineárnej konexie a jej aplikácie

Anotácia: Pojem Lieovej derivácie je v diferenciálnej geometrii a jej aplikáciách veľmi dôležitý a frekventovaný. Preto sa mu venuje každá dobrá učebnica. Okrem iného dáva návod, ako sa počíta pre tenzorové pole ľubovoľného typu. Myšlienka spočíva v (lieovskom) prenose takého poľa proti smeru toku vektorového poľa (t.j. v pull-backu voči toku toho poľa) a porovnaní s pôvodným poľom.

Ukazuje sa však, že pull-back sa dá robiť aj z lineárnej konexie (čo nie je tenzorové pole). To znamená, že sa dá počítať aj Lieova derivácia lineárnej konexie. A na jej základe hovoriť o konexii invariantnej voči toku vektorového poľa.

Cieľ: Cieľom diplomovej práce bude pozrieť sa na túto tematiku podrobnejšie. Odvodiť si explicitné vzorce na výpočet Lieovej derivácie lineárnej konexie a porozmýšľať o nejakých aplikáciách, napríklad vyšetriť pomocou nich Lieovu algebru symetrie niektorých bežných lineárnych konexií.

Literatúra: Leo C. Stein: Notes on the pullback connection, https://duetosymmetry.com/notes/notes-on-the-pullback-connection/

Vedúci:	doc. RNDr. Marián Fecko, PhD.
Katedra:	FMFI.KTF - Katedra teoretickej fyziky
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THESIS ASSIGNMENT

Name and Surname:	Bc. Frederik Ďalak
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Title: Lie derivative of linear connection and its applications

Annotation: In differential geometry and its applications, the concept of Lie derivative is very important and frequent. Therefore, it is discussed in each good textbook of the subject. Among others, recipes of its computing for tensor field of any type is given. The idea consists in (Lie) transport of the field against the direction of the flow of the vector field(i.e. in pull-back with respect to the flow) and comparison with the field itself.
It turns out, however, that one can perform pull-back of linear connection as well (the latter is not a tensor field). This means that it is possible to to introduce (and compute) Lie derivative of linear connection, too. And one can speak of a connestion invariant with respect to the flow of a vector field.

Aim: The aim of the theses is to investigate the matter more closely. Namely derive explicit formulas for computation of the Lie derivative of linear connection and think of some applications. For example determine Lie algebra of symmetry of some common linear connections.

Literature: Leo C. Stein: Notes on the pullback connection, https://duetosymmetry.com/notes/notes-on-the-pullback-connection/

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Dedication

I would like to dedicate this work to my parents for their endless patience and support.

Acknowledgment

I would like to express my deepest gratitude and appreciation to my supervisor doc. RNDr. Marián Fecko, PhD. whom I had the pleasure of working with on the thesis for his invaluable advice and feedback.

Declaration

I declare that I am the sole author of this thesis. To the best of my knowledge, the thesis does not contain material previously published by any other person except where acknowledgment has been made.

Date	Frederik Ďalak

Abstrakt

Autor: Frederik Ďalak

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Je dobre známe, že Lieova derivácia je jedným zo základných prostriedkov používaných v matematickej fyzike. Možno už menej známym faktom je, že okrem tenzorových polí sa dá spočítať aj Lieova derivácia lineárnej konexie ∇ na variete (M, ∇). V práci najskôr predstavíme tento koncept využitím difeomorfizmu f medzi dvomi varietami na zavedenie pullbacku lineárnej konexie, na základe čoho následne definujeme samotnú Lievou deriváciu lineárnej konexie. Sformulujeme podmienky platné pre symetrie lineárnych konexií v rôznych prípadoch a študujeme vlastnosť týchto symetrií zachovávať geodetické krivky. Na záver skúmame aplikácie týchto myšlienok v Newtonovej-Cartanovej teórii gravitácie.

Kľúčové slová: Lieova derivácia, kovariantná derivácia, lineárna konexia

Abstract

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It is well known that one of the fundamental tools used in mathematical physics is the Lie derivative. Perhaps a less known fact is that, apart from tensor fields, one can also compute the Lie derivative of a linear connection ∇ on a manifold (M, ∇). Here we first introduce the concept by showing how to use a diffeomorphism f between two manifolds to compute a pullback of linear connection, based on which the Lie derivative of linear connection itself is defined. We formulate conditions valid for linear connection symmetries in different cases and study the geodesic preserving properties of these symmetries. Finally, we examine the applications of these ideas in Newton-Cartan theory of gravity.

Key words: Lie derivative, covariant derivative, linear connection

Contents

Introduction

Let us begin with a brief history lesson on the subject of the Lie derivative. The notion of Lie differentiation, nowadays known to everyone lucky enough to have at least a basic background in mathematical physics, has its roots in the works of David Hilbert and Élie Cartan [Hilbert, 1915] [Cartan, 1922]. In these early papers, the foundations for the Lie derivative of the metric tensor and differential forms were laid. Later on, in 1931, Wladyslaw Ślebodziński introduced an operator acting on *any geometrical object*, based on the idea of expressing it in infinitesimally shifted coordinates, in a way he described as 'dragging it' along a vector field [Ślebodziński, 1931]. The operator was later called the Lie derivative by David van Dantzig. The general formulation, presented by Ślebodziński, allows one to perform the Lie differentiation, in addition to the largely familiar case of tensor fields, also on any other object which can be expressed in different sets of coordinates [Trautman, 2008].

The linear connection is a structure on a manifold, fully determined by the corresponding coefficients of connection Γ^i_{jk} (or equally well by the forms of connection ω^i_j). Since these are objects with well-defined transformation laws under a change of coordinates, naturally also the *Lie derivative of linear connection* was introduced. It was later studied by Kentaro Yano, Shoshichi Kobayashi, and others [Yano, 1967] [Kobayashi, 1995]. Owing to these works, we know today for instance that the symmetries of linear connection, similarly to those of the metric, constitute a Lie algebra. It was also shown that the Lie derivative of linear connection is represented by a tensor field of type $\binom{1}{2}$. Some of the more recent papers concerned with this topic were published by Tsamparlis, Paliathanasis, Hauer, Jüttler, and other authors [Tsamparlis and Paliathanasis, 2009] [Hauer and Jüttler, 2018].

In the present work, we reformulate the concept of the Lie derivative of linear connection using non-component language. In the first chapter, we familiarize the reader with the pullback connection, deriving its definition and exploring its basic properties. These ideas are then used in the second chapter to derive the definition of the Lie derivative of linear connection. It is first defined as an operator acting on the tensor algebra. Afterwards, the possibility of establishing the corresponding type $\binom{1}{2}$ tensor L_U^{∇} is introduced. We also present a useful formula expressing the L_U^{∇} tensor in terms of the curvature tensor R and the torsion tensor T assigned to the given linear connection ∇. In chapters three and four, the symmetries of linear connection are discussed. We formulate conditions valid for the symmetries of linear connection as demand for vanishing of the L_U^{∇} tensor and also in terms of forms of connection. A useful reformulation of the condition for symmetries, valid specifically in the case of the Levi-Civita connection, is then derived. The fifth chapter is dedicated to the geometrical interpretation of the Lie derivative of linear connection, studying the geodesic preserving properties of linear connection symmetries. Finally, in the sixth chapter, applications of these ideas in Newton-Cartan theory of gravity are examined.

Chapter 1

The pullback connection

The idea of the Lie derivative of linear connection is based on the possibility of introducing a pullback of a linear connection in a similar way as in the case of the pullback of tensor fields [Stein, 2017]. This way one is able to define a linear connection called *the pullback connection* on a manifold with no linear connection previously established if there is another manifold with a linear connection and a diffeomorphism between the two manifolds. In the first chapter, we explain how the pullback connection is introduced and we look at some of its properties.

1.1 Definition of the pullback connection

To introduce the concept of the pullback connection let us consider two manifolds M and N. We will assume that on the manifold N there is a structure of linear connection ∇, whereas on the manifold M there is a priori no linear connection present. Moreover, we consider a diffeomorphic map from M to N $f : M \to N$. As we know, the diffeomorphism f also induces the existence of bijective maps between arbitrary types of tensor fields on both manifolds: pullback f^* and push-forward f_*:

$$f^* : \mathcal{T}^p_q(N) \to \mathcal{T}^p_q(M) \tag{1.1}$$

$$f_* : \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(N) \tag{1.2}$$

The linear connection on (N, ∇) represents a possibility to compute a covariant derivative on the manifold N. This practically means that to any vector field $V \in \mathfrak{X}(N)$ and arbitrary tensor $\tilde{V} \in \mathcal{T}^p_q(N)$ we are able to assign a tensor field $\nabla_V \tilde{V} \in \mathcal{T}^p_q(N)$ called the covariant derivative of $\tilde{V}$ in the direction of V. The question which we would like to answer is: can we come up with a way to establish a linear connection on the manifold M, based on the existence of the linear connection ∇ on N and the diffeomorphism f? In other words, is there a way to introduce a covariant derivative on M ? Fortunately, the answer to these questions is positive (otherwise there would not be much to discuss in the rest of the thesis). The way to establish a covariant derivative on M is actually fairly natural and straightforward. We would like to be able to compute a covariant derivative of a tensor $\tilde{U} \in \mathcal{T}^p_q(M)$ in the direction of a vector $U \in \mathfrak{X}(M)$ resulting in a tensor of the same type $\binom{p}{q}$ on M. Since there is no linear connection present on M, the first step we should take is to compute the pushforward of both U and $\tilde{U}$ resulting in a vector f_*U and a tensor $f_*\tilde{U}$ on the manifold N. Since the resulting objects reside on (N, ∇), it allows us to compute the covariant derivative $\nabla_{f_*U} f_*\tilde{U} \in \mathcal{T}^p_q(M)$. That brings us one step closer to our goal of having a covariant derivative of $\tilde{U}$ in the direction of U. The resulting tensor $\nabla_{f_*U} f_*\tilde{U}$ is a tensor field on the manifold N though. Therefore, we need to use the diffeomorphism f once more to compute the pullback of the created tensor, thus getting our result back to the original manifold M (see Figure 1). Hence, we obtain the tensor field $f^*[\nabla_{f_*V} f_*\tilde{V}]$.

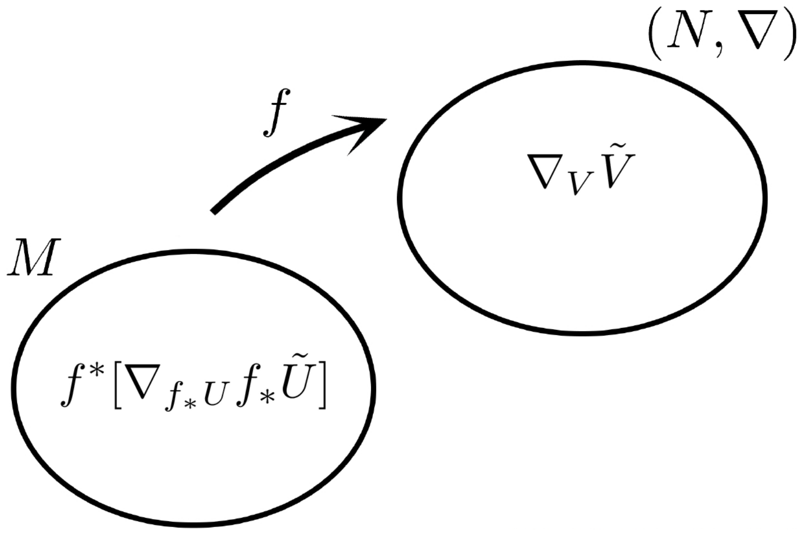


Figure 1: Definition of the pullback connection

It is quite simple to make sure that the term $f^*[\nabla_{f_*U} f_*\tilde{U}]$ actually satisfies all the properties required for it to be a covariant derivative on the manifold M (see Appendix A). This leads us to define the pullback connection $f^*\nabla$ on the manifold M by introducing the covariant derivative $(f^*\nabla)_U$

corresponding to this connection based on the formula we derived above. The covariant derivative represents an operator assigned to every vector field $V \in \mathfrak{X}(M)$ which acts on a tensor $T \in \mathcal{T}^p_q(M)$ in the following way:

$$(f^*\nabla)_V T = f^*[\nabla_{f_* V} f_* T] \tag{1.3}$$

1.2 Coefficients of the pullback connection

To help create a more clear image of what kind of object the pullback connection represents, let us work out the formula describing the coefficients of the pullback connection $f^*\nabla$ based on the known coefficients of the original linear connection ∇. Here we continue working with the geometrical objects set up in the previous section: we consider manifolds M and (N, ∇) while considering a diffeomorphism $f : M \to (N, \nabla)$.

Let y^α be a set of coordinates on the manifold (N, ∇). Then the linear connection ∇ is fully determined by the coefficients $\Gamma^\alpha_{\beta\gamma}$ defined in the standard way:

$$\nabla_{\partial_\gamma} \partial_\beta := \Gamma^\alpha_{\beta\gamma} \partial_\alpha \tag{1.4}$$

Furthermore, let x^i be a set of coordinates on the manifold M. (We may notice that there are as many α indices as there are i ones since both manifolds need to be of the same dimension in order for f to be a *diffeomorphism*.) The diffeomorphism f then provides a set of functions $y^\alpha(x)$ mapping the points of M to the ones in N. We denote the corresponding Jacobi matrices as follows:

$$J^\alpha_i := \frac{\partial y^\alpha}{\partial x^i} \tag{1.5}$$

$$(J^{-1})^i_\alpha := \frac{\partial x^i}{\partial y^\alpha} \tag{1.6}$$

The coefficients of the pullback connection $f^*\nabla$ are then given in analogy with the definition (1.4). We denote these Π^i_{jk}:

$$(f^*\nabla)_{\partial_k} \partial_j := \Pi^i_{jk} \partial_i \tag{1.7}$$

Using the defining formula (1.3) of the covariant derivative $(f^*\nabla)_V$ yields:

$$(f^*\nabla)_{\partial_k} \partial_j := f^*[\nabla_{f_*\partial_k} f_* \partial_j] \tag{1.8}$$

From the obtained expression one may derive the formula for the coefficients Π^i_{jk} in terms of the original $\Gamma^\alpha_{\beta\gamma}$ coefficients by using the Jacobi matrices (1.5) and (1.6) to express the pullback and push-forward in the corresponding terms:

$$\begin{aligned}
(f^*\nabla)_{\partial_k}\partial_j &= f^*\left[\nabla_{J^\gamma_k\partial_\gamma} J^\beta_j \partial_\beta\right] \\
&= f^*\left[J^\gamma_k J^\beta_j{}_{,\gamma}\,\partial_\beta + J^\gamma_k J^\beta_j \Gamma^\alpha_{\beta\gamma}\partial_\alpha\right] \\
&= \left[J^\gamma_k J^\beta_j{}_{,\gamma}\,(J^{-1})^i_\beta + J^\gamma_k J^\beta_j (J^{-1})^i_\alpha \Gamma^\alpha_{\beta\gamma}\right]\partial_i \\
&= \left[\frac{\partial y^\gamma}{\partial x^k}\left(\frac{\partial}{\partial y^\gamma}\frac{\partial y^\beta}{\partial x^j}\right)\frac{\partial x^i}{\partial y^\beta} + \frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\beta}{\partial x^j}\frac{\partial x^i}{\partial y^\alpha}\Gamma^\alpha_{\beta\gamma}\right]\partial_i \\
&= \left[\frac{\partial^2 y^\beta}{\partial x^k \partial x^j}\frac{\partial x^i}{\partial y^\beta} + \frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\beta}{\partial x^j}\frac{\partial x^i}{\partial y^\alpha}\Gamma^\alpha_{\beta\gamma}\right]\partial_i
\end{aligned} \tag{1.9}$$

The obtained result might now be compared to the defining formula (1.7). Doing so reveals that the coefficients of the pullback connection Π^i_{jk} are given by the formula:

$$\Pi^i_{jk} = \left[\frac{\partial^2 y^\alpha}{\partial x^k \partial x^j}\frac{\partial x^i}{\partial y^\alpha} + \frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\beta}{\partial x^j}\frac{\partial x^i}{\partial y^\alpha}\Gamma^\alpha_{\beta\gamma}\right] \tag{1.10}$$

We may notice that the resulting expression looks quite familiar. It is the same formula one would use when converting the coefficients of the original connection ∇ from the y^α coordinates into the x^i coordinates, using the diffeomorphism f which would be the corresponding change of coordinates $f : M[x^i] \to M[y^\alpha]$, mapping M to itself. As we see, computing the coefficients of the connection ∇ in the transformed coordinates, in this case, is the same as using the diffeomorphism f to compute the coefficients of the pullback connection.

1.3 Pullback connection for a vector field flow

For the sake of future reference, let us try to use the formula (1.10) for the case of a specific automorphism of a manifold (M, ∇) given by a flow $\Phi_t : M \to M$ of a vector field U. The relationship (1.10) for the coefficients of the pullback connection is identical to the one corresponding to a change of coordinates. Based on that one may realize that considering a change of coordinates, for which the new coordinates are introduced as the original ones shifted by a flow of a vector field, is exactly the idea which led Ślebodziński to introduce the operator, nowadays known as the Lie derivative (see the Introduction). Only formulated in the language of a vector field flow. Here we attempt to recreate the procedure presented by Ślebodziński, expressing the coefficients of connection in coordinates shifted by an infinitesimal flow Φ_ϵ of the vector field U and computing, to the first order in ϵ, the relationship between the two coordinate expressions. The vector U is given using its components $U^i(x)$ with respect to x^i coordinates:

$$U = U^i(x)\frac{\partial}{\partial x^i} \tag{1.11}$$

Suppose there are coordinates y^i defined as the x^i ones shifted by Φ_ϵ:

$$y^i = x^i + \epsilon\, U^i(x) \tag{1.12}$$

Then to the first order in ϵ the x^i coordinates are expressed using y^i as:

$$x^i = y^i - \epsilon\, U^i(y) \tag{1.13}$$

The corresponding Jacobi matrices of this coordinate change are:

$$\frac{\partial y^a}{\partial x^i} = \delta^a_i - \epsilon U^a{}_{,i}(x) \tag{1.14}$$

$$\frac{\partial x^i}{\partial y^a} = \delta^i_a - \epsilon U^i{}_{,a}(y) = \delta^i_a - \epsilon U^i{}_{,a}(x) + \mathcal{O}(\epsilon^2) \tag{1.15}$$

Once we name the functions expressing the coefficients of connection in the x^i coordinates $\Pi^i_{jk}(x)$ and the ones for the coefficients in the y^a coordinates $\Gamma^a_{bc}(y)$, we may use the formula (1.10) to obtain the following relationship:

$$\Pi^i_{jk}(x) = \left[\frac{\partial^2 y^a}{\partial x^k \partial x^j}\frac{\partial x^i}{\partial y^a} + \frac{\partial y^c}{\partial x^k}\frac{\partial y^b}{\partial x^j}\frac{\partial x^i}{\partial y^a}\Gamma^a_{bc}(y(x))\right] \tag{1.16}$$

To express the difference to the first order in ϵ we need one more identity for the $\Gamma^i_{jk}(y(x))$ term:

$$\Gamma^i_{jk}(y(x)) = \Gamma^i_{jk}(x + \epsilon\, U(x)\,) = \Gamma^i_{jk}(x) + \epsilon\, U^m(x)\Gamma^i_{jk,m}(x) \tag{1.17}$$

We are finally ready to use the obtained relationships (1.12), (1.13), (1.14), (1.15) and (1.17) in the formula (1.16) to get:

$$\begin{aligned}\Pi^i_{jk}(x) =& \Gamma^i_{jk}(x) + \epsilon\left[U^i{}_{,jk} + U^m\Gamma^i_{jk,m} - U^i{}_{,m}\Gamma^m_{jk} + \right.\\ &\left. + U^m{}_{,j}\Gamma^i_{mk} + U^m{}_{,k}\Gamma^i_{jm}\right](x)\end{aligned} \tag{1.18}$$

Therefore the coefficients $\Pi^i_{jk}(x)$ in the coordinates x^i may be expressed by 'plugging' the x^i coordinates into the functions Γ^i_{jk} in place of the y^i coordinates and then adding the infinitesimal term in the brackets. This reduces just to replacing the coordinates in the Γ^i_{jk} functions in the special case when the term in the brackets vanishes:

$$U^i{}_{,jk} + U^m \Gamma^i_{jk},_m - U^i{}_{,m} \Gamma^m_{jk} + U^m{}_{,j} \Gamma^i_{mk} + U^m{}_{,k} \Gamma^i_{jm} = 0 \tag{1.19}$$

As we shall see in Chapter 2, this term is actually nothing else but the expression for the Lie derivative of the connection ∇ in the direction of the vector field U. Further examination reveals that the term in the equation (1.19) represents components of a type $\binom{1}{2}$ tensor (see chapter 2). This also leads to the familiar fact that a difference of two terms Π^i_{jk} and Γ^i_{jk} in the identity (1.18) corresponding to two different sets of linear connection coefficients is given by a tensor field (even though the coefficients themselves do not transform as components of a tensor field).

Chapter 2

Lie derivative of linear connection

Based on the case of tensor fields, we know that if there is a possibility to compute a pullback of a geometric object residing on a manifold M, it also allows us to define the Lie derivative of the object. The concept of the Lie derivative stems from considering a pullback f^* for a specific case when the diffeomorphic map is a flow Φ_t of a vector field U on the manifold M. This way the pullback creates another object on the same manifold M. The next step is to consider an infinitesimal transformation of the given object by computing a pullback Φ_ϵ^* for a small parameter ϵ. The Lie derivative is then defined as the difference between the original object and its pullback to the first order in ϵ.

In this chapter, we apply the recipe for the Lie derivative described above to the case of linear connection. We draw inspiration from the familiar case of tensor fields to define the Lie derivative of linear connection first as an operator acting on the tensor algebra, and then we show that all the information it carries is included in a tensor field of type $\binom{1}{2}$.

2.1 Lie derivative of linear connection as an operator on the tensor algebra $\mathcal{T}(M)$

As we mentioned, to compute the Lie derivative of an object the first ingredient we need is a vector field $U \in \mathfrak{X}(M)$ and the flow $\Phi_t : M \to M$ corresponding to it:

$$\Phi_t \leftrightarrow U \tag{2.1}$$

Now let us briefly remind ourselves of the way these objects are used to define the Lie derivative in the case of tensor fields. For a tensor field $T \in \mathcal{T}^p_q(M)$ we define the Lie derivative of T in the direction of the vector $U \in \mathfrak{X}(M)$ in the following way:

$$\mathcal{L}_U T := \lim_{\epsilon \to 0} \left[\frac{\Phi^*_\epsilon T - T}{\epsilon} \right] \tag{2.2}$$

Here the flow Φ_t is a diffeomorphism satisfying:

$$(\Phi_t)^{-1} = \Phi_{-t} \tag{2.3}$$

Therefore, if there is a linear connection ∇ on M one may also use the flow Φ_t to compute the pullback connection $\Phi^*_\epsilon \nabla$. Consequently, after we look at the defining formula (2.2), there is a fairly clear way to come up with the definition of the Lie derivative of linear connection as well: one could simply replace the letter T denoting the tensor field with the symbol ∇ denoting the linear connection. This motivates the following defining formula for the Lie derivative of linear connection:

$$\mathcal{L}_U \nabla := \lim_{\epsilon \to 0} \left[\frac{\Phi^*_\epsilon \nabla - \nabla}{\epsilon} \right] \tag{2.4}$$

An equivalent expression for $\mathcal{L}_U \nabla$ is given by the following identity valid to the first order in ϵ:

$$\epsilon \mathcal{L}_U \nabla := \Phi^*_\epsilon \nabla - \nabla \tag{2.5}$$

We see that the Lie derivative of the linear connection ∇ in the direction of the vector U is defined as the difference between two linear connections on the manifold M to the first order in ϵ: one of them being the original ∇ connection and the other one is the $\Phi^*_\epsilon \nabla$ connection which is the pullback of ∇ in the direction of U. Based on the comparison with tensor fields this matches our idea of what the Lie derivative of linear connection should represent. The equations (2.4) and (2.5) however are so far just abstract formulas, defining a new geometrical object $\mathcal{L}_U \nabla$ as a difference between two other structures on a manifold. However, there is not an obvious way to directly compute the difference between two structures of linear connection on a manifold.

Usually, when we deal with a linear connection, we commonly describe it in terms of the covariant derivative. So to get a more specific idea of what the newly defined Lie derivative of the connection $\mathcal{L}_U \nabla$ represents we need to express the linear connections ∇ and $\Phi^*_\epsilon \nabla$ using their respective covariant derivatives. To each vector $V \in \mathfrak{X}(M)$ we can assign the operator ∇_V expressing the covariant derivative for the ∇ connection and the operator

$(\Phi_\epsilon^*\nabla)_V$ of the covariant derivative for the pullback connection $\Phi_\epsilon^*\nabla$ which is defined by the formula (1.3). Since both the covariant derivatives reside on the same manifold, and act on the tensor algebra $\mathcal{T}(M)$, we can see that also the Lie derivative of linear connection $\mathcal{L}_U\nabla$ can be expressed as an *operator of the Lie derivative of linear connection* $(\mathcal{L}_U\nabla)_V$, defined by reformulating the equation (2.5) in terms of the covariant derivatives:

$$\epsilon(\mathcal{L}_U\nabla)_V := (\Phi_\epsilon^*\nabla)_V - \nabla_V \tag{2.6}$$

Therefore, to any vector $V \in \mathfrak{X}(M)$ we can assign the $(\mathcal{L}_U\nabla)_V$ operator which acts on a tensor $T \in \mathcal{T}_q^p(M)$ as the difference of the two covariant derivatives to the first order in ϵ:

$$(\mathcal{L}_U\nabla)_V T := \lim_{\epsilon\to 0}\left[\frac{(\Phi_\epsilon^*\nabla)_V T - \nabla_V T}{\epsilon}\right] \tag{2.7}$$

The next step is to find a formula for the operator $(\mathcal{L}_U\nabla)_V$. In other words, to determine how it acts on the tensor algebra $\mathcal{T}(M)$ and what specific operations should one perform on the tensor T to obtain the result $(\mathcal{L}_U\nabla)_V T$. To do this we first express the covariant derivative $(\Phi_\epsilon^*\nabla)_V$ in (2.6) making use of the definition (1.3):

$$\epsilon(\mathcal{L}_U\nabla)_V := \Phi_\epsilon^*\left[\nabla_{\Phi_{\epsilon *}V}\Phi_{\epsilon *}\right] - \nabla_V \tag{2.8}$$

Here we recall a formula expressing the pullback Φ_t^* with respect to a flow Φ_t of a vector field U as the exponent of the Lie derivative:

$$\Phi_t^* = e^{t\mathcal{L}_U} \tag{2.9}$$

Consequently, our result can be obtained by expressing the defining formula (2.7) using the identity (2.9) as:

$$\begin{aligned}
(\mathcal{L}_U\nabla)_V T &:= \lim_{\epsilon\to 0}\left[\frac{(\Phi_\epsilon^*[\nabla_{\Phi_{\epsilon *}V}\Phi_{\epsilon *}T] - \nabla_V T}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{e^{\epsilon\mathcal{L}_U}[\nabla_{e^{-\epsilon\mathcal{L}_U}V}e^{-\epsilon\mathcal{L}_U}T] - \nabla_V T}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{\epsilon[\mathcal{L}_U\nabla_V T - \nabla_V\mathcal{L}_U T - \nabla_{\mathcal{L}_U V}T] + \mathcal{O}(\epsilon^2)}{\epsilon}\right] \\
&= [\mathcal{L}_U\nabla_V - \nabla_V\mathcal{L}_U - \nabla_{\mathcal{L}_U V}]T
\end{aligned} \tag{2.10}$$

Hence, the formula for the operator $(\mathcal{L}_U\nabla)_V$ acting on the tensor algebra $\mathcal{T}(M)$ is:

$$(\mathcal{L}_U\nabla)_V = \mathcal{L}_U\nabla_V - \nabla_V\mathcal{L}_U - \nabla_{\mathcal{L}_U V} =: [\mathcal{L}_U, \nabla_V] - \nabla_{\mathcal{L}_U V} \tag{2.11}$$

2.2 Lie derivative of linear connection as a type $\binom{1}{2}$ tensor field

In this section, we take a closer look at the properties of the $(\mathcal{L}_U\nabla)_V$ operator. We show that a tensor field of type $\binom{1}{2}$ carrying all the information about the Lie derivative of linear connection can be defined and we derive a specific formula for the tensor.

From the fact that the operator $(\mathcal{L}_U\nabla)_V$ is defined by the formula (2.6) as a linear combination of two covariant derivatives, one can extract some of its basic properties. Let us consider a function $f \in \mathcal{F}(M)$, vectors $V, W \in \mathfrak{X}(M)$ and any contraction of tensors C. The $(\mathcal{L}_U\nabla)_V$ operator satisfies the following:

1. it is a derivation of tensor algebra $\mathcal{T}(M)$ preserving the degree of tensors:

$$(\mathcal{L}_U\nabla)_V : \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M) \tag{2.12}$$

2. it commutes with contractions (since both the covariant derivatives have this property):

$$(\mathcal{L}_U\nabla)_V \circ C = C \circ (\mathcal{L}_U\nabla)_V \tag{2.13}$$

3. it vanishes on functions:

$$\epsilon(\mathcal{L}_U\nabla)_V f := (\Phi^*_\epsilon\nabla)_V f - \nabla_V f = Vf - Vf = 0 \tag{2.14}$$

4. it is $\mathcal{F}(M)$-linear with respect to the vector field V:

$$(\mathcal{L}_U\nabla)_{V+fW} = (\mathcal{L}_U\nabla)_V + f(\mathcal{L}_U\nabla)_W \tag{2.15}$$

As we can see, the operator $(\mathcal{L}_U\nabla)_V$ inherited most of its properties from the covariant derivatives in its defining formula. However, there is a difference in the way it acts on functions described by the property 3. which means that the operator $(\mathcal{L}_U\nabla)_V$ itself is not a covariant derivative. A derivation of tensor algebra $\mathcal{T}(M)$ which preserves the tensor degree, commutes with contractions, and vanishes on functions, is fully described by a tensor field of type $\binom{1}{1}$ [Fecko, 2006]. Therefore, based on the properties 1., 2. and 3., we see that *for any* $V \in \mathfrak{X}(M)$, the action of the operator $(\mathcal{L}_U\nabla)_V$ is determined by a type $\binom{1}{1}$ tensor field. Moreover, property 4. tells us that the tensor field of type $\binom{1}{1}$ is assigned to the vector field V $\mathcal{F}(M)$-linearly. In conclusion, this means that it is possible to define a tensor field of type $\binom{1}{2}$ which fully describes the Lie derivative of linear connection $\mathcal{L}_U\nabla$. Here the added vector field slot accounts for the $\mathcal{F}(M)$-linearity in V.

2.2.1 Definition of the L_U^∇ tensor

Let us consider a Lie derivative $\mathcal{L}_U\nabla$ of a linear connection ∇ on (M,∇). The derived properties of the corresponding operator $(\mathcal{L}_U\nabla)_V$ motivate the following definition of a tensor field $L_U^\nabla \in \mathcal{T}_2^1(M)$. For vector fields $X, Y \in \mathfrak{X}(M)$ and a covector $\alpha \in \mathcal{T}_1^0(M)$:

$$L_U^\nabla(Y, X; \alpha) := \langle \alpha\, ,\, (\mathcal{L}_U\nabla)_X Y\rangle \tag{2.16}$$

To show that the tensor L_U^∇ fully determines how the operator $(\mathcal{L}_U\nabla)_X$ acts on $\mathcal{T}(M)$ let us consider a vector basis $e_a \in \mathfrak{X}(M)$ and the dual covector basis $e^a \in \mathcal{T}_1^0(M)$. Based on the properties of the $(\mathcal{L}_U\nabla)_X$ operator the result for $(\mathcal{L}_U\nabla)_X Y$ is given as:

$$(\mathcal{L}_U\nabla)_X Y = X^a Y^b\, (\mathcal{L}_U\nabla)_{e_a} e_b = X^a Y^b\, (L_U^\nabla)^c_{ba} e_c \tag{2.17}$$

Action of $(\mathcal{L}_U\nabla)_X$ on a covector α is then determined by the fact that $(\mathcal{L}_U\nabla)_X$ commutes with contractions and vanishes on functions:

$$\begin{aligned}(\mathcal{L}_U\nabla)_X\, \langle \alpha\, ,\, Y\rangle &\overset{1.}{=} 0\\ &\overset{2.}{=} \langle (\mathcal{L}_U\nabla)_X \alpha\, ,\, Y\rangle + \langle \alpha\, ,\, (\mathcal{L}_U\nabla)_X Y\rangle\end{aligned} \tag{2.18}$$

Which leads to:

$$\begin{aligned}\langle (\mathcal{L}_U\nabla)_X \alpha\, ,\, Y\rangle &= -\,\langle \alpha\, ,\, (\mathcal{L}_U\nabla)_X Y\rangle\\ &= -\alpha_a X^b Y^c\, (L_U^\nabla)^a_{cb}\end{aligned} \tag{2.19}$$

Based on that we may conclude that the tensor L_U^∇ also describes the action of $(\mathcal{L}_U\nabla)_X$ on covectors:

$$(\mathcal{L}_U\nabla)_X\alpha = -\alpha_a X^b (L_U^\nabla)^a_{cb} e^c \tag{2.20}$$

Combined with the fact that $(\mathcal{L}_U\nabla)_X$ is a derivation of the tensor algebra $\mathcal{T}(M)$ which acts according to the Leibniz rule on the tensor product, the formulas (2.17) and (2.20) allow us to compute the action of the $(\mathcal{L}_U\nabla)_X$ operator on an arbitrary tensor field in $\mathcal{T}(M)$ if we have the formula for the tensor L_U^∇. Based on the definition (2.16) it is not difficult to obtain the component expression for L_U^∇. If Γ^i_{jk} are the coefficients of ∇ one gets:

$$(L_U^\nabla)^i_{jk} = U^i{}_{,jk} + U^m\Gamma^i_{jk}{}_{,m} - U^i{}_{,m}\,\Gamma^m_{jk} + U^m{}_{,j}\,\Gamma^i_{mk} + U^m{}_{,k}\,\Gamma^i_{jm} \tag{2.21}$$

We may recognize this is exactly the result (1.19) obtained in Chapter 1. Looking at the result (2.21) it is straightforward to see that in the case of symmetric linear connection ∇, i.e. if the connection coefficients are symmetric in their lower indices: $\Gamma^i_{jk} = \Gamma^i_{kj}$, the tensor L_U^∇ also is symmetric in the lower pair of indices. In the following section 2.2.2, we also derive a useful non-component reformulation of the expression for the L_U^∇ tensor.

2.2.2 Formula for the L_U^∇ tensor

As the notation suggests, the tensor L_U^∇ is an object we can assign to a linear connection ∇ on a manifold (M,∇) and a vector field $U \in \mathfrak{X}(M)$. Therefore, one should be able to express it in terms of some kind of a combination of the vector field U and tensors describing the linear connection ∇. Here we show how to obtain such an expression for L_U^∇. To do so, however, we first need to mention a few objects and operations which correspond to the linear connection ∇ and are essential to derive the result. In this part of the text we work with vector fields X, Y, $Z \in \mathfrak{X}(M)$ and covectors α, $\beta \in \mathcal{T}^0_1(M)$.

Based on the covariant derivative ∇_Z being $\mathcal{F}(M)$-linear with respect to Z one may establish the operator, called the covariant gradient of a tensor $T \in \mathcal{T}^p_q(M)$, as follows:

$$\begin{gathered}\nabla : \mathcal{T}^p_q(M) \to \mathcal{T}^p_{(q+1)}(M)\\ (\nabla T)(X,...Y,Z;\alpha,...,\beta) := (\nabla_Z T)(X,...,Y;\alpha,..,\beta)\end{gathered} \tag{2.22}$$

Going forward we also make use of an identity valid for a covariant gradient of a type $\binom{1}{1}$ tensor field. Let us consider a tensor field $A \in \mathcal{T}^1_1(M)$.

Filling A with X results in a vector $A(X;\bullet)$ and adding α to the other slot results in a function. Applying Y to a function as a differential operator is the same as considering a covariant derivative ∇_Y of the function. Then using the fact that ∇_Y commutes with contractions and acts according to the Leibniz rule on the tensor product allows us to reformulate the term $Y[\,A(X;\alpha)\,]$ in two different ways. First as ∇_Y acting on the tensor product of A, X and the covector α:

$$\begin{aligned} Y[\,A(X;\alpha)\,] &= (\nabla_Y A)(X;\alpha) + A(\nabla_Y X;\alpha) + A(X;\nabla_Y \alpha) \\ &= (\nabla A)(X,Y;\alpha) + \langle \alpha, A(\nabla_Y X;\bullet)\rangle + \langle \nabla_Y \alpha, A(X;\bullet)\rangle \end{aligned} \tag{2.23}$$

And then as ∇_Y acting on the tensor product of the vector $A(X;\bullet)$ and the covector α:

$$Y[\,A(X;\alpha)\,] = \langle \nabla_Y \alpha, A(X;\bullet)\rangle + \langle \alpha, \nabla_Y\,[A(X;\bullet)]\rangle \tag{2.24}$$

Once we compare the equations (2.23) and (2.24), the following identity for $\nabla A \in \mathcal{T}_2^1(M)$ is obtained:

$$(\nabla A)(X,Y;\alpha) = \langle \alpha, \nabla_Y\,[A(X;\bullet)]\rangle - \langle \alpha, A(\nabla_Y X;\bullet)\rangle \tag{2.25}$$

Moving on to the next important objects on our list, we recall that for the linear connection ∇ one also defines the curvature tensor $R \in \mathcal{T}_3^1(M)$ as:

$$R(X,Y,Z;\alpha) = \left\langle \alpha, \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \right\rangle \tag{2.26}$$

The torsion tensor $T \in \mathcal{T}_2^1(M)$ is given as:

$$T(X,Y;\alpha) = \langle \alpha, \nabla_X Y - \nabla_Y X - [X,Y]\rangle \tag{2.27}$$

Let us also take one (more or less unnecessary) step, for mostly aesthetic reasons, which is to define two tensors $T_U \in \mathcal{T}_1^1(M)$ and $R_U \in \mathcal{T}_2^1(M)$, simply by considering the torsion tensor T and the curvature tensor R and inserting the vector U in one of the slots as follows:

$$T_U(X;\alpha) := T(U,X;\alpha) \tag{2.28}$$

$$R_U(Y,X;\alpha) := R(Y,U,X;\alpha) \tag{2.29}$$

The last useful identity on our list is the relationship between the Lie derivative $\mathcal{L}_U$ and the covariant derivative ∇_U given as the following relationship between tensor algebra $\mathcal{T}(M)$ derivations [Fecko, 2006]:

$$\mathcal{L}_U = \nabla_U - \nabla U - T(U, \bullet\, ; \bullet) \tag{2.30}$$

Having thus undergone this preparation, let us examine the definition of the tensor L_U^∇ with these relationships in mind. Expressing L_U^∇ in (2.16) using (2.11) yields:

$$L_U^\nabla(Y, X; \alpha) := \langle \alpha\,,\, \mathcal{L}_U \nabla_X Y - \nabla_X \mathcal{L}_U Y - \nabla_{\mathcal{L}_U X} Y \rangle \tag{2.31}$$

Using the identity (2.25) in the first two terms, one may express it as:

$$\begin{aligned} L_U^\nabla(Y, X;\alpha) = {}& \langle \alpha\,,\, \nabla_U \nabla_X Y \rangle - \langle \alpha\,,\, \nabla_{\nabla_X Y} U \rangle - \langle \alpha\,,\, T(U, \nabla_X Y; \bullet) \rangle - \\ & - \langle \alpha\,,\, \nabla_X \nabla_U Y \rangle + \langle \alpha\,,\, \nabla_X \nabla_Y U \rangle + \langle \alpha\,,\, \nabla_X \left[T(U, Y; \bullet) \right] \rangle - \\ & - \left\langle \nabla_{[U,X]} Y \right\rangle \end{aligned} \tag{2.32}$$

Here we can recognize the appearance of the pair of type $\binom{1}{1}$ tensors: ∇U and T_U:

$$\begin{aligned} L_U^\nabla(Y, X;\alpha) = {}& \langle \alpha\,,\, \nabla_U \nabla_X Y \rangle - \langle \alpha\,,\, \nabla U(\nabla_X Y\,; \bullet) \rangle - \langle \alpha\,,\, T_U(\nabla_X Y; \bullet) \rangle - \\ & - \langle \alpha\,,\, \nabla_X \nabla_U Y \rangle + \langle \alpha\,,\, \nabla_X \nabla U(Y\,; \bullet) \rangle + \langle \alpha\,,\, \nabla_X \left[T_U(Y; \bullet) \right] \rangle - \\ & - \left\langle \nabla_{[U,X]} Y \right\rangle \end{aligned} \tag{2.33}$$

In the resulting terms, one may identify two copies of the identity (2.25): one with the tensor T_U, defined in (2.28), and another one with the tensor ∇U playing the role of A. The remaining terms constitute the curvature tensor R:

$$L_U^\nabla(Y, X; \alpha) = [\nabla\nabla U](Y, X; \alpha) + R(Y, U, X; \alpha) + \nabla T_U(Y, X; \alpha) \tag{2.34}$$

Finally, we may use the tensor R_U, defined by the relation (2.29), to provide a slight cosmetic treatment for the equation (2.34). Thus obtaining the good-looking formula:

$$L_U^\nabla = \nabla\nabla U + R_U + \nabla T_U \tag{2.35}$$

2.3 About the defined geometrical objects

To conclude the chapter, we would like to add a few words about the geometrical objects which were introduced here. We have shown that the Lie derivative of linear connection can be expressed as an operator acting on the tensor algebra $\mathcal{T}(M)$ as well as a type $\binom{1}{2}$ tensor field. In the context of the present work, our aim is to use the Lie derivative of linear connection mostly as a tool to look for the symmetries of linear connections in different cases. As we know, symmetries correspond to the vectors in the direction of which the Lie derivative vanishes. Therefore, in the following chapters, when considering a Lie derivative $\mathcal{L}_U\nabla$ we view both the operator $(\mathcal{L}_U\nabla)_V$ and the corresponding tensor field L_U^∇ mainly as *the objects which need to vanish in order for U to be a symmetry of* ∇. One may notice that the good-looking formula (2.35) allows us to identify what influence the curvature (represented by the R_U term) and the torsion (∇T_U term) have on the condition for symmetries given by the demand for L_U^∇ to vanish.

Chapter 3

Symmetries of linear connection

In the previous chapter, we described the objects which one may use to express the Lie derivative of linear connection. Here we use the acquired knowledge to study the symmetries of linear connection. The core concept is fairly simple: if a Lie derivative $\mathcal{L}_U\nabla$ vanishes then the vector field U is a symmetry of the linear connection ∇. We begin with providing equations expressing the condition for the vector U to be a symmetry of the linear connection ∇. These may be formulated as a condition for vanishing of the tensor L_U^∇ as well as in terms of forms of connection. We also present a proof that the symmetries of linear connection form a Lie algebra.

3.1 Condition for the symmetries of ∇

As we have seen in previous chapter, one of the ways to express the Lie derivative of linear connection is using the tensor L_U^∇. The condition for the Lie derivative to vanish can therefore be formulated simply as a requirement for the tensor L_U^∇ to be equal to zero. Hence, based on the good-looking formula (2.35), one may obtain the corresponding condition as:

$$\nabla\nabla U + R_U + \nabla T_U = 0 \tag{3.1}$$

For the frequent case of linear connections with vanishing torsion, the equation (3.1) is then reduced to:

$$\nabla\nabla U + R_U = 0 \tag{3.2}$$

3.1.1 Symmetries of ∇ in terms of forms of connection

Just as in many other areas of geometry and physics, also in the case of linear connection, the formalism of differential forms serves as an effective tool to describe the geometrical objects appearing in the theory. Therefore, we formulate the condition for a symmetry U of a given linear connection ∇ in terms of forms of connection as well. We start with the case of ∇ with vanishing torsion and then use a natural generalization to obtain the formula also for a general case of linear connection.

Let us, therefore, first consider a linear connection ∇ with vanishing torsion. As we know, one may describe the linear connection ∇ in terms of the coefficients of connection Γ^i_{jk}. Alternatively, the connection ∇ may equally well be described using the forms of connection ω^a_b which may be viewed as a 'matrix of 1-forms' (or $gl(n,\mathbb{R})$-valued 1-form for $n = \dim(M)$). These are 1-forms defined, with respect to the frame field e_a, by the identity:

$$\nabla_V e_b =: \omega^a_b(V) e_a \tag{3.3}$$

To find a symmetry condition in this language, it is sufficient to find the formula also for the forms of the pullback connection $(\Phi^*_\epsilon\nabla)$. Based on the equation (2.6), we find that for a symmetry U the two covariant derivatives, and therefore also the corresponding forms of connection, must be equal. We will consider an infinitesimal pullback with respect to the flow Φ_ϵ of a vector field U:

$$U \leftrightarrow \Phi_t \tag{3.4}$$

We may also express the forms of the pullback connection σ^a_b with respect to the same frame field e^a. These are then given as:

$$(\Phi^*_\epsilon\nabla)_V e_b =: \sigma^a_b(V) e_a \tag{3.5}$$

After we use the definition of the covariant derivative $(\Phi^*_\epsilon\nabla)_V$ (2.6) on the left-hand side of the identity (3.5), we get the following result valid to the first order in ϵ:

$$\sigma^a_b(V) = \langle e^a, \nabla_V e_b + \epsilon\mathcal{L}_U\nabla_V e_b - \epsilon\nabla_V\mathcal{L}_U e_b - \epsilon\nabla_{\mathcal{L}_U V} e_b\rangle \tag{3.6}$$

To reformulate the resulting forms σ^a_b we can use the relationship between the Lie derivative $\mathcal{L}_U$ and the covariant derivative ∇_U (the formula (2.30) from the previous chapter). For the covariant derivative corresponding to a linear connection with vanishing torsion this relationship can be expressed using the covariant gradient ∇U of the vector field U:

$$\mathcal{L}_U = \nabla_U - \nabla U \tag{3.7}$$

To avoid possible confusion with the notation we will denote the covariant gradient ∇U as a tensor A:

$$A^a_b := (\nabla U)^a_b = \langle e^a, \nabla_b U\rangle \tag{3.8}$$

It is once again going to prove useful to view the components of A as a matrix of functions on M. Finally we denote Ω^a_b the forms of curvature for the connection ∇:

$$\Omega^a_b(U,V) = \left\langle e^a, (\nabla_U\nabla_V - \nabla_V\nabla_U - \nabla_{[U,V]})e_b\right\rangle \tag{3.9}$$

These may also be viewed as a matrix of 2-forms. Expressing the Lie derivative $\mathcal{L}_U$ in the equation (3.6) using the relationship (3.7) yields the following result:

$$\begin{aligned}\sigma^a_b(V) = \big\langle e^a, \nabla_V e_b + \epsilon(\nabla_U\nabla_V - \nabla_V\nabla_U - \nabla_{[U,V]})e_b - \epsilon\,\omega^c_b(V)\nabla_c U + \\ +\epsilon\nabla_V(A^c_b e_c)\ \big\rangle = \\ = \omega^a_b(V) + \epsilon\left[\Omega^a_b(U,V) - A^a_c\omega^c_b(V) + \omega^a_c(V)A^c_b + dA^a_b(V)\right]\end{aligned} \tag{3.10}$$

Here is where, the possibility to express the geometrical objects in (3.10) as matrices, proves useful, allowing us to write in the matrix notation:

$$\sigma = \omega + \epsilon(i_U\Omega - A\omega + \omega A + dA) \tag{3.11}$$

If U is a symmetry of ∇, the covariant derivaties ∇_V and $(\Phi^*_\epsilon\nabla)_V$ have to be equal:

$$\mathcal{L}_U\nabla = 0 \iff (\Phi^*_\epsilon\nabla)_V = \nabla_V \iff \sigma = \omega \tag{3.12}$$

Consequently, looking at the equation (3.11) we see that our condition for U is:

$$i_U\Omega - A\omega + \omega A + dA = 0 \tag{3.13}$$

Finally, to generalize the result also for a linear connection with a non-zero torsion tensor T, we have to consider a different relationship between the Lie and the covariant derivative. Instead of the identity (3.7), the formula (2.30) has to be used. However, as it turns out this does not complicate the situation significantly. All one has to do here is consider another matrix of functions given as the components of the tensor T_U defined in (2.28). With respect to the frame field e^a, we get:

$$(T_U)^a_b := \langle e^a, T(U, e_b)\rangle \tag{3.14}$$

Using the identity (2.30) it is now possible to use the analogy with the case of vanishing torsion. One may simply repeat the whole procedure of deriving the equation (3.11), using tensor $A + T_U$ instead of just A. Thus arriving at the formula for the forms σ^a_b of the $\Phi^*_\epsilon\nabla$ connection based on forms ω^a_b of ∇. In matrix notation, this leads to the result:

$$\sigma = \omega + \epsilon\{\, i_U\Omega - (A + T_U)\omega + \omega(A + T_U) + d(A + T_U)\,\} \tag{3.15}$$

And the corresponding equation for symmetries of ∇ is:

$$i_U\Omega - (A + T_U)\omega + \omega(A + T_U) + d(A + T_U) = 0 \tag{3.16}$$

3.2 Lie algebra of the linear connection symmetries

Symmetries of a structure of linear connection ∇ correspond to vector fields U for which the Lie derivative vanishes:

$$\mathcal{L}_U\nabla = 0 \tag{3.17}$$

This means that also the operator $(\mathcal{L}_U\nabla)_X$ vanishes in direction of an arbitrary vector field X:

$$(\mathcal{L}_U\nabla)_X T = 0 \qquad \text{for all } T \in \mathcal{T}(M) \tag{3.18}$$

Here we show that the symmetries of ∇ form a Lie algebra. To prove this, one has to make sure that for any two vector fields U,V which are symmetries of ∇ also their linear combination and their commutator are symmetries of the connection ∇:

$$\mathcal{L}_U\nabla = 0 \quad ; \quad \mathcal{L}_V\nabla = 0 \quad \Longrightarrow \quad \mathcal{L}_{(U+\lambda V)}\nabla = 0 \qquad \text{for } \lambda \in \mathbb{R} \tag{3.19}$$

$$\mathcal{L}_U\nabla = 0 \quad ; \quad \mathcal{L}_V\nabla = 0 \quad \Longrightarrow \quad \mathcal{L}_{[U,V]}\nabla = 0 \tag{3.20}$$

In the case of the linear combination of vector fields it is rather trivial to show that the condition (3.19) is satisfied because the operator $(\mathcal{L}_{(V+\lambda W)})_U$ can be expressed as:

$$(\mathcal{L}_{(U+\lambda V)}\nabla)_X = (\mathcal{L}_U\nabla)_X + \lambda(\mathcal{L}_V\nabla)_X \tag{3.21}$$

In the case of the commutator, the following expression for the operator $(\mathcal{L}_{[U,V]}\nabla)_X$ acting on a tensor $T \in \mathcal{T}(M)$ may be derived:

$$\begin{aligned}(\mathcal{L}_{[U,V]}\nabla)_X T = \quad & \mathcal{L}_U\left[(\mathcal{L}_V\nabla)_X T\right] - \mathcal{L}_V\left[(\mathcal{L}_U\nabla)_X T\right] + \\ & + (\mathcal{L}_V\nabla)_X(\mathcal{L}_V T) - (\mathcal{L}_V\nabla)_X(\mathcal{L}_U T) + \\ & + (\mathcal{L}_U\nabla)_{\mathcal{L}_V X} T - (\mathcal{L}_V\nabla)_{\mathcal{L}_U X} T\end{aligned} \tag{3.22}$$

Based on the condition (3.18) we know, that the operators $(\mathcal{L}_U\nabla)_X$ and $(\mathcal{L}_V\nabla)_X$ vanish in direction of an *arbitrary vector* when acting on an *arbitrary tensor* and therefore if both U and V are symmetries, all six terms on the right in the equation (3.22) vanish. This means that the operator $(\mathcal{L}_{[U,V]}\nabla)_X$ vanishes as well and so the condition (3.20) is satisfied too. Since both the conditions (3.19) and (3.20) are satisfied we may conclude that the symmetries of a linear connection ∇ constitute a Lie algebra.

3.3 Relevance of linear connection symmetries in physics

The content of the thesis up to this point could be summed up in the following way: we have learned how to introduce a pullback of a linear connection, then we have shown how to use it on a manifold (M, ∇) to introduce, except for the original connection ∇, also another connection $\Phi_\epsilon^*\nabla$ and we have gone through a decent amount of trouble to define and study the properties of the operator $(\mathcal{L}_U\nabla)_V$ expressing the difference of these two linear connections, only to then claim that we are actually interested mostly in the case when the two connections ∇ and $\Phi_\epsilon^*\nabla$ are equal and the operator $(\mathcal{L}_U\nabla)_V$ vanishes. To make this look less like nonsense, we dedicate the ending of this chapter to explaining our view of the relevance of the linear connection symmetries for the study of properties of physical theories which use the covariant derivative.

In the geometrical description of physics, the covariant derivative is a widely used tool, the most well-known example being Einstein's theory of general relativity. Nevertheless, it would be hard to imagine a physically relevant situation in which one would need two different covariant derivatives on the same manifold for the description of physics (although, far be it from us to call it impossible). What could be more interesting, from the physicist's point of view though, is learning about the properties of the covariant derivative that is already established on the studied manifold. And that is exactly what finding the symmetries of the given linear connection allows us to do. If we manage to find symmetry vector field U of a linear connection ∇, then for the flow Φ_ϵ of U the covariant derivatives $(\Phi_\epsilon^* \nabla)_V$ and ∇_V are equal which, based on the identity (2.6), leads to the relation for $T \in \mathcal{T}(M)$:

$$\nabla_V T = \Phi_{-\epsilon *} \nabla_{(\Phi_{\epsilon *} V)} (\Phi_{\epsilon *} T) \tag{3.23}$$

This tells us that the flow Φ_ϵ represents a transformation with the following property: let us consider $U = \partial_\varphi$ which generates rotations as an example. If U is a symmetry of ∇ and we need to compute a covariant derivative of T in the direction of V, the result is the same as first rotating both these objects, then computing the derivative for the rotated ones, and rotating it backwards. Moreover, based on the results (1.18) and (1.19) derived in the first chapter (see section 1.3), these flows also generate coordinate transformations which lead to the coefficients of connection given by the same functions of the transformed coordinates as the original ones. That means one could just perform this change of coordinates and continue using the covariant derivative 'without even noticing that the coordinate change happened' and obtain correct results for the covariant derivative. As we shall see in the following chapters, these properties of the connection symmetries lead to uncovering useful facts about the behavior of the covariant derivative itself as well as the geodesic curves on the manifold (M, ∇) which often carry important information about the physics of the theory.

Chapter 4

Symmetries of the Levi-Civita connection

The Levi-Civita connection is the most common case of linear connection (and the only one many people come in contact with). It is canonically present on every Riemannian manifold (M, g) and is fully determined by the metric g and vanishing torsion. Because of its popularity and widespread use, it certainly deserves our special attention. Here we show how to derive a useful reformulation of the condition for symmetries, valid specifically for the Levi-Civita connection, expressing it in terms of the Lie derivative of the metric tensor g [Paliathanasis, 2021]. Consequently, that is going to enable us to make some observations concerning the properties of these symmetries.

4.1 Formula for the Levi-Civita connection symmetries

Here we derive a useful identity satisfied for the Lie derivative of the metric $\mathcal{L}_U g$ in direction of a vector field U which is a symmetry of Levi-Civita connection. We start by briefly recalling a couple of definitions concerned with the Levi-Civita connection and deriving a few identities which we then use to obtain the formula.

The Levi-Civita connection ∇ is uniquely determined by two requirements. The first one says that the lenghts of vectors must be preserved under parallel transport:

$$\nabla g = 0 \tag{4.1}$$

And the second demand is that the torsion tensor vanishes:

$$\nabla_U V - \nabla_V U - [U, V] = 0 \tag{4.2}$$

The corresponding covariant derivative ∇_X for the Levi-Civita connection can be expressed in terms of the metric g by the following defining formula. For vector fields $X, Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned} g(\nabla_X Y, Z) :=& \frac{1}{2}\{Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + \\ &+ g([X,Y],Z) - g([X,Z],Y) - g(X,[Y,Z])\} \end{aligned} \tag{4.3}$$

To find the symmetries of a linear connection means to identify vectors U for which the opperator $(\mathcal{L}_U \nabla)_X$ vanishes. Since it is a derivation of the tensor algebra $\mathcal{T}(M)$ vanishing on functions and commuting with contractions, it is sufficient for us to prove that it vanishes on vector fields. This allows one to formulate the condition for the vector field U to be a symmetry of the Levi-Civita connection ∇ as follows:

$$\begin{aligned} &(\mathcal{L}_U \nabla)_X Y := \mathcal{L}_U \nabla_X Y - \nabla_X \mathcal{L}_U Y - \nabla_{\mathcal{L}_U X} Y = 0 \\ &\forall\, X, Y \in \mathfrak{X}(M) \end{aligned} \tag{4.4}$$

Here we use the fact that the metric tensor g is non-degenerate and reformulate this condition in the following way (which is actually equivalent to lowering the index of L_U^∇ using g):

$$\begin{aligned} &g(\mathcal{L}_U \nabla_X Y, Z) - g(\nabla_X \mathcal{L}_U Y, Z) - g(\nabla_{\mathcal{L}_U X} Y, Z) = 0 \\ &\forall\, X, Y, Z \in \mathfrak{X}(M) \end{aligned} \tag{4.5}$$

Let us continue by listing a few helpful identities for expressions containing the metric tensor g with different combinations of Lie and covariant derivatives of vector fields inserted in both slots. These are going to be essential to obtain our result. Both the Lie derivative and the covariant derivative are derivations of the tensor algebra $\mathcal{T}(M)$ that commute with contractions and act on tensor product according to the Leibniz rule. Based on this fact, one may derive useful identities by applying a vector field as a differential operator to g (or any other tensor for that matter) with inserted combinations of vectors.

This way we obtain the following four relationships:

$$Ug(\nabla_X Y, Z) = (\mathcal{L}_U g)(\nabla_X Y, Z) + g(\mathcal{L}_U \nabla_X Y, Z) + g(\nabla_X Y, \mathcal{L}_U Z) \tag{4.6}$$

$$X(\mathcal{L}_U g)\,(Y, Z) = XUg(Y, Z) - Xg([U, Y], Z) - Xg(Y, [U, Z]) \tag{4.7}$$

$$(\mathcal{L}_U g)\,([X, Y], Z) = Ug([X, Y], Z) - g([U[X, Y]], Z) - g([X, Y], [U, Z]) \tag{4.8}$$

$$[\nabla(\mathcal{L}_U g)]\,(X, Y, Z) = Z(\mathcal{L}_U g)(X, Y) - (\mathcal{L}_U g)(\nabla_Z X, Y) - (\mathcal{L}_U g)(X, \nabla_Z Y) \tag{4.9}$$

Having thus undergone this preparation, we may proceed to deriving our formula. We may express the first term in the condition (4.5) using (4.6) to obtain:

$$\begin{aligned} Ug(\nabla_X Y, Z) - (\mathcal{L}_U g)\,(\nabla_X Y, Z) - g(\nabla_X Y, \mathcal{L}_U Z)- \\ -g(\nabla_X \mathcal{L}_U Y, Z) - g(\nabla_{\mathcal{L}_U X} Y, Z) = 0 \end{aligned} \tag{4.10}$$

This allows us to use the definition of the Levi-Civita connection (4.3) in all the terms except for the one containing the Lie derivative $\mathcal{L}_V g$. Doing that yields a rather unpleasantly looking result consisting of 25 expressions given as:

$$\begin{aligned} &\frac{1}{2}\{\; UXg(Y, Z) + UYg(X, Z) - UZg(X, Y)+ \\ &+ Ug([X, Y], Z) - Ug([X, Z], Y) - Ug(X, [Y, Z])+ \\ &- Xg(Y, [U, Z]) - Yg(X, [U, Z]) + [U, Z]g(X, Y)+ \\ &- g([X, Y], [U, Z]) + g([X, [U, Z]], Y) + g(X, [Y, [U, Z]])+ \\ &- Xg([U, Y], Z) - [U, Y]g(X, Z) + Zg(X, [U, Y])+ \\ &- g([X, [U, Y]], Z) + g([X, Z], [U, Y]) + g(X, [[U, Y], Z])+ \\ &- [U, X]g(Y, Z) - Yg([U, X], Z) + Zg([U, X], Y)+ \\ &- g([[U, X], Y], Z) + g([[U, X], Z], Y) + g([U, X], [Y, Z])\; \}- \\ &- (\mathcal{L}_U g)\,(\nabla_X Y, Z) = 0 \end{aligned} \tag{4.11}$$

To make the result look less intimidating we may use the equations (4.7) and (4.8) (and utilize the Jacobi identity for some of the vector field commutators). That reduces the condition (4.11) to:

$$\begin{aligned}
&\frac{1}{2}\{\ X(\mathcal{L}_U g)\,(Y,Z) + (\mathcal{L}_U g)\,([X,Y],Z)+ \\
&+ Y(\mathcal{L}_U g)\,(X,Z) - (\mathcal{L}_U g)\,([X,Z],Y)+ \\
&- Z(\mathcal{L}_U g)\,(X,Y) - (\mathcal{L}_U g)\,(X,[Y,Z])\,\} - \\
&- (\mathcal{L}_U g)\,(\nabla_X Y, Z) = 0
\end{aligned} \tag{4.12}$$

The next step stems from vanishing of the torsion tensor in (4.2) which implies one may express the commutator in terms of a combination of covariant derivatives of the correspnding vectors. After doing that in (4.12) we can make use of the relationship (4.9) valid for the covariant gradient of Lie derivative of metric $\nabla(\mathcal{L}_U g)$. Consequently, our symmetry condition is expressed in terms of the $\nabla(\mathcal{L}_U g)$ tensor as:

$$[\nabla(\mathcal{L}_U g)]\,(Y,Z,X) + [\nabla(\mathcal{L}_U g)]\,(Z,X,Y) - [\nabla(\mathcal{L}_U g)]\,(X,Y,Z) = 0 \tag{4.13}$$

Or in components:

$$(\mathcal{L}_U g)_{ij;k} + (\mathcal{L}_U g)_{jk;i} - (\mathcal{L}_U g)_{ki;j} = 0 \tag{4.14}$$

In the resulting equation (4.14) we may notice that expressing the same relationship for a permutation of indices: $ijk \mapsto kij$ leads to:

$$(\mathcal{L}_U g)_{ki;j} + (\mathcal{L}_U g)_{ij;k} - (\mathcal{L}_U g)_{jk;i} = 0 \tag{4.15}$$

After computing a sum of the expressions (4.14) and (4.15) we get the result. The condition for a symmetry U of the Levi-Civita connection ∇ is given by the formula:

$$(\mathcal{L}_U g)_{ij;k} = 0 \tag{4.16}$$

Or going back to the non-component language:

$$\nabla \mathcal{L}_U g = 0 \tag{4.17}$$

4.2 Killing fields and homothetic vectors

The obvious take away from the obtained condition for the Levi-Civita connection symmetries (4.17) is that it is clearly satisfied for the Killing vectors ξ, i.e. those which satisfy:

$$\mathcal{L}_{\xi} g = 0 \tag{4.18}$$

We can therefore see that the Lie algebra of Killing fields is a subalgebra of the Lie algebra of symmetries of the Levi-Civita connection ∇. This is an expected and a very natural result. Since geodesics (at least those of the Levi-Civita connection) are curves connecting points with the line of the shortest possible lenght, isometries automatically preserve them by preserving all the distances on the manifold (M, g).

Moreover, the formula (4.17) allows us to also examine tha case of conformal transformations $\chi \in \mathfrak{X}(M)$. These are automorphisms of M which satisfy the condition:

$$\mathcal{L}_{\chi} g = f g \tag{4.19}$$

Here $f \in \mathcal{F}(M)$ is an arbitrary function on M. A quick analysis reveals that these constitute symmetries of ∇, if and only if f is a constant. In case of constant f these are called the homothetic vectors.

In conclusion, this reveals that the Lie algebra of the Levi-Civita connection symmetries contains the subalgebras of Killing fields and homothetic vectors of (M, g) [Paliathanasis, 2021].

Chapter 5

Geometrical interpretation: preservation of geodesics

Having thus learned *how* to look for the symmetries of a given linear connection ∇ we may proceed to ask the next, perhaps even more important question: *why* should one be interested in finding these? In this chapter, we show that the pullback generated by a symmetry of ∇ acts on the covariant derivative in a particularly useful manner. This, among other things, leads to the fact that symmetries of linear connection preserve the geodesic curves on the given manifold (M, ∇).

5.1 Symmetries act naturally on the covariant derivative

Consider a vector field U which is a symmetry of the connection ∇. Then a brief look at the equation (2.7) reveals that the covariant derivatives $(\Phi_\epsilon^* \nabla)_V$ and ∇_V in this case have to be equal. For the corresponding flow Φ_ϵ of the field U that leads to an identity valid for any tensor $T \in \mathfrak{T}(M)$:

$$\nabla_V T = \Phi_\epsilon^* \left[\nabla_{\Phi_{\epsilon *} V} \Phi_{\epsilon *} T \right] \tag{5.1}$$

Here one may act on the equation (5.1) with another pullback $\Phi_{-\epsilon}^*$ and use the relation: $\Phi_{-\epsilon}^* = \Phi_{\epsilon *}$. Leading to the formula:

$$\Phi_\epsilon^* [\nabla_V T] = \nabla_{\Phi_\epsilon^* V} \Phi_\epsilon^* T \tag{5.2}$$

This is called a *natural behavior* of the pullback Φ_ϵ^* corresponding to a symmetry U when acting on the covariant derivative. In other words, the symmetries are, in this sense, compatible with the covariant derivative. When computing a pullback of a tensor $\nabla_V T$, the pullback acts separately on the vector V and the tensor T.

Based on what we learned so far, we only know that the equation (5.2) clearly holds for the *infinitesimal* value of the parameter ϵ. Nevertheless, one may act on the result with another pullback Φ_ϵ^* which once again acts naturally. Then, due to the composition property of the flow Φ_t, we may iterate this procedure resulting in the fact that the natural behavior is exhibited by the flow Φ_t for an *arbitrary value of the parameter* t:

$$\Phi_t^*[\nabla_V T] = \nabla_{\Phi_t^* V} \Phi_t^* T \tag{5.3}$$

5.2 Affinely parametrised geodesics

The fact that symmetries exhibit the property (5.2) derived above is particularly useful once we consider the geodesic equation. The affinely parametrized geodesics on a manifold (M, ∇) are the curves γ for which their tangent vector $\dot{\gamma}$ is parallel to itself. This condition may be expressed as follows:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \tag{5.4}$$

Computing a pullback Φ_ϵ^* of the geodesic equation (5.4) and using the property (5.2) yields an interesting result:

$$\nabla_{\Phi_\epsilon^* \dot{\gamma}} \Phi_\epsilon^* \dot{\gamma} = 0 \tag{5.5}$$

The equation (5.5) reveals that if the tangent vector $\dot{\gamma}$ to the original curve γ satisfied the condition (5.4), so does the transformed vector $\Phi_\epsilon^* \dot{\gamma}$. This actually is the tangent vector to the curve $\Phi_{-\epsilon} \circ \gamma$ created by shifting the original γ in the direction of the vector field flow (the negative sign of the parameter is not an issue here, since it may be chosen arbitrarily). Meaning that the flows generated by symmetry vectors of the connection ∇ always transform a geodesic curve into another geodesic curve.

5.2.1 Preservation property determines symmetries of linear connection

Once we have learned that all the symmetries of linear connection have the property of preserving the affinely parametrized geodesics, it would be interesting to know if the statement may be reversed: are all the transformations preserving the affinely parametrized geodesic equation also symmetries of the linear connection ∇ ? As it turns out, the answer to this question is positive. To prove this statement we may consider a vector field $X \in \mathfrak{X}(M)$ which is *not* a symmetry of ∇ and its flow Ψ_ϵ. Once we shift a geodesic γ using Ψ_ϵ, we obtain the curve $\Psi_\epsilon \circ \gamma$. Consequently, we may express the analogical term $\nabla_{\Psi_{\epsilon *}\dot\gamma}\Psi_{\epsilon *}\dot\gamma$ term corresponding to the tangent vector to our shifted curve. Using the relationship (2.6) between the covariant derivatives ∇_V and $(\Psi_\epsilon^* \nabla)_V$ yields the following relationship:

$$\nabla_{(\Psi_{\epsilon *}\dot\gamma)}(\Psi_{\epsilon *}\dot\gamma) = \Psi_{\epsilon *}\nabla_{\dot\gamma}\dot\gamma + \epsilon \Psi_{\epsilon *}(\mathfrak{L}_X \nabla)_{\dot\gamma}\dot\gamma \tag{5.6}$$

Here the first term on the right containing $\nabla_{\dot\gamma}\dot\gamma$ vanishes since γ, being an affinely parametrized geodesic, satisfies the condition (5.4). In the second term though, the operator $(\mathfrak{L}_X \nabla)_{\dot\gamma}$ does not vanish since X is not a symmetry of ∇ (and $\Psi_{\epsilon *}$ is a linear isomorphism, so it can't make it vanish as well). Hence we conclude that the only continuous transformations preserving the equation (5.4) are the symmetries of the linear connection ∇. This may be formulated as an equivalence of the two statements:

flow of $U \in \mathfrak{X}(M, \nabla)$ preserves $\nabla_{\dot\gamma}\dot\gamma = 0 \iff U$ is a symmetry of ∇

5.3 Non-affinely parametrised geodesics

As it turns out though, examining the geodesic preservation transformations even further, yields more interesting results. Let us have a torsionless connection, implying symmetric coefficients $\Gamma^i_{jk} = \Gamma^i_{kj}$ and hence also symmetric tensor L^∇_U:

$$(L^\nabla_U)^i_{jk} = (L^\nabla_U)^i_{kj} \tag{5.7}$$

Once again we consider here a curve γ and the corresponding tangent vector given in components as:

$$\gamma \leftrightarrow x^i(t) \quad ; \quad \dot\gamma \leftrightarrow \dot x^i(t) \tag{5.8}$$

Except for the geodesic curves with affine parameterization, the geodesic may also be parametrized non-affinely, leading to its covariant derivative of the tangent vector given as:

$$\nabla_{\dot\gamma}\dot\gamma = \sigma\dot\gamma \tag{5.9}$$

Here σ is an arbitrary function of the curve parameter. First of all, we may realize the following: if a certain vector field U is a symmetry of the linear connection ∇, then it does also preserve the equation of non-affinely parametrized geodesic. That may be seen from the following equation gained by computing the pullback of (5.9) (Φ_t is the flow of U):

$$\nabla_{(\Phi_{\epsilon *}\dot{\gamma})}(\Phi_{\epsilon *}\dot{\gamma}) = \Phi_{\epsilon *}\nabla_{\dot{\gamma}}\dot{\gamma} = \Phi_{\epsilon *}\sigma\,(\Phi_{\epsilon *}\dot{\gamma}) \tag{5.10}$$

We find that the original geodesic γ in (5.9) is just shifted to another non-affinely parametrized geodesic $\Phi_\epsilon \circ \gamma$ with the tangent vector $\Phi_{\epsilon *}\dot{\gamma}$ and the original function σ is just replaced by the transformed function $\Phi_{\epsilon *}\sigma$. This means that, indeed, the symmetry U also preserves the non-affinely parametrized geodesic.

However, one may now ask the question if the symmetries of ∇ are the *only* transformations preserving the non-affinely parametrized geodesics as well. As it turns out, the answer here is negative. There is a separate class of projective transformations, which are generally all the transformations that preserve the non-affinely parametrized geodesic equations [Tsamparlis and Paliathanasis, 2009]. For the projective transformations generated by a vector $V \in \mathfrak{X}(M)$, one gets a separate condition on the form of the tensor L_V^∇ expressed in components in the following way:

$$(L_V^\nabla)^i_{jk} = \delta^i_j \beta_k + \delta^i_k \beta_j \tag{5.11}$$

Here $\beta \in \mathcal{T}^0_1(M)$ represents an arbitrary 1-form. Let us verify if indeed these preserve the geodesic equations. We may consider a geodesic γ given in (5.9), a vector field V satisfying (5.11) and we denote Ψ_t its corresponding flow. Consequently, one can once again apply the relation (2.6) to express the corresponding term $\nabla_{\Psi_{\epsilon *}\dot{\gamma}}\Psi_{\epsilon *}\dot{\gamma}$ as:

$$\begin{aligned}\nabla_{(\Psi_{\epsilon *}\dot{\gamma})}(\Psi_{\epsilon *}\dot{\gamma}) &= \Psi_{\epsilon *}\nabla_{\dot{\gamma}}\dot{\gamma} + \epsilon\Psi_{\epsilon *}(\mathcal{L}_V\nabla)_{\dot{\gamma}}\dot{\gamma} = \\ &= \Psi_{\epsilon *}\sigma\,(\Psi_{\epsilon *}\dot{\gamma}) + \epsilon\Psi_{\epsilon *}(\mathcal{L}_V\nabla)_{\dot{\gamma}}\dot{\gamma}\end{aligned} \tag{5.12}$$

The first expression obtained on the right this time does not vanish but is instead expressed using (5.9) and to reformulate the second term, we use the relationship (2.17) between the tensor L_V^∇ and the $(\mathcal{L}_V\nabla)_{\dot{\gamma}}$ operator. Then using the component expressions (5.8):

$$\begin{aligned}(\mathcal{L}_V\nabla)_{\dot{\gamma}}\dot{\gamma} &= (L_V^\nabla)^i_{jk}\dot{x}^j\dot{x}^k\partial_i = \left[\delta^i_j\beta_k + \delta^i_k\beta_j\right]\dot{x}^j\dot{x}^k\partial_i \\ &= 2\beta(\dot{\gamma})\dot{\gamma}\end{aligned} \tag{5.13}$$

Pluging the result in (5.13) into the equation (5.12) then leads to the identity:

$$
\begin{aligned}
\nabla_{(\Psi_{\epsilon *}\dot{\gamma})}(\Psi_{\epsilon *}\dot{\gamma}) &= \Psi_{\epsilon *}\sigma\,(\Psi_{\epsilon *}\dot{\gamma}) + 2\epsilon\Psi_{\epsilon *}\left[\,\beta(\dot{\gamma})\dot{\gamma}\,\right] = \\
&= \Psi_{\epsilon *}\left[\sigma + 2\epsilon\beta(\dot{\gamma})\right](\Psi_{\epsilon *}\dot{\gamma})
\end{aligned}
\tag{5.14}
$$

Since the term $\Psi_{\epsilon *}\left[\sigma + 2\epsilon\beta(\dot{\gamma})\right]$ also is a function of the curve parameter t we may conclude that the geodesic γ whose non-affine parametrisation was given by the function σ was transformed into another non-affinely parametrized by the $\Psi_{\epsilon *}\left[\sigma + 2\epsilon\beta(\dot{\gamma})\right]$ function.

One may notice, however, that these transformations *do not* constitute symmetries of the linear connection ∇ because the L_U^{∇} tensor does not vanish. From the point of view of physics, we are more interested in actual symmetries of the linear connection, as those are the transformations that actually preserve the covariant derivative which we use to describe the physical situation. Therefore, those tell us more about the properties of our theory. The transformations given as (5.11) do not preserve the covariant derivative, but instead shift it into one which differs by the action of the non-zero operator $(\mathcal{L}_V\nabla)_X$. For $X \in \mathfrak{X}(M)$ we get:

$$
\nabla_X = \Psi_\epsilon^* \nabla_{\Psi_{\epsilon *}X}\Psi_{\epsilon *} + (\mathcal{L}_V\nabla)_X \neq \Psi_\epsilon^* \nabla_{\Psi_{\epsilon *}X}\Psi_{\epsilon *} \tag{5.15}
$$

For that reason, we did not pay as much attention to these transformations here as we did to the proper symmetries of linear connection ∇. They definitely deserve to be mentioned in context of the geodesic preservation though. For the sake of completeness, we also add that the geodesic equations might be preserved by certain non-continuous transformations as well, such as the reflections in Euclidean spaces. These cannot be expressed as a continuous flow of a vector field and therefore cannot be studied using the Lie derivative.

5.4 Example: E^n

Finally, let us illustrate the geodesic preserving transformations mentioned above in the example of Euclidean space E^n. Our goal in this section is to first work out the Lie algebra of symmetries of the Levi-Civita connection in E^n and secondly, we shall also look at the corresponding projective transformation given by the condition (5.11) from the previous section. Therefore, we consider a manifold (E^n, g, ∇) where ∇ is the Levi-Civita connection. To look for the symmetries of the connection ∇ we may use the good-looking formula (2.35). Since E^n is not curved and the Levi-Civita connection is torsionless, the corresponding two terms in the formula (2.35) vanish and the condition for a symmetry U, given by the demand for L_U^{∇} to vanish, simply becomes:

$$\nabla \nabla U = 0 \tag{5.16}$$

Furthermore, after one chooses to work with the Cartesian coordinates x^i, this leads to all the coefficients of the Levi-Civita connection vanishing as well. Therefore, when expressed in components, the condition for the vector U^i reads:

$$U^i{}_{,jk} = 0 \tag{5.17}$$

Which leads to U given as:

$$U^i = k^i_j x^j + q^i \tag{5.18}$$

Here $k^i_j \in \mathbb{R}$ and $q^i \in \mathbb{R}$ represent constant coefficients. As expected based on the results of chapter four, the symmetries clearly contain translations and rotations which are the isometries of Euclidean spaces. We may realize that this result is meaningful also because geodesics (at least those of the Levi-Civita connection) are the curves on the manifold which connect every two points with the line of the shortest possible length. Therefore, isometries automatically preserve these curves by preserving all the distances on the manifold. If each pair of points on the geodesic are in the shortest distance from each other, then isometries map them to another pair of points in the same (also the shortest) distance which induces the geodesic preservation. The isometries are spanned by the vectors (5.18) with the additional condition:

$$k^i_j = -k^j_i \tag{5.19}$$

Except for the isometries, there are other symmetries that do not satisfy (5.19). These also transform each geodesic into another geodesic, but they do so while changing the distances between points. Once again verifying the results of the fourth chapter, we may identify among these the (only) homothetic vector in Euclidean space. This is the vector given by the following coefficients:

$$k^1_1 = k^2_2 = \ldots = k^n_n \tag{5.20}$$

While all the remaining constants k^i_j vanish. The rest of the symmetry vectors are symmetries only of the connection ∇ (not of the metric g).

Let us now move on to the projective transformations $V \in \mathfrak{X}(M)$ of the linear connection ∇. The condition for these may be obtained by combining the condition (5.11) with the simple form of the tensor L_V^{∇} we see in (5.17). It is given in components as follows:

$$V^i{}_{,jk} = \delta^i_j \beta_k + \delta^i_k \beta_j \tag{5.21}$$

Where $\beta \in \mathcal{T}^0_1(M)$ may be an arbitrary 1-form. Solving these equations leads to the Lie algebra of projective transformations:

$$V^i = k^i_j x^j + (p_j x^j) x^i + q^i \tag{5.22}$$

Here k^i_j, p_j and q^i again represent constant coefficients.

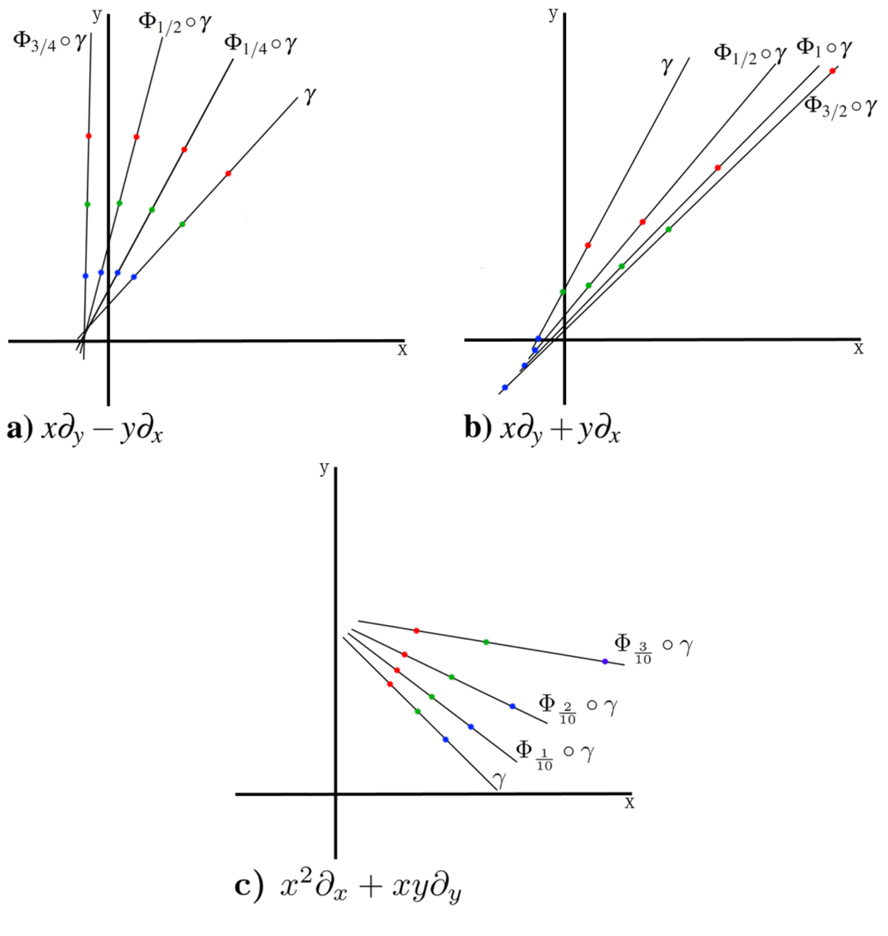


Figure 2: Geodesic transformations in E^2 corresponding to: **a)** isometry $x\partial_y - y\partial_x$ **b)** non-isometric symmetry of linear connection $x\partial_y + y\partial_x$ **c)** projective transformation $x^2\partial_x + xy\partial_y$ (points of the same color are images of the same point on the shifted curves).

We may recognize that the result (5.22) again contains the subalgebra of isometries satisfying the condition (5.19), the homothetic vector (5.20) and the rest of the proper symmetries of ∇ (5.18). In addition to that, there are now $n = \dim(M)$ extra vector fields spanned by the constants p_i which represent the proper projective symmetries (the geodesic preserving properties of different types of transformations are illustrated for the case of E^2 in Figure 2). Since in Euclidean space we are dealing with the Levi-Civita connection, for which the curve parameter of the affinely parametrized geodesic may be identified with the actual length of the curve, the difference between various types of transformations is manifested in the way in which points on the shifted geodesics are mapped. We may notice that for an isometry (figure 2 a)), each couple of points remains in constant distance after shifting the geodesic. For the non-isometric symmetries (figure 2 b)) the distances do change but the geodesic remains affinely parametrized. Therefore the distance between each pair of shifted points is the same multiple of the distance of the original two points for every couple of points on γ (in other words: if the blue point and the red one are in the same distance from the green point, this remains the case on the shifted curves as well, although the distance itself may change). In the case of a projective transformation (figure 2c)), the resulting geodesic is non-affinely parametrized which means not even these ratios of distances remain the same.

Chapter 6

Newton-Cartan structures

The geometrical properties of the Lie derivative of linear connection, discussed in the previous chapters, make it an exceptionally useful instrument for a physicist to have in his toolbox as well. The structure of linear connection plays an important role in many theories of modern physics [Misner et al., 1973]. Therefore, the study of the corresponding symmetries is a topic that is discussed by physicists as well [Duval and Horváthy, 2009]. To illustrate one of the possible physical applications of these ideas, we use the example of Newton-Cartan structures. We begin with a short introduction of the physical situation, comparing the relativistic and the classical spacetime geometries and briefly presenting to the reader the basic concepts behind the Newton-Cartan theory of gravity. Consequently, we investigate the corresponding linear connection of the Newton-Cartan theory and find its symmetries, allowing us to see their importance for the geometrical description of classical physics.

In the whole chapter, we work with a four-dimensional spacetime manifold M with coordinates x^μ ; $\mu \in \{0, 1, 2, 3\}$ where $x^0 = t$ is the time coordinate and x^i ; $i \in \{1, 2, 3\}$ are the cartesian spatial coordinates.

6.1 Relativistic and Newtonian spacetime

Although we mainly aim to explore the geometrical structures in classical spacetime, one of the best ways to appreciate their importance is in contrast to their more familiar relativistic counterparts. Let us, therefore, begin with comparing the basic geometrical objects in relativistic and classical spacetimes [Künzle, 1972]. In the well-known relativistic case, a metric may be introduced on the spacetime manifold M based on the requirement for its invariance under transformations of the Lorentz group. This condition yields the famous Minkowski metric expressed using the $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ matrix as:

$$g = \eta_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{6.1}$$

The symmetry group of (M, g) is then the Poincaré group, consisting of spacetime translations, rotations, and boosts. Their generators form the familiar Poincaré algebra of vectors $X \in \mathfrak{X}(M)$:

$$X^\mu = A^\mu_\nu x^\nu + B^\mu \tag{6.2}$$

The coefficients A^μ_ν and B^μ are constants satisfying: $A^i_j = -A^j_i$, $A^0_i = A^i_0$, $A^0_0 = 0$. Thus, we see that in the relativistic case, we arrive at a very intuitive result. The obtained metric g represents a structure on the spacetime manifold whose symmetry group consists of exactly those transformations that correspond to the coordinate changes which should leave the laws of physics invariant according to special relativity. The complete description of physics in the relativistic spacetime is given also by the Levi-Civita connection ∇ corresponding to the metric g. As we have seen in Chapter 4 though, the condition for the symmetries of the metric is stronger that the one for the symmetries of the corresponding Levi-Civita connection. Therefore, the symmetries of the relativistic structure (M, g, ∇) are determined only by the symmetries of the metric g.

Inspired by the simple beauty of relativistic geometry, one might hope to replicate this result in classical spacetime. As it turns out though, it is quite a bit more intricate in this case. In classical physics, we consider the principle of Galilean relativity, meaning essentially that the laws of physics should remain invariant under the coordinate changes associated with switching inertial reference frames of the classical observers. The corresponding non-relativistic counterparts of the Poincaré group transformations are those of the Galilei group, generated by spacetime translations, rotations, and Galilei boosts. The algebra of generating vectors is the Galilei algebra, consisting of the vectors $V \in \mathfrak{X}(M)$:

$$V = s\partial_t + \left[k^i_j\, x^j + l^i\, t + q^i\right] \partial_i \tag{6.3}$$

Here s, l^i, q^i and $k^i_j = -k^j_i$ are constant coefficients. Contrary to the relativistic case though, one finds that no metric can be introduced in Newtonian spacetime which would be invariant under the Galilei group transformations. However, it is possible to introduce the *Galilei structure* (M, h, θ). It consists of $h \in \mathcal{T}^2_0(M)$ which is a *degenerate* symmetric tensor of type $\binom{2}{0}$ and a 1-form $\theta \in \mathcal{T}^0_1(M)$ which determines a submanifold $T = M/\mathrm{ker}(\theta)$. The submanifold T then represents the absolute Newtonian time, flowing at the exact same rate for all the observers. These tensors satisfy:

$$h^{\mu\nu}\theta_\nu = 0 \tag{6.4}$$

The pair of tensors partially substitutes the role of a 'metric' in Newtonian spacetime: the 1-form θ presents a possibility to measure time intervals and h may be used to raise the indices of the spatial components of tensors. For the purpose of our investigation here, it is sufficient to work with the most common, standard case of the Galilei structure which consists of $M = \mathbb{R} \times \mathbb{R}^3$ and the pair of tensors is defined as follows:

$$h = \delta^{ij} \partial_i \otimes \partial_j \tag{6.5}$$

$$\theta = \mathrm{d}t \tag{6.6}$$

Let us now examine the symmetries of the standard Galilei structure given by (6.5) and (6.6). These are simply vectors $W \in \mathfrak{X}(M)$ preserving h and θ:

$$\mathcal{L}_W h = 0 \quad ; \quad \mathcal{L}_W \theta = 0 \tag{6.7}$$

Solving the equations (6.7) results in the infinite-dimensional Coriolis algebra [Duval, 1993]:

$$W = \sigma \partial_t + \left[\omega^i_j(t)\, x^j + \kappa^i(t)\right] \partial_i \tag{6.8}$$

Where: $\omega^i_j = -\omega^j_i$. Here σ is a constant, but ω^i_j and κ^i are abitrary functions of time. Comparing the result to the vectors of the Galilei algebra in (6.3), we see that the Galilei transformations constitute a subalgebra of the Coriolis algebra. Nevertheless, we may notice that by considering the symmetries of the Galilei structure (M, h, θ), we *do not* reach a result similar to the relativistic case, where there is a structure (M, g) with its symmetry group consisting purely of the Poincaré transformations. As we shall see, to find such a structure in Newtonian spacetime, one needs to also introduce a linear connection.

6.2 Newton-Cartan theory

Similarly to general relativity, also in the classical case, it is possible to understand the gravitational phenomena using the language of geometry. The corresponding theory is called the Newton-Cartan theory of gravity, introduced by Élie Cartan in 1923 [Cartan, 1923]. Cartan's idea is based on the possibility to endow Newtonian spacetime with a linear connection ∇. That results in obtaining the *Newton-Cartan structure* (M, h, θ, ∇). The corresponding connection ∇ is the Newton-Cartan connection. Cartan developed this concept inspired by the (at the time recently published) theory of general relativity. His linear connection in classical spacetime leads to, just as in the relativistic case, trajectories of the moving points of mass being represented by the geodesic curves in (M, h, θ, ∇). Here we briefly sum up the key components of the Newton-Cartan theory of gravity.

So let us try to recreate Cartan's train of thought as he derived the theory. When describing the trajectory of a moving massive object in classical spacetime, one may naturally adopt the Newtonian time $x^0 = t$ as the curve parameter along its spacetime path: $\gamma(t) \leftrightarrow x^\mu(t)$. This convenient parametrization then leads to the following identities valid for the derivatives of the $x^0(t)$ component along the curve γ:

$$\dot{x}^0 = 1 \quad ; \quad \ddot{x}^0 = 0 \tag{6.9}$$

Since Cartan's aim was to identify the point mass trajectories with the geodesics of the connection ∇, he simply choose to seek such coefficients of connection Γ^i_{jk} that would make the geodesic equation equal to the equation of Newton's gravitational law:

$$\ddot{x^i} = -\partial_i \varphi \iff \ddot{x^\mu} + \Gamma^\mu_{\nu\rho} \dot{x^\nu} \dot{x^\rho} = 0 \tag{6.10}$$

Here $\varphi \in \mathcal{F}(M)$ is a function representing the Newtonian gravitational potential. The condition (6.10), along with considering a vanishing torsion, lead to a uniquely determined linear connection ∇ given by the following three non-zero coefficients:

$$\Gamma^i_{00} = \partial_i \varphi \tag{6.11}$$

The rest of the $\Gamma^\mu_{\nu\rho}$ coefficients vanish. One may easily check that the connection ∇ defined by (6.11) indeed leads to the Newtonian equations of motion being equivalent to the equation of (affinely parametrized) geodesic. We have thus succeeded at constructing a geometrical formulation of classical Newtonian gravity.

6.3 Symmetries of the flat Newton-Cartan structure

As we have learned in section 6.1, considering the Galilei structure (M, h, θ) did not bring about an analogy of the relativistic result, where the Minkowski metric (6.1) represents a structure which has a symmetry group consisting purely of the Poincaré group transformations. The Newton-Cartan structure though, consists, in addition to h and θ also of the linear connection ∇. Therefore, let us now work out the symmetries of the Newton-Cartan structure (M, h, θ, ∇). We consider here the simplest, so-called flat case. This means that our spacetime manifold once again is $M = \mathbb{R} \times \mathbb{R}^3$ and the tensors h and θ are defined as in the equations (6.5) and (6.6). Moreover, the gravitational potential φ vanishes. This actually describes the situation with no gravitational effects, so it represents a classical analogy to the spacetime of special relativity with the Minkowski metric. Based on (6.11) that leads to a linear connection ∇ with all the connection coefficients vanishing:

$$\Gamma^\mu_{\nu\rho} = 0 \tag{6.12}$$

For a vector field U to be symmetry of the flat Newton-Cartan structure (M, h, θ, ∇), there is a demand that it preserves h, θ and the connection ∇. Therefore, U has to satisfy the condition (6.7) (meaning it also has to be a symmetry of the corresponding Galilei structure). In addition to that though, here also the condition for U to preserve the linear connection ∇ has to be considered. Thus the demand for U to be a symmetry of (M, h, θ, ∇) is to satisfy the following three equations:

$$\mathcal{L}_U h = 0 \quad ; \quad \mathcal{L}_U \theta = 0 \quad ; \quad \mathcal{L}_U \nabla = 0 \tag{6.13}$$

In order for U to preserve h and θ, it has to be in the Coriolis algebra, therefore it has to have the form (6.8). Moreover, the condition for U to preserve the linear connection ∇, in case of all the coefficients $\Gamma^{\mu}_{\nu\rho}$ vanishing, is just analogical to the condition obtained in case of euclidean space E^n:

$$U = [a^{\mu}_{\nu} x^{\nu} + b^{\mu}]\, \partial_{\mu} \tag{6.14}$$

Here a^{μ}_{ν} and b^{μ} are constant, but otherwise arbitrary coefficients. Once we combine the conditions (6.8) and (6.15), the resulting symmetries of the flat Newton-Cartan structure (M, h, θ, ∇) are exactly the Galilei group transformations (6.3) [Duval, 1993].

In conclusion, to find a Newtonian structure that has, as its symmetry group, the Galilean transformations, we need to consider the Newton-Cartan structure (M, h, θ, ∇), not just the Galilei structure (M, h, θ). In other words, to find a non-relativistic analog of the Minkowski metric (6.1), representing a structure with its symmetry group given by the corresponding spacetime transformations, one has to also consider the Newton-Cartan connection ∇. This presents a quite illuminating example of a physical situation in which considering a linear connection and its symmetries is essential to obtain the desired result (the symmetry algebras of geometric objects, discussed in this chapter so far, are summed up schematically in Figure 3).

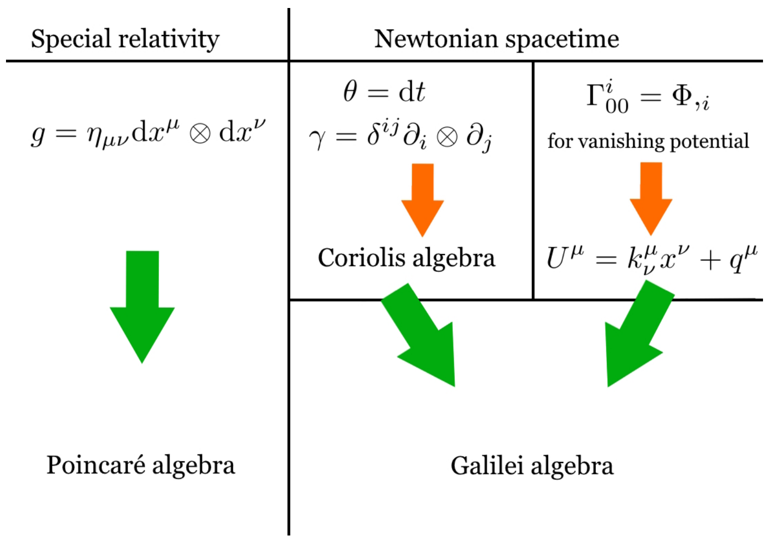


Figure 3: Symmetry algebras in relativistic and Newtonian spacetimes.

6.4 Symmetries depending on potential

After introducing the Newton-Cartan connection ∇ and explaining its relevance for the geometrical description of classical gravitational physics, we may further examine the symmetries of ∇ also in other cases than the one corresponding to vanishing gravitation. Before we start though, let us think about the meaning of these symmetries. Since we are looking for proper symmetry vectors U of ∇ for which $\mathcal{L}_U \nabla = 0$, these vectors generate flows preserving the affinely parametrized geodesic equations of ∇. Since those represent the trajectories of moving massive objects (based on (6.10)), we see that such flows will map one possible trajectory to another one. To find these transformations, we begin by formulating the corresponding equation. Because we are dealing here with a case of torsionless connection, one can obtain the condition for a symmetry vector field U based on the equation (3.2):

$$\nabla\nabla U + R_U = 0 \tag{6.15}$$

Going forward we consider a gravitational potential Φ depending on (cartesian) spatial coordinates x^i only, leading to vanishing time derivative $\Phi_{,0}$. We derive the equations for a general case of time-independent potential Φ here first and then we study some of the most common cases. To obtain our equations for a symmetry U we first need the Riemann curvature tensor. Using the coefficients of the Newton-Cartan connection (6.11) we get:

$$R^i_{0j0} = -R^i_{00j} = \partial_j \partial_i \Phi \tag{6.16}$$

Similarly, we may use the (6.11) coefficients to compute the components of the tensor $\nabla\nabla U$, resulting in (not the best looking) identity:

$$\begin{aligned}(\nabla\nabla U)^\mu_{\nu\rho} = {} & U^\mu{}_{,\nu\rho} + \delta^\mu_i \delta^0_\nu U^0{}_{,\rho}\, \Gamma^i_{00} + \delta^\mu_i \delta^0_\nu U^0 \Gamma^i_{00}{}_{,\rho} - \\ & - \delta^0_\nu \delta^0_\rho U^\mu{}_{,i}\, \Gamma^i_{00} + \delta^0_\rho \delta^\mu_i U_{,\nu}\, \Gamma^i_{00}\end{aligned} \tag{6.17}$$

That allows us to finally formulate the equations for a symmetry U (the summation convention holds also for two identical lower indices here):

$$U^0{}_{,ij} = U^0{}_{,i0} = U^i{}_{,jk} = 0 \tag{6.18}$$

$$U^0{}_{,00} = U^0{}_{,i}\, \Phi_{,i} \tag{6.19}$$

$$U^i{}_{,0j} = -U^0{}_{,j}\, \Phi_{,i} \tag{6.20}$$

$$U^i{}_{,00} = U^i{}_{,j}\, \Phi_{,j} - 2U^0{}_{,0}\, \Phi_{,i} - U^j \Phi_{,ij} \tag{6.21}$$

One may notice that the first condition (6.18) is the same for any function representing the time-independent potential Φ. Solving it leads to the general time and spatial components of the vector field U expressed as:

$$U^0 = \theta(t) + \gamma_k\, x^k \tag{6.22}$$

$$U^i = a^i(t) + b^i_j(t)\, x^j \tag{6.23}$$

As we can see, the coefficients $\theta(t)$, $a^i(t)$ and $b^i_j(t)$ are arbitrary functions of time, while γ_k are constants. The formulae (6.22) and (6.23) represent the basic form of the symmetry vector U for time-independent potential. Further restrictions are then applied on the parameters $\theta(t)$, $a^i(t)$, $b^i_j(t)$ and γ_k based on the other three equations (6.19), (6.20) and (6.21), which depend on the specific function representing the gravitational potential $\Phi(x^i)$. In the rest of this section, we examine some of the most common specific cases of functions $\Phi(x^i)$ and find the corresponding symmetry algebras.

6.4.1 If everything is falling, then nothing is falling

After one examines the additional conditions for the symmetry vector U stemming from the relationships (6.19), (6.20) and (6.21), it becomes clear that the possible solutions differ heavily based on one specific distinction: whether the potential Φ is a linear function of all three spatial coordinates (leading to $\Phi_{,i} = \text{const.}$) or not. Here we dissect the case of linear potential to explain the prominent role it plays. Let us, therefore, consider a potential Φ, known from elementary physics classes, given as:

$$\Phi = g_i x^i \tag{6.24}$$

The constant coefficients g_i are the well-known components of the constant gravitational acceleration vector $\vec{g}$. The conditions (6.19), (6.20) and (6.21) imposed on the general form of the vector U with components (6.22) and (6.23) then lead to:

$$\ddot{\theta}(t) = \gamma_i g_i \tag{6.25}$$

$$\dot{b}^i_j(t) = -\gamma_j g_i \tag{6.26}$$

$$\ddot{a}^i(t) = b^i_j(t) g_j - 2\dot{\theta}(t) g_i \tag{6.27}$$

We may notice that the resulting system of equations only involves one spacetime variable which is the time t. Solving these equations leads to the symmetry algebra of the Newton-Cartan connection ∇ for the linear potential (6.24). After one considers constant (but otherwise arbitrary) coefficients: α, β, γ_k, c^i_j, d^i and h^i, the components of the symmetry vector U are given as follows:

$$U^0 = \frac{1}{2}\gamma_k g_k t^2 + \gamma_k x^k + \alpha t + \beta \tag{6.28}$$

$$U^i = -\frac{1}{2}g_i g_k \gamma_k t^3 - g_i \gamma_k x^k t + \frac{1}{2}c^i_k g_k t^2 + c^i_k x^k - \alpha g_i t^2 + d^i t + h^i \tag{6.29}$$

One may realize that the resulting symmetry algebra is of exactly the same dimension as the symmetry algebra (6.15) corresponding to the case of vanishing (or constant) potential. This result makes a lot of sense after one performs a simple change of coordinates in Newtonian spacetime and expresses the symmetry algebra afterward. Let us consider a different set of spacetime coordinates $\tilde{x}^\mu$, established by leaving the time component unchanged: $\tilde{t} = t$, while introducing the spatial coordinates $\tilde{x}^i$ by shifting the original ones in the following way:

$$\tilde{x}^i = x^i + \frac{1}{2}g_i t^2 \tag{6.30}$$

These are clearly the coordinates of an observer in Newtonian spacetime who is 'falling' under the influence of gravity (it is also a coordinate change which leads to vanishing coefficients in (6.11)). Computing the components of the symmetry vector field U after the coordinate change (6.30) yields the following result:

$$\tilde{U}^0 = \alpha\tilde{t} + \gamma_k \tilde{x}^k + \beta \tag{6.31}$$

$$\tilde{U}^i = c^i_j \tilde{x}^j + d^i \tilde{t} + h^i \tag{6.32}$$

This is, of course, nothing else but the symmetry algebra (6.15) for the vanishing potential with renamed constant coefficients. To make sense of this result, it is sufficient to realize that the situation in which all the massive objects are equally accelerated by gravitation and move with a constant acceleration $\vec{g}$, from the point of view of the observer who is accelerated in such way as well, is just equivalent to the situation in which there is no gravitation at all. The case of linear potential is then linked by a simple coordinate change to the case of vanishing (or constant) gravitational potential Φ.

6.4.2 Spherically symmetrical potential

The next case we examine is the gravitational potential proportionate to the distance from the center of coordinates $r = \sqrt{x^2 + y^2 + z^2}$. We consider here the potential given as a function:

$$\Phi = r^m \tag{6.33}$$

The case of $m = -1$ then represents the familiar gravitational potential describing, in Newtonian theory, the gravitation around a massive object placed in the origin of the spatial coordinates (moreover, we consider here only the case of non-constant Φ: $m \neq 0$). Let us, once again, begin with imposing the three conditions (6.19), (6.20) and (6.21) on the general form of the symmetry vector U given by identities (6.22) and (6.23). This time for the case of the spherically symmetrical potential (6.33). After considering the conditions we obtain the equations:

$$\ddot{\theta}(t) = \gamma_i \, m r^{m-2} x^i \tag{6.34}$$

$$\dot{b}^i_j(t) = -\gamma_j \, m r^{m-2} x^i \tag{6.35}$$

$$\begin{aligned} \ddot{a}^i(t) = & \, b^i_j(t) \, m r^{m-2} x^j - 2\dot{\theta}(t) \, m r^{m-2} x^i - \\ & - \left[a^i(t) + b^i_j(t)\right] \left[m r^{m-2} \delta^{ij} + m(m-2) r^{m-4} x^i x^j\right] \end{aligned} \tag{6.36}$$

Let us now find the solution to the acquired system of equations. All the three equations (6.34) and (6.35) and (6.36) demand a function of time to be equal to a function of spatial coordinates. Thus the only possibility is that both sides of the equation must vanish, leading to:

$$\gamma_i = 0 \tag{6.37}$$

$$\theta(t) = \alpha t + \beta \tag{6.38}$$

$$a^i(t) = d^i t + h^i \tag{6.39}$$

$$b^i_j(t) = c^i_j = \text{const.} \tag{6.40}$$

Here α, β, c^i_j, d^i and h^i are arbitrary constants. The resulting conditions (6.37), (6.38), (6.39) and (6.40), as a matter of fact, hold for an arbitrary form of the (time-independent) gravitational potential, other than the cases of linear or constant Φ which were discussed in the previous sections. The obtained conditions lead to the equations (6.34) and (6.35) being satisfied automatically and also to the vanishing of the left side of (6.36). All that is left to do now is to make sure that the non-trivial right side of (6.36) vanishes as well. Using (6.37), (6.38), (6.39) and (6.40), one may now reformulate the last equation (6.36) to obtain the following formula:

$$\begin{aligned} 0 = & -2\alpha m r^{m-2} x^i - c^j_k m(m-2) r^{m-4} x^i x^j x^k - \\ & - \left[d^j t + h^j\right] \left[m r^{m-2} \delta^{ij} + m(m-2) r^{m-4} x^i x^j\right] \end{aligned} \tag{6.41}$$

The task to find symmetries is then reduced to identifying the appropriate combinations of constants which lead to satisfying the equation (6.41). The first obvious observation is that it is solved for an arbitrary value of the constant β, introduced in the equation (6.38), since it does not show up in the condition at all. This corresponds to the time translation symmetry of the theory which is obvious, since the considered potential Φ is time-independent. Furthermore, one may find after quick examination that the constants d^i and h^i must vanish:

$$d^i = h^i \stackrel{!}{=} 0 \tag{6.42}$$

The constants c^j_k clearly satisfy the condition (6.41) in case they are antisymmetric in their indices (since they are summed over with $x^j x^k$ term):

$$c^i_j \stackrel{!}{=} -c^j_i \tag{6.43}$$

These correspond to the rotational symmetry of the theory which is, once again, no surprise since we consider a spherically symmetrical gravitational potential Φ. What is a bit more interesting though, is that the condition (6.41) is satisfied for one more combination of the constants which is:

$$c^1_1 = c^2_2 = c^3_3 = \frac{\alpha}{\left(1 - \frac{m}{2}\right)} \tag{6.44}$$

This combination of constants corresponds to the vector field $\left(1 - \frac{m}{2}\right) t\partial_t + x^i \partial_i$ or in spherical coordinates: $\left(1 - \frac{m}{2}\right) t\partial_t + r\partial_r$. To make sense of this result let us consider the most familiar case of gravitational potential $1/r$ by choosing $m = -1$. The corresponding symmetry vector field then is $3/2\, t\partial_t + r\partial_r$. One may already recognize, based on the factor $3/2$ representing the ratio of transforming space and time, that this symmetry vector has to do with Kepler's third law of planetary motion. Indeed, after we compute the vector field flow $\Psi_\lambda : M \to M$ corresponding to this symmetry we get:

$$\begin{aligned} t &\mapsto t e^{3/2\lambda} \\ r &\mapsto r e^{\lambda} \end{aligned} \tag{6.45}$$

Based on the geodesic preserving property of the flow Ψ_λ (discussed in chapter 5) we know that each trajectory of a moving massive body in Newtonian spacetime, represented by a geodesic of the Newton-Cartan connection ∇, is transformed into another possible trajectory. That leads to the fact that for the coordinates t and r on these trajectories, as they transform under the flow (6.45), the following ratio remains constant:

$$\frac{t^2}{r^3} = \text{const.} \tag{6.46}$$

Although we are a few centuries late with this discovery of Kepler's third law [Kepler, 1619], this represents an interesting example of how the formalism of the Lie derivative of linear connection may help us uncover symmetries of the solutions for geodesic equations which are not clearly visible at the first sight. We can also see that we obtain, from the symmetry (6.44), a transformation corresponding only to the scaling of space and time, while the information about the form of possible planetary orbits which may be scaled in that way is not contained here (this point of view on Kepler's third law is illustrated in a comic version in Appendix B). We may also see, based on this result, what the analogy of Kepler's third law would look like for the potential given by a different function, provided we pick a different value of the parameter m.

The complete Lie algebra of the symmetries of the Newton-Cartan connection for the spherically symmetrical potential (6.33) is then given as:

$$U^0 = \alpha \left(1 - \frac{m}{2}\right) t + \beta \tag{6.47}$$

$$U^i = c^i_j x^j + \alpha x^i \tag{6.48}$$

With constant coefficients α, β and $c^i_j = -c^j_i$.

Appendices

Appendix A

Properties of the pullback connection

The pullback connection $f^*\nabla$ on a manifold M is introduced, based on a diffeomorphism $f : M \to (N, \nabla)$, by defining the corresponding covariant derivative for $V \in \mathfrak{X}(M)$ and $T \in \mathcal{T}(M)$ as:

$$(f^*\nabla)_V T := f^* \left[\nabla_{f_* V} f_* T\right] \tag{A.1}$$

Here we examine the properties of the covariant derivative defined in (A.1) to make sure that the defining formula actually satisfies all the conditions required for it to be a covariant derivative. We list the requirements for an operator to be a covariant derivative [Fecko, 2006] and for each one, we provide proof that it is satisfied for the operator $(f^*\nabla)_V$. Let us consider tensors A, $B \in \mathcal{T}(M)$, vectors V, $W \in \mathfrak{X}(M)$, a function $\varphi \in \mathcal{F}(M)$ and a constant $\lambda \in \mathbb{R}$. The properties of the covariant derivative are:

1. it acts linearly (satisfied because the operator $(f^*\nabla)_V$ is defined as a composition $f^* \circ \nabla_{f_* V} \circ f_*$ and all of these operators act linearly):

$$(f^*\nabla)_V (A + \lambda B) = (f^*\nabla)_V A + \lambda (f^*\nabla)_V B \tag{A.2}$$

2. it preserves the degree of tensors (once again satisfied because all the operators f^*, $\nabla_{f_* V}$ and f_* have this property):

$$(f^*\nabla)_V : \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M) \tag{A.3}$$

3. it acts on the tensor product according to the Leibniz rule (we use the fact that for a diffeomorphism: $f_*^{-1} = f^*$):

$$\begin{aligned}(f^*\nabla)_V(A\otimes B) &= f^*\left[\nabla_{f_*V} f_*(A\otimes B)\right] = f^*\left[\nabla_{f_*V}(f_*A\otimes f_*B)\right] = \\ &= f^*\left[(\nabla_{f_*V} f_*A)\otimes f_*B\right] + f^*\left[f_*A\otimes(\nabla_{f_*V} f_*B)\right] = \\ &= (f^*\nabla)_V A\otimes B + A\otimes(f^*\nabla)_V B\end{aligned} \tag{A.4}$$

4. on functions it is equal to the Lie derivative:

$$\begin{aligned}(f^*\nabla)_V\varphi &= f^*\left[\nabla_{f_*V} f_*\varphi\right] = f^*\left[f_*V f_*\varphi\right] = f^*\left\langle \mathrm{d}f_*\varphi\,,\, f_*V\right\rangle = \\ &= f^*\left\langle \mathrm{d}(f^{-1})^*\varphi\,,\,(f^{-1})^*V\right\rangle = f^*(f^{-1})^*\left\langle \mathrm{d}\varphi\,,\,V\right\rangle = \\ &= V\varphi =: \mathcal{L}_V\varphi\end{aligned} \tag{A.5}$$

5. it commutes with contractions (again satisfied because it is satisfied for the operators f^*, ∇_{f_*V} and f_*), for any contraction C:

$$(f^*\nabla)_V\circ C = C\circ(f^*\nabla)_V \tag{A.6}$$

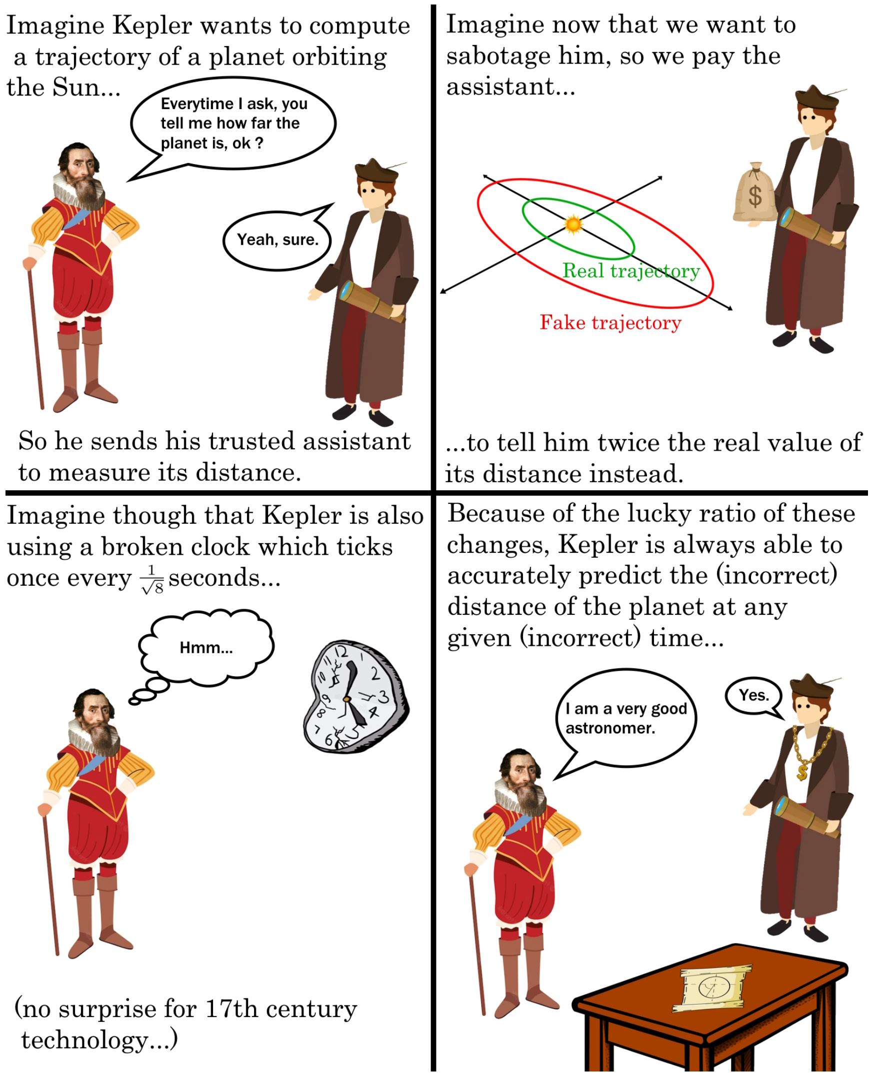
6. it is $\mathcal{F}(M)$-linear with respect to the vector field V:

$$\begin{aligned}(f^*\nabla)_{(V+\varphi W)}A &= f^*\left[\nabla_{f_*(V+\varphi W)} f_*A\right] \\ &= f^*\left[\nabla_{f_*V} f_*A + f_*\varphi\nabla_{f_*V} f_*A\right] \\ &= (f^*\nabla)_V A + \varphi(f^*\nabla)_W A\end{aligned} \tag{A.7}$$

Since all of these are satisfied, we may conclude that $(f^*\nabla)_V$ is a covariant derivative, and therefore the pullback connection $f^*\nabla$ is a well-defined linear connection.

Appendix B

How two wrongs make a right



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