Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Heisenberg group and sub-Riemannian geometry

Bachelor thesis

2026

Timotej Abrhan

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Study Programme:	Physics
Field of Study:	Physics
Department:	Department of Theoretical Physics
Supervisor:	doc. RNDr. Marián Fecko, PhD.

Bratislava, 2026
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Univerzita Komenského v Bratislave
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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Timotej Abrhan
Študijný program: fyzika (Jednoodborové štúdium, bakalársky I. st., denná forma)
Študijný odbor: fyzika
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Sekundárny jazyk: slovenský

Názov: Heisenberg group and sub-Riemannian geometry
Heisenbergova grupa a subriemannovská geometria

Anotácia: Subriemannovská geometria sa líši od (staršej a známejšej) riemannovskej tým, že metrický tenzor (základný objekt v riemannovskej geometrii) nedefinuje skalárny súčin pre všetky vektory, ale len pre vektory z istého podpriestoru dotykového priestoru variety (pre iné skalárny súčin nie je definovaný). Geodetika na subriemannovskej variete je (podobne ako na riemannoskej variete) čiara extremálnej dĺžky, ale tu musí byť jej každý dotykový vektor z toho podpriestoru, v ktorom je definovaný skalárny súčin (aby sa dal vyrátať jeho kvadrát a v konečnom dôsledku aj dĺžka krivky).
Ukazuje sa, že táto teória ma veľmi zaujímavé praktické aplikácie (napríklad v robotike).

Heisenbergova grupa je Lieova grupa, ktorej Lieova algebra je definovaná ,,kanonickými komutačnými vťahmi" známymi z kvantovej mechaniky (z ktorých sa technicky odvodzujú slávne Heisenbergove vzťahy neurčitosti).
Pre jednu súradnicu a k nej kanonicky združenú hybnosť je teda trojrozmerná.

Ukazuje sa, že táto Lieova grupa (chápaná ako varieta) prirodzene poskytuje štruktúru istej subriemannovskej variety. Navyše táto subriemannovská varieta sa dá využiť na riešenie tzv. ,,úlohy Dido" (variačnej úlohy známej už zo staroveku - ohraničiť čo najväčšiu plochu čiarou danej dĺžky): riešenie tejto úlohy sa dá preformulovať na nájdenie subriemannovskej geodetiky.

Cieľ: Zoznámiť sa so základnými myšlienkami subriemannovskej geometrie a ohmatať si jej konkrétnu verziu danú Heisenbergovou grupou. Overiť, že geodetiky riešia úlohu Dido. A samozrejme všetko pekne a zrozumiteľne spísať.

Literatúra: M.Fecko: Differenciálna geometria a Lieove grupy pre fyzikov, Iris 2004, 2008, 2018
R.Montgomery: A Tour of Subriemannian Geometries, Their Geodesics and Applications, AMS 2002

Vedúci: doc. RNDr. Marián Fecko, PhD.
Katedra: FMFI.KTF - Katedra teoretickej fyziky
Vedúci katedry: doc. RNDr. Tomáš Blažek, PhD.

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doc. RNDr. Tomáš Blažek, PhD.
garant študijného programu

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študent vedúci práce

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THESIS ASSIGNMENT

Name and Surname: Timotej Abrhan
Study programme: Physics (Single degree study, bachelor I. deg., full time form)
Field of Study: Physics
Type of Thesis: Bachelor´s thesis
Language of Thesis: English
Secondary language: Slovak

Title: Heisenberg group and sub-Riemannian geometry

Annotation: Sub-Riemannian geometry differs from (the older and better known) Riemannian geometry in that the metric tensor (the fundamental object in Riemannian geometry) does not define the scalar product for all vectors, but only for vectors from a certain subspace of the tangent space of the manifold (for others the scalar product is not defined). A geodesic on a sub-Riemannian manifold is (similarly to a Riemannian manifold) a line of extreme length, but here each tangent vector must be from the subspace in which the scalar product is defined (so that its square and ultimately the length of the curve can be calculated).
It turns out that this theory has very interesting practical applications (for example, in robotics).

The Heisenberg group is a Lie group whose Lie algebra is defined by the "canonical commutation relations" known from quantum mechanics (from which the famous Heisenberg uncertainty relations are technically derived). For one coordinate and the momentum canonically associated with it, it is thus three-dimensional.

It turns out that this Lie group (regarded as a manifold) naturally provides the structure of a certain sub-Riemannian manifold. Moreover, this sub-Riemannian manifold can be used to solve the so-called "Dido problem" (a variational problem known since ancient times - to enclose the largest possible area by a line of given length): the solution of this problem can be reformulated to find a sub-Riemannian geodesic.

Aim: To get acquainted with the basic ideas of sub-Riemannian geometry and to get a feel for its specific version given by the Heisenberg group. To verify that geodesics solve the Dido problem. And of course to write everything down nicely and clearly.

Literature: M.Fecko: Differential geometry and Lie groups for physicists, CUP 2006
R.Montgomery: A Tour of Subriemannian Geometries, Their Geodesics and Applications, AMS 2002

Supervisor: doc. RNDr. Marián Fecko, PhD.

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Assigned:	26.09.2025
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Guarantor of Study Programme

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Student

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Declaration: I declare that I am the sole author of this thesis. To the best of my knowledge, the thesis does not contain material previously published by any other person except where acknowledgment has been made.

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Timotej Abrhan

Abstrakt

Úloha Dido je známa už od staroveku z Vergíliovej epickej básne Aeneas, kde kráľovná Dido musela ohraničiť najväčšiu možnú plochu pre svoje mesto kožou jedného voľa. Táto úloha bola v 17.tom storočí matematikmi tej doby prevedená do reči variačného počtu a pomocou tohto nového matematického odvetvia sa im podarilo rigorózne dokázať, že riešenie tejto úlohy sú časti kružnice, čo sa už v staroveku predpokladalo. Omnoho neskôr sa ukázalo, že riešenie tejto úlohy sa dá dostať aj úplne iným spôsobom a to pomocou diferenciálnej geometrie. Presnejšie vieme túto úlohu previesť na problém nájdenia subriemannovských geodetík na Heisenbergovej grupe. V tejto práci zavedieme teoretické základy pre hľadanie subriemannovských geodetík, ukážeme ako vyjadriť Heisenbergovu grupu ako subriemannovskú varietu, spočítame geodetiky na nej a porovnáme ich s klasickým riešením úlohy Dido.

Kľúčové slová: subriemannovská geometria, distribúcie, geodetiky, Heisenbergova grupa, úloha Dido

Abstract

The Dido problem has been known since ancient times from Vergilius' epic poem Aeneas, in which queen Dido must have enclosed the largest area for her city by the skin of one ox. This problem has been reformulated in the 17th century by the mathematicians of that time in the language of the calculus of variations and with the help of this new mathematic field they managed to rigorously prove that the solution to this problems are parts of a circle, which has been assumed in the ancient times. Much later it has been shown that we can get the same solution in a completely different way and that is through differential geometry. More precisely we can convert this problem to the problem of finding sub-Riemannian geodesics on the Heisenberg group. In this thesis we will lay theoretical basics for finding sub-Riemannian geodesics, we will show how we can look at the Heisenberg group as a sub-Riemannian manifold, compute geodesics on it and finally compare this solution to the classical solution of the Dido problem.

Keywords: sub-Riemannian geometry, distributions, geodesics, Heisenberg group, Dido problem

Preface

This thesis is focused on a purely mathematical problem with only a very small connection to physics, but its main point is to help us understand the mathematical method correctly and then later we could use it in a way which is physically interesting, but more computationally and structurally difficult to understand. The example on which we will show how the method works was specifically chosen to be quite simple and understandable, as well as solvable by more conventional methods, so we can then compare the two solutions.

Sub-Riemannian geometries are a much bigger part of mathematics than any bachelor thesis could contain, as we can see in [5]. This book contains most of what is written in this thesis and much more. It is written in a more complicated language, but also is much more rigorous, as is expected. A very good shortened introduction to sub-Riemannian geometry is given in a diploma thesis [6], which is written in a very good way and was my own first taste of sub-Riemannian geometry and helped me understand the basics.

There are many physical problems which utilize sub-Riemannian geometry, for example, it appears in thermodynamics, where it was used to rigorously prove the existence of enthropy and temperature, which is why sub-Riemannian spaces are often also called Carnot-Carathéodory spaces and today they are used for optimizing paths of heat engines. Another important physical usage for sub-Riemannian geometry is for non-holonomic constraints in classical mechanics, these are constraints on velocity which cannot be intergrated to constraints on position, for example a sphere rolling without slipping or a person riding a bicycle. But these more physical problems would be much less educational as a first view on sub-Riemannian geometry and would not be that easy to understand, which is why the Dido problem was used as the example in this thesis.

Contents

Introduction

In basic differential geometry we have a basic object given by the pair of a manifold and the metric tensor. When the metric tensor can compute the length of curves in any direction it is called Riemannian geometry.

However, some manifolds have metric tensors which can only compute the length of so called horizontal curves, these are curves which belong to something that we call a distribution.

Figure 1: Two paths between Bratislava and Malacky

We can imagine a distribution and sub-Riemannian geodesics through one simple example: Imagine you are trying to get from the city of Bratislava to the city of Malacky, everyone who knows the surroundings of our capital knows, that the shortest path, in our language a Riemannian geodesic, would lead through the Sitina tunnel and we would be there in a short while (except for the traffic jams). But now, let us imagine we have a horrible situation, the National Highway Company has closed up tunnel Sitina and we suddenly have only one direction in which we can go and that is one line, either to the west or to the east. Let us say we try going east and we arrive to the city of Pezinok and, who would've guessed, another road block, our chances of catching our imaginary appointment quickly fade away as now the only path open is north to south. And thus, through this series of roadblocks we finally (after about 2 hours) arrive at the city of Malacky. We all feel that this wasn't the shortest path and, of course, it most likely wasn't, we would have been extremely lucky to find the exact shortest path by ourselves. Now let us imagine a special GPS, which would show us the new shortest path, considering all the roadblocks and no roadblocks can surprise us then, using this magical GPS we would have finally had the real new shortest path, with the roadblocks accounted for. And so, to convert this example to the language used in the rest of the theses: Tunnel Sitina is a Riemannian geodesics, our intuition and knowledge of the surroundings is the metric tensor, the series of roadblocks is a distribution, the new shortest path is a sub-Riemannian geodesic and our magical GPS is a cometric, which can be chosen in such a way that the distribution is encoded within it. So, all in all, a sub-Riemannian object, which was originally encoded by the triplet: manifold, metric tensor and distribution, can now be only a pair of objects: manifold and cometric and that very much simplifies the situation.

Now that we all have an idea of what sub-Riemannian geometry is, let us now segway to the second part of the thesis, where we will apply it on a quite simple, but interesting problem. This problem is known as the Dido problem and it comes from the ancient poem Aeneas from the famed Vergilius. In this epic we come across a character, a queen named Dido. She was fleeing from her homeland and found refuge in the north of Africa, where the local king tried to trick her, so he wouldn't have to deal with her anymore. When she asked him for some land for her stranded people, he agreed, but as he did not want to part with too much of his precious land, he told her that he will give her a plot of land which could be encompassed by the hide of an ox. But Dido was a very intelligent woman who was not about to get tricked by anyone. She agreed to the terms of the king, because she saw a loophole in them. She did not simply stretched the ox hide on the ground and claimed the land beneath. She had the brilliant idea to cut the hide into thin stripes, which she used to enclose a much larger area, specifically an entire hill and thus the legendary city of Carthage was found.

This interesting story sparked an interest between ancient Greek mathematicians, as it is a quite interesting and non trivial question – how should Dido shape the thin stripes, so she could get the largest area possible. And thus, the famed Dido problem was formed. Let us now state it in a more mathematical and less mythical way. We have a line of given length and we want to find the shape in which it would encompass the largest area possible. The ancient Greeks correctly assumed that it would be the circle or, in another formulation where the end points of the line have to share a line between them, parts of a circle. Although they had the correct solution, they simply weren't able to prove it.

The breakthrough came much later, in the 18th century with two giants of mathematics – Leonhard Euler and Joseph-Louis Lagrange, which were sort of the father figures to a new field of mathematics called the calculus of variations and they are a namesake of the Euler-Lagrange equations, which are used to find the extremes of functionals. It is through these equations that the Dido problem was finally and truly solved.

Let us again move a tiny bit forward in time, where it was discovered that we can use differential geometry to solve this ancient problem. This brings us little by little to the other part of this thesis, as it can be solved by finding sub-Riemannian geodesics on a specific Lie group, which has been called the Heisenberg group, as it has very interesting implications in quantum mechanics. Some might say that using this quite difficult mathematical apparatus of sub-Riemannian geometry would be like hunting a deer with a bazooka, quite unnecessary firepower. And they would, of course, be correct. And yet, it is such an interesting finding that in such a different field of mathematics, very much divorced from the Dido problem as we know it and quite some way off from the calculus of variations, the exact same correct result just falls out of the sky, we just have to look at the solution for some time and we can find it.

Chapter 1

Theoretical introduction

The theoretical basics, which are stated in this chapter, are given much more rigor in the literature that I have used. The basics, the first two sections and parts of the third are explained very well in [1]. The other part of the third section is covered in more detail in [2].

1.1 Distribution on a manifold

The main difference between sub-Riemannian and Riemannian geometry is that the metric tensor cannot make a scalar product of any two vectors in a point, only of those that are in a subspace of its tangent space. This subspace is called a distribution on the point x:

$$\mathcal{D}_x \subset T_x\mathcal{M} \tag{1.1}$$

When we want to define a distribution on a coordinate patch $\mathcal{O} \subset \mathcal{M}$ of a manifold, not in just one point, then we can write:

$$\mathcal{D} = \bigcup_{x \in \mathcal{O}} \mathcal{D}_x \tag{1.2}$$

A distribution of dimension k on the whole manifold can be given, for example, in 3 ways:
1. k linearly independent vector fields
2.(n-k) linearly independent covector fields
3.$\binom{2}{0}$ type tensor
For the vector fields e_a we see that they create a basis of the distribution

$$\forall \mathbf{u} \in \mathcal{D} : \mathbf{u} = u^a e_a \tag{1.3}$$

The covector fields θ^i are sometimes called annihilators of the distribution because if we apply them to any vector from the distribution, then the result will be 0.

$$\forall \mathbf{u} \in \mathcal{D} \quad \& \quad \forall \theta^i : \langle \theta^i, \mathbf{u} \rangle = 0 \tag{1.4}$$

$$\mathcal{D} = \bigcap_{i=1}^{n-k} Ker\{\theta^i\} \tag{1.5}$$

The $\binom{2}{0}$ tensor h defines the distribution when we can state that:

$$\mathcal{I}m(h) = \mathcal{D} \tag{1.6}$$

We use the tensor as a linear map from covectors to vectors and we see that if we raise any covectors index using it, we get a vector from the distribution. Let us now look at how the tensor should look like, for this to be true. Let $e_\alpha = (e_a, e_i)$ be the basis of our manifold, e_a is the basis of the distribution and e_i the basis of its complement. Now, we can see that our tensor looks like this in this basis decomposition:

$$h = h^{ab} e_a \otimes e_b + h^{ai} e_a \otimes e_i + h^{ia} e_i \otimes e_a + h^{ij} e_i \otimes e_j \tag{1.7}$$

In matrix form:

$$\begin{pmatrix} h^{ab} & h^{ai} \\ h^{ia} & h^{ij} \end{pmatrix} \tag{1.8}$$

Let us now look at what the tensor does to a covector $p = p_a e^a + p_i e^i$, where e^a, e^i are the dual bases to our adapted basis.

$$h(p) = (p_b h^{ba} + p_i h^{ia}) e_a + (p_i h^{ai} + p_j h^{ji}) e_i \tag{1.9}$$

From our condition that the image of the tensor is the distribution, we can easily see that $h^{ji} = h^{ai} = 0$. Now we know that our distribution can be described by any tensor of rank k that looks like this:

$$h = h^{ab} e_a \otimes e_b + h^{ia} e_i \otimes e_a \tag{1.10}$$

And now we can simplify it a bit more, because we have some freedom in the tensor by putting $h^{ab} = h^{ba}; h^{ia} = 0$. So in matrix form:

$$\begin{pmatrix} h^{ab} & h^{ai} \\ h^{ia} & h^{ij} \end{pmatrix} \to \begin{pmatrix} h^{ab} & 0 \\ 0 & 0 \end{pmatrix} \tag{1.11}$$

And now we have our distribution described by a symmetric $\binom{2}{0}$-type tensor of rank k.

Example :
Let $\theta^3 = dz + xdy - ydx = dz + r^2 d\varphi$ be a given covector field in the sense of the definition given before as the annihilator of a distribution $\forall V \in \mathcal{D} : \langle \theta^3, V \rangle = 0$

i) Let $V = a\partial_x + b\partial_y + c\partial_z$ be a vector field. Find a relation between functions a, b, c such that $V \in \mathcal{D}$

ii) Find basis vectors e_1, e_2 for $\mathcal{D}$.
Solution :
For the first part we need to solve

$$\langle \theta^3, V \rangle = \langle dz + xdy - ydx, a\partial_x + b\partial_y + c\partial_z \rangle = 0 \tag{1.12}$$

From the definition of the canonical pairing and the dual basis, we can easily simplify the expression to:

$$c = ay - bx \implies V = a\partial_x + b\partial_y + ay\partial_z - bx\partial_z = a(\partial_x + y\partial_z) + b(\partial_y - x\partial_z) \tag{1.13}$$

From the last expression we can easily see that basis vectors for the distribution are:

$$e_1 = \partial_x + y\partial_z; e_2 = \partial_y - x\partial_z \tag{1.14}$$

1.1.1 Integrablility of distributions

We need to consider the integrability of a distribution. When it is integrable, then our distribution basically creates a submanifold, which is Riemannian by definition. When it is non integrable, then we need to use the sub-Riemannian formalism. Luckily there is a simple criterion, which can help us determine whether a distribution is integrable or not. It is called the Frobenius' criterion and it exists in two versions, although we will use only one of them (the simpler one).

Theorem (Frobenius' criterion):

$$\mathcal{D} \text{ is integrable} \iff (U, V \in \mathcal{D} \implies [U, V] \in \mathcal{D}) \tag{1.15}$$

Or in components if $\{e_a\}$ is a basis of $\mathcal{D}$ then:

$$[e_a, e_b] = c^c_{ab}(x) e_c \tag{1.16}$$

With the knowledge of this criterion, we can now easily check, that the distribution given in our previous example is non-integrable:

$$[e_1, e_2] = [\partial_x + y\partial_z, \partial_y - x\partial_z] \tag{1.17}$$

$$\partial_x\partial_y + y\partial_z\partial_y - \partial_z - x\partial_x\partial_z - xy\partial_z^2 - \partial_y\partial_x - \partial_z - y\partial_y\partial_z + x\partial_z\partial_x + xy\partial_z^2 = -2\partial_z \tag{1.18}$$

$$\partial_z \notin \mathcal{D} \tag{1.19}$$

So we can see that the commutator of the basis vectors of our distribution is not in $\mathcal{D}$, thus the distribution is non-integrable.

1.2 Fiber bundles

Another theoretical concept we will need to define is the fiber bundle. We are now aware of the concept of a tangent space $T_x\mathcal{M}$, where vectors (velocities) exist and also the cotangent space $T_x^*\mathcal{M}$, where covectors (momenta) exist. We know that for each point on the manifold $\mathcal{M}$ there is its own tangent/cotangent space. We can think of these spaces as fibers and from those fibers, by stitching them together, we can create another space and this space is called a fiber bundle. We can see that it has to be $2n$ dimensional, for any n dimensional manifold, given by two sets of n coordinates (x^i, v^i) where v^i are the coeficients in the coordinate basis decomposition of the vector $\mathbf{v} = v^i\partial_i$ or (x^i, p_i) where p_i are the coeficients in the dual coordinate basis decomposition of the covector $\mathbf{p} = p_i dx^i$, the first n-tuple tells us, where we are on the manifold and the second tells us about the particular vector or covector there.

$$T\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x\mathcal{M} \tag{1.20}$$

We have met fiber bundles in our studies before, specifically in theoretical mechanics as the phase space ($T^*\mathcal{M}$).

Let us define the fiber bundle more specifically: there has to be a typical fiber $\mathcal{F}$ to which fibers over all the points $\mathcal{F}_x$ are dipheomorphic, in our specific case it is the space $\mathbb{R}^n$. Then there has to be the base space, in our case the manifold $\mathcal{M}$ and the total space $\mathcal{B} = \bigcup_{x\in\mathcal{M}} \mathcal{F}_x$ in our case it is for example the tangent bundle $T\mathcal{M}$. Then we need to define a projection from the total space to the basis space

$$\pi : \mathcal{B} \to \mathcal{M} \tag{1.21}$$

To understand the projection, we can look at what it does to a point on the fiber bundle $T\mathcal{M}$. Every point of the tangent bundle is a part of a fiber - a tangent space, given by the point x on the manifold. Our projection takes a vector from $T\mathcal{M}$ and gives us the point in which the tangent space exists.

$$\pi : T\mathcal{M} \to \mathcal{M};\ \pi(v) \mapsto x \text{ where } v \in T_x\mathcal{M} \subset T\mathcal{M} \tag{1.22}$$

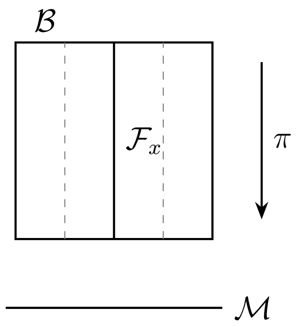

Figure 1.1: Schematic representation of a fiber bundle.

So, having a defined fiber bundle means knowing what $\mathcal{B}, \mathcal{M}$ and π are. An easy way to visualize tangent and cotangent spaces, and also the notion of a typical fiber, in our case $\mathbb{R}^n$, is to think of a north pole of an n-dimensional sphere. For example, when it is a one dimensional sphere S^1, which basically is a circle, then it is the tangent line in that point and we can easily see that it is very similar to $\mathbb{R}$. When we look at what we regularly consider a sphere, denoted S^2, then we can imagine a tangent plane to the north pole of a sphere. We see that it is a flat plane, very similar to the Euclidean plane $\mathbb{R}^2$.

1.3 Geodesics

1.3.1 Riemannian geodesics

In Riemannian geometry, an object is given by the pair $(\mathcal{M}, g)$ so by a manifold and a metric tensor, which induces scalar product. A geodesic is a curve of extremal length (in our case the shortest). We know that the length of a curve γ from point A to point B is given by the length functional:

$$l[\gamma] = \int_{t_A}^{t_B} \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt \tag{1.23}$$

This is the formula for the length of any curve between points A and B, so we can see that the shortest path between them must be

$$d(A, B) = \inf(l[\gamma]) \tag{1.24}$$

There is a different way to look at geodesics, which is also very useful. We need to realize that geodesics are the equivalent of a straight line and we can ask ourselves how do we describe this motion. After a quick thought we can see that it is the uniform linear motion i.e. motion with no acceleration. This motion can be be easily described on a manifold (as we already know from Theoretical mechanics) by the free motion Lagrangian (only the kinetic part)

$$L(q,\dot{q}) = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j = \frac{1}{2} g(\dot{\gamma},\dot{\gamma}) \tag{1.25}$$

It can be shown that the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.26}$$

which minimize the action of the system are equivalent to the geodesic equations, which look like this in general and in coordinates.

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \implies \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0 \tag{1.27}$$

Where $\nabla_{\dot{\gamma}}\dot{\gamma}$ is the covariant derivative of the curve's velocity vector field with respect to itself and $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol (connection coefficient) induced by the partial derivatives of the metric tensor (You can read more about Christoffel symbols and covariant derivatives in chapter 15 of [1]). In this thesis we will only be considering the Euler-Lagrange equations as our geodesic equations.

So the task of finding Riemannian geodesics comes down to solving Euler-Lagrange equations given by the metric tensor on a given manifold.

There is another way in which we can look at this task and that is by looking at the cometric tensor instead of the metric tensor. So now our object is given by the pair $(\mathcal{M}, h)$ where h is the cometric defined by the relation $g_{ik}h^{kj} = \delta^i_j \implies h^{ij} = (g_{ij})^{-1}$.

We know that as the free motion Lagrangian is given by the metric tensor then the free motion Hamiltonian is given by its inverse, the cometric.

$$H(q,p) = \frac{1}{2} h^{ab}(q)p_a p_b \tag{1.28}$$

It is easy to draw the conclusion that when the Lagrange equations were equivalent to the equations of the geodesics, so would the Hamiltonian equations[1]:

$$\dot{p} = -\frac{\partial H}{\partial q};\ \dot{q} = \frac{\partial H}{\partial p} \tag{1.29}$$

That assumption turns out to be correct, so we have two equivalent ways of computing the geodesics, either we can use the metric or the cometric and that leads to computing the Lagrange or Hamilton equations, respectively.

[1]We have to compute both $p(t)$ and q(t), but only $q(t)$ are important for our solution.

1.3.2 Sub-Riemannian geodesics

Sub-Riemannian geodesics are given by the triplet of objects $(\mathcal{M}, g, \mathcal{D})$. In the previous section, in which we defined distributions, we found that a distribution can be expressed in terms of a symmetric $\binom{2}{0}$-type tensor. We still have some freedom in choosing the exact tensor h^{ab} and a good way to choose it would be to make it satisfy the relation:

$$h^{ac}g_{cb} = \delta^a_b \tag{1.30}$$

Let us recall that we got the original tensor h to the form

$$h = \begin{pmatrix} h^{ab} & 0 \\ 0 & 0 \end{pmatrix} \tag{1.31}$$

by choosing a particular frame field which annulled the h^{ia} term in the tensor, which can seem as a problem because now this relation is left a bit ambiguous as it seems to be dependent on the coordinates so let us introduce a more rigorous relation.

Let us define the metric tensor g that satisfies the coordinate (and frame) free relation[2]:

$$g(h(\alpha), h(\beta)) := h(\alpha, \beta) \quad \text{in frame components} \quad h^{ac}g_{cd}h^{bd} = h^{ab} \tag{1.32}$$

Now we can see that this relation is satisfied regardless of the chosen frame field and our tensor h which satisfies these relations is the cometric on our sub-Riemannian manifold.

And now we can define the sub-Riemannian object by the pair $(\mathcal{M}, h)$, which simplifies the situation. As h is the cometric, we can clearly see that the first option, i.e. the computation of the Lagrange equations to find the geodesics, will be inaccessible to us in this problem, but the second option still exists.

For the particulars of later computations, it will be beneficial for us to define something called a canonical lift. It can be defined in a very similar way for covariant tensors, but for sub-Riemannian geometry the lift of purely contravariant tensors will suffice.

[2] $h(\alpha), h(\beta)$ are by definition from $\mathcal{D}$, so our metric tensor is only defined on $\mathcal{D}$, which is exactly what we wanted.

Definition :
For each strictly contravariant tensor B of type $\binom{k}{0}$ we can define the function $\mathring{B}$ on $T^*\mathcal{M}$ such that:

$$\mathring{B}(p) = B_x(p, ..., p) \tag{1.33}$$

In so called canonical coordinates (x^i, p_i) on $T_x^*\mathcal{M}$ it is written as:

$$B^{i...j}(x)\partial_i \otimes ... \otimes \partial_j \implies \mathring{B}(x,p) = B^{i...j}(x)p_i...p_j \tag{1.34}$$

Where p_i are the coefficients from the coordinate basis decomposition of the momentum covector at point x.

$$\mathbf{p} = p_i dx^i\Big|_{x\in\mathcal{M}} \tag{1.35}$$

We can rewrite the Hamiltonian using this function for the sub-Riemannian cometric.

$$H(x,p) = \frac{1}{2}\mathring{h} \tag{1.36}$$

We have already encountered this function in theoretical mechanics, although we did not properly define it back then. Let us now show how this function works on the coordinate basis vector fields and frame vector fields and then we can generalize it for the cometric and write it in different ways, depending on what kind of basis we need.

$$\begin{aligned} \mathring{\partial}_\mu &= p_\mu && \mu\text{-th canonical momentum} \\ \mathring{e}_a &= e_a^\mu(x)p_\mu = P_a(x,p) && \text{a-th canonical momentum} \end{aligned} \tag{1.37}$$

Where e_a is the frame field basis and $e_a^\mu(x)$ is the transition matrix to the frame field basis from the coordinate basis. And then we can write the sub-Riemannian Hamiltonian in these ways depending on the situation:

$$\begin{aligned} H(x,p) = \frac{1}{2}\mathring{h} &= \frac{1}{2}h^{ab}P_aP_b \\ &= \frac{1}{2}h^{ab}(x)(e_a^\mu(x)p_\mu)(e_b^\nu(x)p_\nu) \\ &= \frac{1}{2}h^{\mu\nu}(x)p_\mu p_\nu \end{aligned} \tag{1.38}$$

The geodesic equations are equivalent (see section 1.3.3) to the Hamilton equations:

$$\dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x} \tag{1.39}$$

for this specific Hamiltonian. And the solutions are automatically horizontal and also extremal.

1.3.3 Proof that the method works

Let us have a frame (e_a, e_i) adapted to the distribution $\mathcal{D}$ on a sub-Riemannian manifold $(\mathcal{M}, h)$ and let us define a Riemannian metric G in such a way, that our frame (e_a, e_i) is orthonormal. Then we can write the metric in cometric easily like this:

$$G = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \delta_{ij} \end{pmatrix} \quad G^{-1} = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \delta^{ij} \end{pmatrix} \tag{1.40}$$

We know that for a Riemannian manifold we can use the Hamiltonian formalism for computing geodesics. There we can write the Hamiltonian and it looks like this:

$$H(x,p) = \frac{1}{2}\mathring{G}^{-1} = \frac{1}{2}(P_a P_a + P_i P_i) \tag{1.41}$$

But in this case we can see that the geodesics given by this Hamiltonian have no reason to belong to the distribution, so as of now we cannot proclaim them to be geodesics of the sub-Riemannian manifold. Let us introduce parameter λ which will make the non horizontal curves less viable - artificially make them longer. So our new metric and cometric will look like this:

$$G_\lambda = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \lambda^2\delta_{ij} \end{pmatrix} \quad G_\lambda^{-1} = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \frac{1}{\lambda^2}\delta^{ij} \end{pmatrix} \tag{1.42}$$

Now that with $\lambda^2 >> 1$ every movement which is not horizontal is very long so only curves which are horizontal to the distribution in all points are viable candidates for a minimum, we can also see this on the adjusted Hamiltonian:

$$H_\lambda(x,p) = \frac{1}{2}\left(P_a P_a + \frac{1}{\lambda^2} P_i P_i\right) \tag{1.43}$$

And then in the limit $\lambda^2 \to \infty$ we get exactly what we wanted:

$$H(x,p) = \frac{1}{2}(P_a P_a) \tag{1.44}$$

which is the Hamiltonian corresponding to the given sub-Riemannian manifold $(\mathcal{M}, h)$

Chapter 2

Heisenberg group

This chapter is mainly a continuation and a short recap of exercise 10.2.6 from [1] , which is given in more detail in [3].

Heisenberg group H is a Lie group of 3×3 upper triangular matrices with 1s on the diagonal.

$$A(a,b,c) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

It can be shown by simple computation that these matrices satisfy the composition law:

$$A(a,b,c)A(\tilde{a},\tilde{b},\tilde{c}) = A(a+\tilde{a}, b+\tilde{b}, c+\tilde{c}+a\tilde{b}) \tag{2.2}$$

From the third term we can easily see that these matrices are non commutative, because of the $a\tilde{b}$ term. Let us now compute its Lie algebra $\mathfrak{h}$. From (2.1) we can easily see that the Lie algebra must have the form:

$$X(x,y,z) = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3}$$

So a basis of the algebra is:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

Let us now look at the commutation relations between the basis elements of the algebra, and it will give us a small insight to why it is named the Heisenberg group. The commutation relations are easily computed:

$$[E_1, E_2] = E_3 \quad [E_1, E_3] = 0 \quad [E_2, E_3] = 0 \tag{2.5}$$

And when we look at these commutators, it can be seen that these commutation relations are the same as the Heisenberg commutation relations in quantum mechanics

$$[\hat{x}, \hat{p}] = i\hbar\hat{1} \quad [\hat{x}, i\hbar\hat{1}] = 0 \quad [\hat{p}, i\hbar\hat{1}] = 0 \tag{2.6}$$

If we want to get the commutation relations for more canonical variables, we can just promote x and z to n-dimensional vectors, so the modified matrix in block form looks like this:

$$\tilde{X}(\mathbf{x}, \mathbf{y}, z) = \begin{pmatrix} 0 & \mathbf{x}^T & z \\ 0 & 0_n & \mathbf{y} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

Then the commutation relations look almost exactly like the more variable Heisenberg commutation relations. Because of these properties, the Heisenberg group and its representation can be used to show equivalence between the Heisenberg and the Schrodinger pictures in quantum mechanics, if anyone would like to know more about how specifically it can be done, it is done very precisely in [4].

Let us now look at how we could relate the coordinates on our Lie group to coordinates on our Lie algbera. There is a very obvious way to do so. We can simply write:

$$A(a, b, c) = \mathbb{1} + X(x, y, z) \tag{2.8}$$

and in matrix form,

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

So now the we can see the coordinates satisfy a very simple relation:

$$a(x, y, z) = x \quad b(x, y, z) = y \quad c(x, y, z) = z \tag{2.10}$$

2.1 Exponential coordinates

We can now look at a more complicated but also useful way of relating the coordinates. From the characteristic equation of our general Lie algebra matrix $X(x, y, z)$ we can see that all of the eigenvalues are zero, which satisfies the condition for a nilpotent matrix, so we can see that the exponential of the matrix e^X will be fairly simple. Let us now calculate the powers of the matrix.

$$X = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \quad X^2 = \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.11}$$

So we can now easily see that our exponential e^X will have only 3 terms:

$$e^X = \mathbb{1} + X + \frac{1}{2!}X^2 \tag{2.12}$$

After a quick calculation, we get the following:

$$e^X = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

So, possible new coordinates on the group could be (x, y, z) with the following relation to old (a, b, c):

$$a(x, y, z) = x \quad b(x, y, z) = y \quad c(x, y, z) = z + \frac{xy}{2} \tag{2.14}$$

Let us now check how the composition law looks like with these new exponential coordinates, at first we will get the first two coordinates out of the way, as there is not much happening there and it is quite straight forward and they basically stay the same because a and b just turn into x and y. The main problem is with the c coordinate which changes into $z + \frac{xy}{2}$. It is important to note that the third coordinate of the composition must also be of the form $z + \frac{xy}{2}$ so after we compute the third coordinate as is, we must then subtract the term $\frac{xy}{2}$ with the new x and y which as we already said will be $x + \tilde{x}, y + \tilde{y}$ let us look how it affects the third coordinate in the composition law[1]:

$$\begin{aligned} c + \tilde{c} + a\tilde{b} &= z + \tilde{z} + \frac{xy}{2} + \frac{\tilde{x}\tilde{y}}{2} + x\tilde{y} = \\ = z + \tilde{z} + \frac{1}{2}\left(xy + 2x\tilde{y} + \tilde{x}\tilde{y}\right) &= z + \tilde{z} + \frac{1}{2}(xy + x\tilde{y} + \tilde{x}y + \tilde{x}\tilde{y} + x\tilde{y} - \tilde{x}y) = \\ &= z + \tilde{z} + \frac{1}{2}(x + \tilde{x})(y + \tilde{y}) + \frac{1}{2}(x\tilde{y} - \tilde{x}y) \end{aligned} \tag{2.15}$$

[1] $(x + \tilde{x})(y + \tilde{y}) = xy + x\tilde{y} + \tilde{x}y + \tilde{x}\tilde{y}$ and in the second line we added and subtracted the term $\tilde{x}y$.

So now after subtracting the term $\frac{1}{2}(x+\tilde{x})(y+\tilde{y})$ we can see that the composition relation within the group with exponential coordinates is:

$$(x, y, z) \circ (\tilde{x}, \tilde{y}, \tilde{z}) = (x+\tilde{x}, y+\tilde{y}, z+\tilde{z}+\frac{1}{2}(x\tilde{y}-\tilde{x}y)) \tag{2.16}$$

2.2 Generators of fundamental fields

Another information about the group that we will need to find are the generators of the fundamental fields of right translation on the group. Let us recall what the right translation on a group is:

$$\begin{gathered} R_g h = hg \quad h, g \in G \\ R_{g_1,g_2} h = hg_1 g_2 = (R_{g_1} h) g_2 = R_{g_2} R_{g_1} h \end{gathered} \tag{2.17}$$

Now, we need to look at the way we can compute the generators of the right action (i.e. fundamental fields on basis of the Lie algebra). We know that the representation of the group from the action is given by the simple formula[2] $\rho(g) = R_g^*$ and we also know that the derived representation of Lie algebra is given by the formula $\rho'(X) = \xi_X$. Let us recall that the way to find the way a derived representation from a given representation is basically analogical to the method used for finding the Lie algebra of a given Lie group:

$$\rho'(X) = \frac{d}{dt}\left(\rho\left(e^{tX}\right)\right) \quad \text{at } t = 0 \tag{2.18}$$

In a more simplified and for our purposes beneficial form we can write:

$$\rho\left(\mathbb{1} + \varepsilon X\right) = \mathbb{1} + \varepsilon \rho'(X) \tag{2.19}$$

There is also an easy way to check whether we computed the generators of the fundamental fields correctly, because they preserve the commutators of the Lie algebra, so they have the same structural constants:

$$\begin{gathered} \xi_{[E_i,E_j]} = [\xi_{E_i}, \xi_{E_j}] \\ [E_i, E_j] = c_{ij}^k E_k \\ [\xi_{E_i}, \xi_{E_j}] = c_{jk}^i E_k \end{gathered} \tag{2.20}$$

[2] R_g^* means the pullback of the right action.

So let us now use these methods for finding the generators of the fundamental fields for our right translation. As we only need the basis we will be using $X = E_i, \quad i = 1, 2, 3$:

$$\begin{aligned} R^*_{\mathbb{1}+\varepsilon E_1} g(x,y,z) &= g(x,y,z)\tilde{g}(\varepsilon,0,0) = h(x+\varepsilon, y, z - \frac{1}{2}\varepsilon y) \\ R^*_{\mathbb{1}+\varepsilon E_2} g(x,y,z) &= g(x,y,z)\tilde{g}(0,\varepsilon,0) = h(x, y+\varepsilon, z + \frac{1}{2}\varepsilon x) \\ R^*_{\mathbb{1}+\varepsilon E_3} g(x,y,z) &= g(x,y,z)\tilde{g}(0,0,\varepsilon) = h(x, y, z+\varepsilon) \end{aligned} \tag{2.21}$$

Now, to get these expressions to the correct form, so we could clearly see our fundamental fields, we will use Taylor expansion on our function h for all three of our expressions.

$$\begin{aligned} h(x+\varepsilon, y, z - \frac{1}{2}\varepsilon y) &= h + \varepsilon\partial_x h - \frac{1}{2}\varepsilon y \partial_z h = (\mathbb{1} + \varepsilon(\partial_x - \frac{1}{2}y\partial_z))h(x,y,z) \\ h(x, y+\varepsilon, z + \frac{1}{2}\varepsilon x) &= h + \varepsilon\partial_y h + \frac{1}{2}\varepsilon x \partial_z h = (\mathbb{1} + \varepsilon(\partial_y + \frac{1}{2}x\partial_z))h(x,y,z) \\ h(x,y,z+\varepsilon) &= h + \varepsilon\partial_z h = (\mathbb{1} + \varepsilon\partial_z)h(x,y,z) \end{aligned} \tag{2.22}$$

Now we can see that our derived representations and thus our generators are:

$$\begin{aligned} \xi_{E_1} &= \partial_x - \frac{1}{2}y\partial_z \\ \xi_{E_2} &= \partial_y + \frac{1}{2}x\partial_z \\ \xi_{E_3} &= \partial_z \end{aligned} \tag{2.23}$$

Let us check that our generators preserve the commutation relations of the Lie algebra of the Heisenberg group.

$$\begin{aligned} &[\xi_{E_1}, \xi_{E_2}] = \\ &(\partial_x\partial_y + \frac{1}{2}\partial_z + \frac{1}{2}x\partial_x\partial_z - \frac{1}{2}y\partial_z\partial_y - \frac{1}{4}yx\partial_z\partial_z) - (\partial_y\partial_x - \frac{1}{2}\partial_z - \frac{1}{2}y\partial_y\partial_z + \frac{1}{2}x\partial_z\partial_x - \frac{1}{4}xy\partial_z\partial_z) \\ &= \partial_z = \xi_{E_3} \end{aligned} \tag{2.24}$$

The other commutators are very simple and we can just look and see that they are both equal to zero, so it is obvious that the structure constants are all zero except for $c^3_{12} = -c^3_{21} = 1$, which is the same as for the structure constants of our Lie algebra.

We can use this triple of fundamental fields as a (non-holonomic) frame field (since they are linearly independent in each point). It will be very beneficial for us to also compute its dual co-frame field. We know that the dual basis must satisfy the relation: $\langle e_i, e^j\rangle = \delta_i^j$. We will use a general 1-form $F^i = A_i dx + B_i dy + C_i dz$ and satisfy the conditions for these constants which arise from this form being the dual basis to our new basis [3]:

$$\begin{aligned}
\langle F^1, e_j\rangle = \delta_j^1 \implies & A_1 - \frac{1}{2}yC_1 = 1 \\
& B_1 = 0 \\
& C_1 = 0 \\
\langle F^2, e_j\rangle = \delta_j^2 \implies & A_2 + \frac{1}{2}xC_2 = 0 \\
& B_2 + \frac{1}{2}yC_2 = 1 \\
& C_2 = 0 \\
\langle F^3, e_j\rangle = \delta_j^3 \implies & A_3 - \frac{1}{2}yC_3 = 0 \\
& B_3 + \frac{1}{2}xC_3 = 0 \\
& C_3 = 1
\end{aligned} \tag{2.25}$$

The constants can be easily found from these conditions and now we can see what our dual basis will look like:

$$\begin{aligned}
e^1 &= dx \\
e^2 &= dy \\
e^3 &= dz - \frac{1}{2}\left(xdy - ydx\right)
\end{aligned} \tag{2.26}$$

We have one simple way in which we can check that we found the dual basis to the generators correctly, because we know that each vector of the dual basis must be a left-invariant covector field. Left invariant tensor fields are the fields that do not change when a group element g acts on them by the pullback of the left translation:

$$(L_g)^*\omega = \omega \tag{2.27}$$

Let us at first look how left translation by an arbitrary but fixed element $g(a, b, c)$ acts on the coordinates (x, y, z) and on the covector basis dx, dy, dz:

$$\begin{aligned}
& x \to x + a \quad dx \to dx \\
& y \to y + b \quad dy \to dy \\
& z \to z + c + \frac{1}{2}(ay - bx) \quad dz \to dz + \frac{1}{2}(ady - bdx)
\end{aligned} \tag{2.28}$$

[3]We used the linearity of the canonical pairing and the fact that $\langle \partial_{x_i}, dx^j\rangle = \delta_i^j$.

We can clearly see that the first two covectors of the dual basis are left invariant, the only one which might seem problematic is the third covector, so let us check whether it is truly left invariant:

$$\begin{gathered}L_g^*(e^3) = dz + \frac{1}{2}(ady - bdx) - \frac{1}{2}((x+a)dy - (y+b)dx) = \\ dz + \frac{1}{2}ady - \frac{1}{2}ady - \frac{1}{2}bdx + \frac{1}{2}bdx - \frac{1}{2}(xdy - ydx) = e^3\end{gathered} \tag{2.29}$$

So now we can see that we have found the covectors of the dual basis correctly as they really are left invariant covector fields.

2.3 Finding the sub-Riemannian Hamiltonian

Let us now look at a distribution given by the vector field which is defined by the first two generators of the fundamental fields, which we have renamed to e_1 and e_2. We can clearly see that the same 2-dimensional distribution is given by the 1-form (or covector field) which we have named e^3 because it functions as the annihilator of these two vector fields. Now we have to check whether the distribution we have needs to be treated in a sub-Riemannian way or it can be solved using basic riemannian formalism.

We know from theory about distributions in this thesis, this question comes down to the question whether our distribution is integrable. We already know a simple criterion, which we have stated before and that is the Frobenius' criterion, in this thesis it is equation (1.15,1.16). Fortunately, we have already shown that the distribution is not integrable, because we have already computed the commutator of these two vectors:

$$[e_1, e_2] = \partial_z \notin \mathcal{D} \tag{2.30}$$

Now that we know that the sub-Riemannian formalism is needed, we need to find the sub-Riemannian Hamiltonian, which will give us the hamilton equations that will give us the geodesic we are looking for.

Because we do not want to complicate our lives too much, we can choose the most simple cometric which will satisfy the relation that $\mathcal{I}m(h) = \mathcal{D}$ and this can be easily done in this way that the cometric will look like this:

$$h = e_1 \otimes e_1 + e_2 \otimes e_2 \tag{2.31}$$

So the Hamiltonian will look like this:

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2}\mathring{h} = \frac{1}{2}(p_1^2 + p_2^2) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z(xp_y - yp_x) + \frac{1}{4}p_z^2(x^2 + y^2)\right) \tag{2.32}$$

The fact that the basis vector fields of our distribution are left invariant vector fields carries with it a very nice simplification for the calculations of the geodesics. We can check that for left invariant tensor fields it holds, that they are entirely given by their value at the unity element of the group, because they can be then easily pushed to any given point by left translation. This is a very important observation, when we consider, that the Hamiltonian is given by these vector fields and thus is also left invariant. So if we calculate the geodesics starting at the unity element of the Heisenberg group, we can then easily change the starting point by right action on the resulting geodesics and it will still have the properties of a geodesic, only starting in a different point[4].

$$\gamma : \mathbb{R} \to H \quad R_g : H \to H \quad \gamma \circ R_g : \mathbb{R} \to H \tag{2.33}$$

[4] H in the following equation means the Heisenberg group, not the Hamiltonian.

Chapter 3

Dido problem - two different solutions

In this chapter we will use our new formalism to solve an interesting problem which appears in the calculus of variations. It is called the Dido problem and it is based on an interesting old Greek myth. In the myth, the legendary Phoenician queen Dido, founder of Carthage, is fleeing her homeland in Tyre. After a while she arrives in North Africa, where she bargains with the locals for some land where she and her people could seek refuge. She knew that the locals didn't want to give up too much land, so she asked them to give her only as much land as could be enclosed by the hide of an ox and they reluctantly agreed so they wouldn't have to bother with her anymore. Only later did they realize that they have been tricked. Dido cut up the oxhide into finde stripes, practically creating a very long rope which she used to enclose an entire hill on which she was able to have found the city of Carthage. Dido's problem is a very interesting problem and we will look at both the classic solution and also the new solution using sub-Riemannian geodesics of the Heisenberg group.

3.1 Classical solution of the Dido problem

In this first section we will be using the classical approach to solving the Dido problem, the calculus of variations. The "Lagrangian" of the functional of length is very well known, but let us state it one more time:

$$\mathscr{L} = \sqrt{\dot{x}^2 + \dot{y}^2} \tag{3.1}$$

this is the standard parametrization of a line integral. Let us now look at the second part, which is the area functional. It can be easily computed as a surface integral, but for us it would be beneficial if we could express it as a line integral. We already know of one theorem which allows us to compute areas of a certain 2-dimensional surface using a line integral. We will only need the 2-dimensional version, so let us state the theorem:

$$\textbf{Theorem}(\text{Green}) : \oint_{\partial\mathcal{S}} f dx + g dy = \int_{\mathcal{S}} (\partial_x g - \partial_y f) dx dy \tag{3.2}$$

From it we can see that we can compute the area of $\mathcal{S}$ using a line integral, we only need to satisfy the relation $\partial_x g - \partial_y f = 1$. We also want to be able to compute the area given by an arbitrary curve between any two points and enclosed by going from the origin to the first point and from the second point back to the origin, so we satisfy the conditions of Green's theorem, but we do not want the lines to and from the origin to change the value of our line integral. So we need to find such functions, which satisfy the following condition:

$$\int f dx + g dy = 0 \quad \forall \text{ all curves parametrized as } x(t) = k_1 t; y(t) = k_2 t \quad \forall k_1, k_2 \in \mathbb{R} \tag{3.3}$$

Let us now rewrite this problem and start solving it[1]:

$$\int (k_1 f + k_2 g) dt = 0 \tag{3.4}$$

As this rule must hold for any point in the plane, we can see that the integrand must be zero. We also need to get rid of the k_1, k_2 constants. We can do this quite simply by taking the functions of this form:

$$f(x, y) = -y h(x, y) \quad g(x, y) = x h(x, y) \quad \text{where } h(x, y) \text{ is an arbitrary function} \tag{3.5}$$

Now we can check that the constants truly vanish after applying the parametrization to the functions:

$$k_1 f + k_2 g = -k_1 k_2 t h(k_1 t, k_2 t) + k_2 k_1 t h(k_1 t, k_2 t) = 0 \tag{3.6}$$

So now we can see that functions of this form do satisfy our condition which annuls the line integrals on straight lines to and from the origin.

We still have some freedom in the particular choosing of the arbitrary function $h(x, y)$ and this will be used to satisfy the condition from Green's theorem so we

[1]We have got this expression from differentiating dx using the chain rule: $dx(t) = \frac{dx}{dt} dt = k_1 dt$ and analogically for y.

compute the areas using the line integral. Let us now rewrite this condition for f and g in their new form.

$$\begin{aligned} \partial_x g - \partial_y f = 1 \implies & h + x\partial_x h - (-h - y\partial_y h) = 1 \\ & 2h + x\partial_x h + y\partial_y h = 1 \end{aligned} \tag{3.7}$$

Now we can see what the function must look like, but we also see one very simple solution, which is $h(x,y) = \frac{1}{2}$ and so the integrand looks like this:

$$f dx + g dy = -yh dx + xh dy = \frac{1}{2}(x dy - y dx) \tag{3.8}$$

Now that we have found the functional that calculates the area using the line integral, we can begin solving the Dido problem using the formalism of the calculus of variations. We can look at this from two angles which both give us the same answer. Either we have a line of given length and we try to maximize the area, which is the classical Dido problem from the story, or we have a given finite area and we try to minimize the length of the boundary. We will be using Euler-Lagrange equations, but because we have these specific constraints, we will also have to use the method of Lagrange multipliers and for reasons of simplicity I will only consider the second approach as the calculation is virtually the same and they do give the same results. Let us reintroduce our functionals which will appear in the integral.

$$\begin{aligned} &\mathscr{L}(x,\dot{x},y,\dot{y},t) = \sqrt{\dot{x}^2 + \dot{y}^2} \\ &\mathscr{A}(x,\dot{x},y,\dot{y},t) = \frac{1}{2}(x\dot{y} - y\dot{x}) \\ &\mathscr{F} = \mathscr{L} + \lambda\mathscr{A} \end{aligned} \tag{3.9}$$

Let us take our function for the Dido problem:

$$\mathscr{F} = \sqrt{\dot{x}^2 + \dot{y}^2} + \frac{\lambda}{2}(x\dot{y} - y\dot{x}) \tag{3.10}$$

We can easily see that the partial derivatives are as follows:

$$\begin{aligned} \frac{\partial\mathscr{F}}{\partial\dot{x}} &= \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \frac{\lambda}{2}y \quad & \frac{\partial\mathscr{F}}{\partial x} &= \frac{\lambda}{2}\dot{y} \\ \frac{\partial\mathscr{F}}{\partial\dot{y}} &= \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{\lambda}{2}\dot{x} \quad & \frac{\partial\mathscr{F}}{\partial y} &= -\frac{\lambda}{2}\dot{x} \end{aligned} \tag{3.11}$$

Now we can rewrite the equations like this:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathscr{F}}{\partial\dot{x}}\right)&=\frac{\partial\mathscr{F}}{\partial x}\\ \frac{d}{dt}\left(\frac{\partial\mathscr{F}}{\partial\dot{y}}\right)&=\frac{\partial\mathscr{F}}{\partial y}\end{aligned}\tag{3.12}$$

In our case it would translate to:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathscr{F}}{\partial\dot{x}}\right)&=\frac{\lambda}{2}\dot{y}=\frac{d}{dt}\left(\frac{\lambda}{2}y\right)\\ \frac{d}{dt}\left(\frac{\partial\mathscr{F}}{\partial\dot{y}}\right)&=-\frac{\lambda}{2}\dot{x}=\frac{d}{dt}\left(-\frac{\lambda}{2}x\right)\end{aligned}\tag{3.13}$$

And we know that this holds for functions[2]:

$$\frac{df}{dt}=\frac{dg}{dt}\implies f=g+C\tag{3.14}$$

Therefore, it must be true that:

$$\begin{aligned}\frac{\partial\mathscr{F}}{\partial\dot{x}}&=\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}-\frac{\lambda}{2}y=\frac{\lambda}{2}y\implies\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}=\lambda y+C=\lambda\left(y+\frac{C}{\lambda}\right)=\lambda v\\ \frac{\partial\mathscr{F}}{\partial\dot{y}}&=\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}+\frac{\lambda}{2}x=-\frac{\lambda}{2}x\implies\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}=-\lambda x+\tilde{C}=-\lambda\left(x-\frac{\tilde{C}}{\lambda}\right)=-\lambda u\end{aligned}\tag{3.15}$$

We can see that the transformation of coordinates $u=x-\frac{\tilde{C}}{\lambda}$ and $v=y+\frac{C}{\lambda}$ does not change differentials, specifically $du=dx$ and $dv=dy$. We can rewrite these equations using this known identity $\sqrt{\dot{u}^2+\dot{v}^2}dt=ds$:

$$\frac{\dot{u}}{\sqrt{\dot{u}^2+\dot{v}^2}}=\frac{du}{\sqrt{\dot{u}^2+\dot{v}^2}dt}=\frac{du}{ds}\equiv u'\tag{3.16}$$

The same is true for v. So now our set of equations looks like this:

$$\begin{aligned}u'&=\lambda v\\ v'&=-\lambda u\end{aligned}\tag{3.17}$$

And now we can use the trick $z=u+iv$:

$$z'=u'+iv'=\lambda v-i\lambda u=-i\lambda(u+iv)=-i\lambda z\tag{3.18}$$

[2]These functions f,g are different from the functions in (3.8)

Now we can solve for z:

$$z = Ae^{-i\lambda s} \tag{3.19}$$

Let us now take the square of the magnitude of both sides and then substitute back for u and v:

$$|z|^2 = u^2 + v^2 = \left(x - \frac{\tilde{C}}{\lambda}\right)^2 + \left(y + \frac{C}{\lambda}\right)^2 = |A|^2 \tag{3.20}$$

Let us now simply rename the constants: $\frac{C}{\lambda} = -b$; $\frac{\tilde{C}}{\lambda} = a$; $|A| = R$ and we can finally write:

$$(x-a)^2 + (y-b)^2 = R^2 \tag{3.21}$$

Which is the general formula for a circle with arbitrary radius and center.

3.2 Dido problem through the Heisenberg group

The Dido problem occurs in the Euclidean plane, where we can compute area using a line integral (like we showed in the previous section (3.8)). In chapter 2 we computed the dual basis to the generators of the right action (2.26) and we can see that the second part of e^3 is exactly this integrand of Lagrangian that we found as the desired function in (3.8).

Let us now introduce a trick. We shall amend our Euclidean plane to 3D in such a way that the area dS between the curve and the origin [3] and let us now say that the curve rises by dS in the third dimension. This is exactly what annulling the whole 1-form e^3. Now that we have used e^3 as the annihilator of our new distribution, we know that all curves in it will have the desired quality. When we now add a metric to the distribution (so a cometric to the whole 3D space) in such a way that the length of the 3D sub-Riemannian curve is equal to the length of its 2D projection and this can be done in a way that e_1 and e_2 are orthonormal.[4] So the metric has to be δ_{ab} for $a, b = 1, 2$ and thus the cometric (from (1.32)) has to be δ^{ab}.This is the way, in which we found the cometric in equation (2.31).

Let us now, in the sense of this sub-Riemannian geometry, find the sub-Riemannian geodesics connecting two points A and B. These points have some ”heights” (z components) and the difference in them is by definition the area under the 2D projection of the curve and thus the area will always be the same, as the difference in ”height” is strictly given by the points themselves. Finding the geodesic now means

[3] Connected by imaginary straight lines from the origin to the endpoints of the curve.

[4] Their 2D projections in our case are ∂_x, ∂_y, which are orthonormal.

finding the shortest line with a given area (let us recall that we measure the 3D curve the same as its 2D projection). And this is basically the dual Dido problem, the original is fixed length and maximizing the area and as we discussed in section 3.1, they give the same results.

In the end of the chapter about the Heisenberg group we have derived the specific formula for the sub-Riemannian Hamiltonian given by the distribution $\mathcal{D}$ on the group, which was specifically given by the generators of the fundamental fields with the exponential coordinates. Let us recall what the Hamiltonian looked like exactly in equation (2.32):

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z(xp_y - yp_x) + \frac{1}{4}p_z^2(x^2+y^2)\right) \tag{3.22}$$

We already know that the Hamiltonian equations given by this are equivalent to the sub-Riemannian geodesic equations. Firstly let us write the equations:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = p_x - \frac{y}{2}p_z & \dot{p_x} &= -\frac{\partial H}{\partial x} = -\frac{1}{2}p_z p_y - \frac{1}{4}p_z^2 x \\ \dot{y} &= \frac{\partial H}{\partial P_y} = p_y + \frac{x}{2}p_z & \dot{p_y} &= -\frac{\partial H}{\partial y} = \frac{1}{2}p_x p_z - \frac{1}{4}p_z^2 y \\ \dot{z} &= \frac{\partial H}{\partial p_z} = \frac{1}{2}(xp_y - yp_x) + \frac{1}{4}p_z(x^2+y^2) & \dot{p_z} &= -\frac{\partial H}{\partial z} = 0 \end{aligned} \tag{3.23}$$

One equation can be solved immediately and that is obviously the equation for $\dot{p_z}$ and we can see that $p_z = const. \equiv p$

Now we can begin solving the other equations, at first let us take the second time derivative of coordinates x and y:

$$\begin{aligned} \ddot{x} &= \dot{p_x} - \frac{p}{2}\dot{y} = -\frac{p}{2}p_y - \frac{p^2}{4}x - \frac{p}{2}p_y - \frac{p^2}{4}x = -p\left(p_y + \frac{p}{2}x\right) = -p\dot{y} \\ \ddot{y} &= \dot{p_y} + \frac{p}{2}\dot{x} = \frac{p}{2}p_x - \frac{p^2}{4}p_x + \frac{p}{2}p_x - \frac{p^2}{4}y = p\left(p_x - \frac{p}{2}y\right) = p\dot{x} \end{aligned} \tag{3.24}$$

Having simplified the equations, we can use the substitution $u = \dot{x}$ and $v = \dot{y}$ to lower the rank of the differential equation.

$$\dot{u} = -pv \quad \dot{v} = pu \tag{3.25}$$

Let us use a trick from the lecture of theoretical mechanics and set a new complex variable $z = u + iv$ and express $\dot{z}$ from our equations:

$$\dot{z} = \dot{u} + i\dot{v} = -p(v - iu) = ip(u + iv) = ipz \tag{3.26}$$

From this expression we can clearly see what our z must be:

$$z = Ae^{ipt} \tag{3.27}$$

Now we need to go back to the original variables, so we can write:

$$z = \dot{q} = \dot{x} + i\dot{y} \implies \frac{dq}{dt} = Ae^{ipt} \tag{3.28}$$

And now we can solve for q:

$$q - C = A\frac{2i}{\lambda}e^{ipt} \tag{3.29}$$

Where $C \in \mathbb{C}$; $\quad C = a + ib$ and $A\frac{i}{p}$ is a complex number, so we can write it in this way $A\frac{i}{p} = Re^{i\varphi}; R \in \mathbb{R}; \varphi \in [0, 2\pi)$. So now we have an equation in this form:

$$q - C = Re^{i(pt+\varphi)} \tag{3.30}$$

And if we now take look at the squared norm of the complex numbers on both sides we get to the desired equation:

$$\begin{aligned} |q - C|^2 &= R^2 \\ (x-a)^2 + (y-b)^2 &= R^2 \end{aligned} \tag{3.31}$$

Which is the general formula for a circle with arbitrary center and radius, which is what we wanted to get. As it is the exact same as the classical solution of the Dido problem.

Let us now also write the specific functions $x(t) = \mathrm{Re}(q)$ and $y(t) = \mathrm{Im}(q)$ as they will be very helpful in calculating the rest of the equations:

$$x(t) = R\cos(pt + \varphi) + a \quad y(t) = R\sin(pt + \varphi) + b \tag{3.32}$$

And now let us look at their first time derivatives:

$$\dot{x}(t) = -Rp\sin(pt + \varphi) \quad \dot{y}(t) = Rp\cos(pt + \varphi) \tag{3.33}$$

We can now look at solving the differential equations given for p_x and p_y and we will be calculating them in a similar way, which means taking the second time derivatives of both of them:

$$\begin{aligned} \ddot{p}_x &= -\frac{p}{2}\dot{p}_y - \frac{p^2}{4}\dot{x} = -\frac{p^2}{2}p_x + \frac{p^3}{4}y = -\frac{p^2}{2}\left(p_x - \frac{p}{2}y\right) = -\frac{p^2}{2}\dot{x} \\ \ddot{p}_y &= \frac{p}{2}\dot{p}_x - \frac{p^2}{4}\dot{y} = -\frac{p^2}{2}p_y - \frac{p^3}{4}x = -\frac{p^2}{2}\left(p_y + \frac{p}{2}x\right) = -\frac{p^2}{2}\dot{y} \end{aligned} \tag{3.34}$$

Now we can solve for p_x:[5]

$$\begin{aligned}
\ddot{p}_x &= \frac{p^3}{2} R\sin(pt+\varphi) \\
\dot{p}_x &= \frac{p^2}{2} R\int \sin(pt+\varphi)pdt = \frac{p^2}{2}R\int \sin u du = -\frac{p^2}{2}R\cos(pt+\varphi) + K_x \\
p_x &= -\frac{p}{2}R\int \cos(pt+\varphi)pdt + \int K_x dt = -\frac{p}{2}R\sin(pt+\varphi) + K_x t + K_x'
\end{aligned} \tag{3.35}$$

Now we can solve for p_y in a very similar fashion:

$$\begin{aligned}
\ddot{p}_y &= -\frac{p^3}{2}\cos(pt+\varphi) \\
\dot{p}_y &= -\frac{p^2}{2}R\int \cos(pt+\varphi)pdt = -\frac{p^2}{2}R\sin(pt+\varphi) + K_y \\
p_y &= -\frac{p}{2}R\int \sin(pt+\varphi)pdt + \int K_y dt = \frac{p}{2}R\cos(pt+\varphi) + K_y t + K_y'
\end{aligned} \tag{3.36}$$

And now for the final equation for z we don't have to do a second time derivative, but we will have to play with the formula for a bit to see the solution:

$$\dot{z} = \frac{1}{2}(xp_y - yp_x) + \frac{p^2}{4}(x^2+y^2) = \frac{x}{2}(p_y + \frac{p}{2}x) - \frac{y}{2}(p_x - \frac{p}{2}y) = \frac{1}{2}(x\dot{y} - y\dot{x}) \tag{3.37}$$

Now we can simply put in x, y and their derivatives into the equation and then simplify, so we get an exact formula for $\dot{z}(t)$:

$$\begin{aligned}
\dot{z} &= \frac{1}{2}\Big(R^2 p\cos^2(pt+\varphi) + aRp\cos(pt+\varphi) - bRp\sin(pt+\varphi) + R^2 p\sin^2(pt+\varphi)\Big) \\
\dot{z} &= \frac{R^2 p}{2} + \frac{Rp}{2}(a\cos(pt+\varphi) - b\sin(pt+\varphi))
\end{aligned} \tag{3.38}$$

And now we can simply integrate the previous expression to get the exact formula for $z(t)$

$$z(t) = \frac{R}{2}(a\sin(pt+\varphi) + b\cos(pt+\varphi)) + \frac{R^2 p}{2}t + K_z \tag{3.39}$$

Now we have successfully solved all the hamilton equations given by our subriemannia Hamiltonian and it is important to comment that the exact expression for our geodesic is given by all three coordinate functions, but the analog for the Dido problem is given by its projection to the (x, y) plane and thus we can disregard z and we can rejoice that we have found the exact solution which was given by the classic solution to the Dido problem, which we calculated using the calculus of variations.

[5] Substitution $u = pt + \varphi \quad du = pdt$.

Conclusion

In this thesis we have laid down the theoretical basics for the method of finding sub-Riemannian geodesics.

We have then looked at the Heisenberg group as a manifold, using exponential coordinates from its Lie algebra, which have given us a distribution on it as given by e_1 and e_2 (the two out of the three generators of right translation on the group) by finding the generators of the fundamental fields with these coordinates.

Using Frobenius' criterion it was shown that the distribution is non-integrable, thus the sub-Riemannian formalism must be used.

Then, the dual basis to the basis given by the generators was computed, which gave us the annihilator 1-form, which defines the same distribution and we have proved that this 1-form is left-invariant. This made the computations easier, as we only needed to compute the geodesic from the origin, as it can simply be pushed to another point using left action of the group and the new geodesic will still satisfy the geodesic equations.

We have then computed the sub-Riemannian Hamiltonian, which was used in the later Hamilton equations for the system.

The solution to the Dido problem using the classical way, i.e., using the calculus of variations, was computed and we found that the solutions are in the form of a circle with arbitrary starting point and radius.

Finally, we have computed the Hamilton equations and have obtained the exact same solution for the x and y coordinates of our geodesics, which proves that these two methods are equivalent and that we are able to use sub-Riemannian geodesics to solve the Dido problem.

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