Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

ABC of non-commutative geometry

Bachelor Thesis

2026

Marek Sliva

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Study Programme:	Physics
Field of Study:	Physics
Department:	Department of Theoretical Physics
Supervisor:	doc. RNDr. Marián Fecko, PhD.

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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Marek Sliva
Študijný program: fyzika (Jednoodborové štúdium, bakalársky I. st., denná forma)
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Názov: ABC of non-commutative geometry
ABC nekomutatívnej geometrie

Anotácia: Nekomutatívna diferenciálna geometria je matematická disciplína, ktorá prenikla zhruba od 90-tych rokov aj do matematickej fyziky. Myšlienka bola nahradiť (bežnú komutatívnu) algebru funkcií na varietách (napríklad súradnice) prvkami nejakej nekomutatívnej algebry (takže potom „súradnice nekomutujú"). Táto myšlienka je pre (matematickú) fyziku zaujímavá tým, že kandiduje na vysvetlenie štruktúry časopriestoru na ultrakrátkych vzdialenostiach (na Planckovej škále). V tomto prístupe sa totiž v istom zmysle strácajú „presné" polohy bodov na príslušnej „variete", podobne, ako sa to deje vďaka Heisenbergovym vzťahom neurčitosti vo „fázovom priestore" v kvantovej mechanike. A predstava je, že presne to sa má stať na veľmi malých vzdialenostiach aj s bodmi priestoru a času.

Cieľ: Cieľom práce je vyrátať, ako tu vyzerajú „vektorové polia". Potom skúsiť nejaký príklad na pôsobenie Lieovej grupy a vyrátať príslušné generátory (fundamentálne vektorové polia). A samozrejme všetko pekne a zrozumiteľne spísať.

Literatúra: M.Fecko: Diferenciálna geometria a Lieove grupy pre fyzikov, Iris 2004, 2008, 2018

Vedúci: doc. RNDr. Marián Fecko, PhD.
Katedra: FMFI.KTF - Katedra teoretickej fyziky
Vedúci katedry: doc. RNDr. Tomáš Blažek, PhD.

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THESIS ASSIGNMENT

Name and Surname:	Marek Sliva
Study programme:	Physics (Single degree study, bachelor I. deg., full time form)
Field of Study:	Physics
Type of Thesis:	Bachelor´s thesis
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Title: ABC of non-commutative geometry

Annotation: Non-commutative differential geometry is a mathematical discipline that has penetrated mathematical physics since the 1990s. The idea was to replace the (common commutative) algebra of functions on manifolds (for example, coordinates) with elements of some non-commutative algebra (so that then "coordinates do not commute"). This idea is interesting for (mathematical) physics because it is a candidate for explaining the structure of space-time at very short distances (on the Planck scale). In this approach, the "exact" positions of points on the relevant "manifold" are lost in a sense, similar to what happens thanks to Heisenberg's uncertainty relations in "phase space" in quantum mechanics. And the idea is that this is exactly what should happen at very small distances with points in space and time.

Aim: The goal of the work is to calculate what the "vector fields" look like here. Then try an example of the action of the Lie group and calculate the corresponding generators (fundamental vector fields). And of course, write everything down nicely and clearly.

Literature: M.Fecko: Differential geometry and Lie groups for physicists, CUP 2006

Supervisor:	doc. RNDr. Marián Fecko, PhD.
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Abstrakt

Cieľom tejto práce je jednoduchým spôsobom popísať, ako nekomutatívna geometria pristupuje k situácii, keď sa komutatívna algebra funkcií na variete nahradí nekomutatívnou algebrou. Práca sa zameriava na popísanie analógu vektorových polí a explicitný výpočet fundamentálnych vektorových polí akcie konjugáciou Lieovej grupy $SO(3)$ na komplexných maticiach rozmeru 2×2 a 3×3, čím sa získajú ich invariantné podpriestory a identifikujú bázové matice ako vlastné vektory operátorov momentu hybnosti.

Kľúčové slová: nekomutatívna geometria, maticové algebry, inveriantné podpriestory

Abstract

The aim of this thesis is to describe in a straightforward manner how non-commutative geometry approaches the situation where the commutative algebra of functions on a manifold is replaced by a non-commutative algebra. The thesis focuses on describing the analogue of vector fields and explicitly calculating the fundamental vector fields of the Lie group $SO(3)$ acting by conjugation on complex matrices of dimensions 2×2 and 3×3, thereby obtaining their invariant subspaces and identifying the basis matrices as eigenvectors of the angular momentum operators.

Keywords: non-commutative geometry, matrix algebras, invariant subspaces

Preface

This thesis aims to describe as simply as possible some fundamental concepts in non-commutative geometry. We hope it will be accessible to students who have completed courses in Quantum Theory (1), (2) and Mathematical Physics, that is, to third-year physics students. Since introductions to non-commutative geometry are usually mathematically very formal, this thesis intends to serve as a helpful starting point for those approaching it for the first time.

Contents

Introduction

It is known that one can associate to a (real) smooth manifold M the commutative associative algebra of smooth functions on that manifold $\mathcal{F}(M)$. Because each manifold has points where functions on it are evaluated, we can define point-wise operations on $\mathcal{F}(M)$. So for all $x \in M$, $\lambda \in \mathbb{R}$, and $f, g \in \mathcal{F}(M)$:

$$(f+g)(x) = f(x) + g(x), \tag{1}$$

$$(fg)(x) = f(x)g(x), \tag{2}$$

$$(\lambda f)(x) = \lambda f(x). \tag{3}$$

The algebra is closed with regard to these operations, because all of them are smooth. Addition and multiplication of (real) numbers as results of the functions at points are commutative, associative and there exists an identity function (everything maps to 1); thus, it is really an algebra constructed out of the manifold.

A non-trivial fact is that it can go backwards, so a manifold is fully determined by the commutative algebra of smooth functions on it (Gelfand–Naimark theorem). If we have a non-commutative algebra (most naturally matrices), such a manifold does not exist. Despite this, it is still interesting to study these algebras because this approach has many applications, e.g. it is one of the methods of a regularization of QFT and a useful tool for numerical modeling. For more details see for example [6], [9]. We will not do any complex theory, just observe some elementary properties.

The idea of non-commuting coordinates is present in quantum theory. For example, the position operator and momentum operator do not commute due to the Heisenberg uncertainty principle. The phase space (positions, momenta) can be considered to be fuzzy.

There is a hypothesis that at the Planck scale, it is impossible to distinguish points below a minimal length scale, so the spacetime behaves as a "fuzzy manifold", meaning there does not exist a precise location of points. In the classical limit (Planck's constant goes to 0), the fuzziness disappears. If the spacetime behaves as a fuzzy manifold, it has some consequences that may serve to experimentally find some limits of that hypothesis (for example, see [1], [8]).

In numerical models, it is impossible to work with infinite-dimensional objects because computers have limited memory. As an infinite-dimensional object, imagine, for instance, a Euclidean space which has assigned a number in each point. A common approach is to use a lattice, that is, to pick only finitely many points of the space. This approach was used in QFT where some singularities can be avoided by replacing a continuous space by a lattice. Even though it brings wanted results, it breaks the isometries of the space which becomes discrete. The approach of the non-commutative geometry (fuzzy sphere) preserves a full rotational isometry. Despite the origin of the approach in QFT, it may be used to study classical phenomena, see [10].

Euclidean space has a maximal possible number of isometries (depending on the dimension, one can, for example, translate objects in 1D). An isometry is a transformation of the manifold that does not deform objects on it, i.e. all lengths and angles of it are the same as before the transformation. To define a usual geometry (or to measure lengths and angles) it is necessary to have a metric tensor, so we have to work with Riemannian manifolds. We will briefly remind how lengths and angles are defined.

Let (M, g) be a Riemannian manifold and $f : M \to M$ a diffeomorphism (so that we do not need to discuss the existence and properties of the corresponding pushforward). The length of a curve γ on M can be defined as

$$l[\gamma, g] = \int_{t_1}^{t_2} \mathrm{d}t \, \sqrt{g(\dot{\gamma}, \dot{\gamma})}. \tag{4}$$

Now when f transforms γ, it means that its length maps to $l[f \circ \gamma, g] = l[\gamma, f^* g]$. [3]

▼

To prove it we need to know how pullbacks and pushforwards work.

The pullback of functions on a manifold M:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f^*h"'] & M \arrow[d, "h"] \\ & \mathbb{R} \end{tikzcd}$$

so $f^* h = h \circ f$.

The pushforward of vectors:

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"] & TM \arrow[d, "\pi_M"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

This simply says that $T_x M \ni \dot{\gamma} \mapsto f_* \dot{\gamma} \in T_{f(x)} M$ and $\gamma \mapsto f \circ \gamma$, where π_M is a natural projection from the set of pairs (vector, point on M in which the vector sits) to the point on M (in other words, the vector is forgotten), $T_x M$ is a tangent space on M in the point x and $(Tf)(v) := f_* v$, where v on the left-hand side is considered as a point of TM and on the right-hand side as a vector in a point $x \in M$. Another definition is that $f_* \dot{\gamma} = (f \dot{\circ} \gamma)$ because the vector $\dot{\gamma}$ transformed by the pushforward has to be the same as when f transforms γ and then the tangent vector is computed.

Now a short computation

$$\begin{aligned} l[f \circ \gamma, g] &= \int_{t_1}^{t_2} \mathrm{d}t \sqrt{g((f \dot{\circ} \gamma), (f \dot{\circ} \gamma))} = \int_{t_1}^{t_2} \mathrm{d}t \sqrt{g(f_* \dot{\gamma}, f_* \dot{\gamma})} = \\ &= \int_{t_1}^{t_2} \mathrm{d}t \sqrt{(f^* g)(\dot{\gamma}, \dot{\gamma})} = l[\gamma, f^* g]. \end{aligned} \tag{5}$$

The angle between vectors v and w is defined as

$$\cos \alpha = \frac{g(v, w)}{\sqrt{g(v, v)} \sqrt{g(w, w)}} \tag{6}$$

and it is obvious that when we transform vectors by pushforwards, at the end we will obtain a pullback of the metric tensor as before. [3]

This can be interpreted straightforwardly. First of all, let us imagine metric tensor as a dwarf sitting in each point of the Riemannian manifold. When we ask about the length of a curve, we first have to identify dwarfs that lie under the curve. These dwarfs can do only three things: measure infinitesimal lengths of curves above them, measure angles between vectors that have their common starting point above them and report to us the results of their measurements. Then we find the sum of the lengths from the dwarfs and define it to be a length of the curve.

We have a curve and are interested in how its length changes after a transformation the result of which is that the curve may be moved elsewhere. The computation shows that it is the same question as what is the length of the curve that stays in its position but when dwarfs under it move (or their measuring devices are deformed).

▲

We now see that transformations that do not change the length have to obey: $f^* g = g$ (measuring devices of dwarfs are invariant with regard to f or they do not move). Because metric tensors are symmetric, this is a system of $n(n+1)/2$ conditions where n is the number of coordinates. The transformations with this property are called isometries of the Riemannian manifold.

One may ask here: is it appropriate to write a page about isometries in a thesis that is about something different? No, it is not. But if we want to reflect the importance of isometries we will not have any space to do the things that are in the thesis assignment so we have to stop somewhere.

After this subtle indication of the importance of isometries one may expect that we want to have as many of them as possible. Useful tools used for finding isometries obtained as a continuous deformation of an identity are Killing equations. The most important fact is that the algebra of Killing vectors (solutions of Killing equations) is always finite-dimensional contrary to the dimension of the algebra of all vector fields. Moreover, it can be shown that for a n-dimensional manifold the highest possible number of isometries is $n(n+1)/2$ which is the actual number, e.g., in (pseudo)Euclidean manifolds and on spheres. [3]

For this reason, we will use a sphere S^2 as our space. It is natural to study actions of the group $SO(3)$ of rotations of 3D Euclidean space because a sphere is rotationally invariant so this is a group of its isometries.

One may argue that a lattice may have some isometries, too. This is true, but they are not continuous, so only the discrete transformations that map points of the lattice to other points are accepted. The approach of non-commutative geometry allows some continuous symmetries and there is still an advantage that we work with something that is finite-dimensional.

As was suggested, we replace the commutative algebra of functions with something non-commutative. Because matrices in general do not commute, functions may be replaced by square matrices of a fixed size.

One of our aims is to find an analogue of vector fields. In differential geometry, a vector field V on a manifold M means that there is a vector at each point x of the manifold. Alternatively, it is an operator that maps a function on a manifold $\mathcal{F}(M)$ to another function on M in a way that the vector field applied to the function gives a function defined on each point of the manifold by applying a vector to the original function.

$$V : \mathcal{F}(M) \to \mathcal{F}(M);\ f \mapsto Vf, \quad (Vf)(x) := V_x(f) \tag{7}$$

Smooth vector fields $\mathfrak{X}(M)$ are those that assign a smooth function to every smooth function.

Vectors on a manifold can be treated as differential operators that are evaluated at points. Equivalently, they are equivalence classes of curves with respect to the relation of being tangent at a point or linear functionals on $\mathcal{F}(M)$ that act on products according to the Leibniz rule. This means that there is a correspondence between the vector field and a derivation of the algebra of smooth functions on the manifold because it is a set of linear mappings that obey the Leibniz rule. Thanks to the Gelfand-Naimark theorem it is sufficient to study the algebra of smooth functions on a manifold instead of the manifold itself. This leads us to the study of derivations of the non-commutative algebra, because a fuzzy manifold is not a manifold in the standard sense and is hard to study.

We will use the notation $|j,m\rangle$ for a basis, but in the literature (for instance [4]) it is common to use $Y_{lm}^{(n)}$ for a matrix algebra, where l stands for j.

It is known from quantum mechanics that the action of $SO(3)$ on the algebra of functions on a sphere leads to a decomposition of the space of functions into irreducible components, for which spherical harmonics Y_{lm} form a basis. In analogy to Fourier series, spherical harmonics are an orthonormal basis in the space of functions on a sphere. Functions $Y_{lm}(\theta, \phi)$ are eigenfunctions of L_3 and L^2 that obey equations

$$L_3 Y_{lm}(\theta, \phi) = m Y_{lm}(\theta, \phi) \tag{8}$$

and

$$L^2 Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi), \tag{9}$$

where $l \in \mathbb{N}$ is a non-negative integer and $m \in [-l, l] \cap \mathbb{Z}$. We will use a convention $\hbar = 1$.

These functions can be expressed in a way

$$Y_{lm}(\theta, \phi) = C_{lm} P_l^{|m|}(\cos\theta) e^{im\phi}, \tag{10}$$

where

$$P_l^m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x), \quad l \in \mathbb{N},\ m \in \mathbb{N},\ m \leq l,\ |x| \leq 1 \tag{11}$$

are associated Legendre functions to Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \left(x^2 - 1\right)^l, \quad l \in \mathbb{N}. \tag{12}$$

C_{lm} are normalization constants so that

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \, \sin\theta \; Y_{lm}^* Y_{lm} = 1, \tag{13}$$

which is a convention that they are normalized on the unit sphere.

Any square-integrable function on the sphere $f(\theta, \phi) \in L^2(S^2)$ can be decomposed as

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi), \tag{14}$$

where

$$a_{lm} = \int_{S^2} \mathrm{d}\Omega \, Y_{lm}^* f(\theta, \phi). \tag{15}$$

More details can be found in Chapter 4.9 of [7]. Our aim is to find a similar decomposition of "functions" on a fuzzy sphere.

In this thesis, we will start by introducing some properties of the algebra $M_n(\mathbb{C})$, continue by describing its derivations and automorphisms, introduce a scalar product, and find eigenvectors of the operators J_3 and J^2.

Chapter 1

Algebra $M_n(\mathbb{C})$

Technical definitions of an algebra can be found in Appendix A: Algebra. We define $M_n(\mathbb{C})$ as an algebra of all $n \times n$ complex matrices, so a set of matrices equipped with standard matrix multiplication and addition, together with the possibility to multiply any matrix from it by a complex number. Because of the properties of matrix multiplication, this algebra is non-commutative. A useful point of view is that the matrices form a vector space where we define a multiplication. A natural question may arise: which basis is practical to use?

1.1 Weyl basis

The easiest way is to take as basis matrices those that have a 1 in a single position and zeroes elsewhere. The Weyl basis is exactly this. Its elements are labeled by two indices which define the column and row. When using matrix elements, elements of the Weyl basis can be defined:

$$(E^i_j)^k_l = \delta^i_l \delta^k_j. \tag{1.1}$$

Any matrix $B \in M_n(\mathbb{C})$ can be written as

$$B = B^i_j E^j_i. \tag{1.2}$$

Now make a short computation.

$$E^k_l E^b_a = (E^k_l E^b_a)^d_c E^c_d = E^c_d (E^k_l)^d_f (E^b_a)^f_c = E^c_d \delta^k_f \delta^d_l \delta^b_c \delta^f_a = E^b_l \delta^k_a \tag{1.3}$$

Here we have to emphasize that indices of E just choose a single basis matrix, so this notation may be a bit misleading, because δ^k_a at the end equals 1 for $a = k$ regardless of the dimension of the space, 0 otherwise.

One short computation is not enough; we will show another using the previous one, because commutators will appear frequently (all indices are fixed and do not imply a summation).

$$[E^b_a, E^j_i] = E^b_a E^j_i - E^j_i E^b_a = E^j_a \delta^b_i - E^b_i \delta^j_a \tag{1.4}$$

The basis can be illustrated with 2×2 matrices:

$$E_1^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.5}$$

One may notice that the indices of row and column are interchanged, but it is because we want B_j^i in (1.2) to be in the standard way (upper index for rows).

Chapter 2

Derivations of $M_n(\mathbb{C})$

A vector field on a manifold prescribes a vector at each point of the manifold. Vectors can be interpreted as differential operators evaluated at points or as linear functionals on the algebra of smooth functions on the manifold that act on multiplication according to the Leibniz rule. This means vector fields can be regarded as derivations of that algebra.

By derivation $\mathcal{D}$, we mean any linear operator that acts on a product according to the Leibniz rule:

$$\mathcal{D}(AB) = \mathcal{D}(A)B + A\mathcal{D}(B). \tag{2.1}$$

Linearity of $\mathcal{D}$ means

$$\mathcal{D}(A + \lambda B) = \mathcal{D}(A) + \lambda\mathcal{D}(B) \tag{2.2}$$

and consequently

$$\mathcal{D}(B) = B^i_j\mathcal{D}(E^j_i) = B^i_j\mathcal{D}^{jk}_{il}E^l_k, \tag{2.3}$$

where the first equality enables us to work only with the Weyl basis (1.1) and the second reflects the fact that it is a linear operator[1].

The Leibniz rule:

$$\begin{aligned}\mathcal{D}(E^j_iE^l_k) &\overset{1}{=} \mathcal{D}(E^l_i\delta^j_k) = \delta^j_k\mathcal{D}^{lx}_{iy}E^y_x \\ &\overset{2}{=} \mathcal{D}(E^j_i)E^l_k + E^j_i\mathcal{D}(E^l_k) = \mathcal{D}^{jp}_{iq}E^l_p\delta^q_k + E^y_i\delta^j_x\mathcal{D}^{lx}_{ky} = \mathcal{D}^{jp}_{ik}E^l_p + \mathcal{D}^{lj}_{ky}E^y_i.\end{aligned} \tag{2.4}$$

We do not need to work with whole matrices; it is sufficient to write equations for their general $()^a_b$ element which results in

$$\delta^j_k\mathcal{D}^{la}_{ib} \overset{!}{=} \mathcal{D}^{ja}_{ik}\delta^l_b + \mathcal{D}^{lj}_{kb}\delta^a_i. \tag{2.5}$$

[1]It has four indices because each basis matrix is labeled by 2 indices and we want it to be an operator from the space of matrices to itself.

If both sides of an equation are multiplied by the same number, its solutions remain the same.[2]

The resulting equation is not easily solvable, so we will discuss some partial conditions. It seems like a good idea to decrease the number of indices. As the first step we will multiply the equation (2.5) by δ^b_j which leads to

$$\mathcal{D}^{lj}_{kj} \overset{!}{=} 0 \tag{2.6}$$

that holds for any i and a so δ^a_i was omitted. It means that whenever the upper and lower second indices coincide, the result is 0.

Now we multiply the equation (2.5) by δ^b_a and using (2.6) obtain

$$\mathcal{D}^{jl}_{ik} \overset{!}{=} -\mathcal{D}^{lj}_{ki}. \tag{2.7}$$

Multiplication of (2.5) by deltas seems to be helping us, so let us multiply it by δ^k_j.

$$\delta^j_j \mathcal{D}^{la}_{ib} \overset{!}{=} \mathcal{D}^{ja}_{ij}\delta^l_b + \mathcal{D}^{lj}_{jb}\delta^a_i = \mathcal{D}^{ja}_{ij}\delta^l_b - \mathcal{D}^{jl}_{bj}\delta^a_i \tag{2.8}$$

Now consider that $\delta^j_j = n$, where n is the dimension of the linear space on which the matrices act (i.e. it is the same n as in $M_n(\mathbb{C})$). As can be seen, each element of $\mathcal{D}$ on the right side has two indices the same, which means that it can be treated as an object with only two indices (something like a trace or a contraction of a tensor). So let

$$D^p_o := \mathcal{D}^{jp}_{oj}. \tag{2.9}$$

$$n\mathcal{D}^{la}_{ib} \overset{!}{=} D^a_i \delta^l_b - D^l_b \delta^a_i \tag{2.10}$$

$$\mathcal{D}^{la}_{ib} \overset{!}{=} \frac{1}{n}(D^a_i \delta^l_b - D^l_b \delta^a_i) = A^a_i \delta^l_b - A^l_b \delta^a_i \tag{2.11}$$

In the last step matrix A was constructed out of the matrix D, which is arbitrary, taking into account that n is just a non-zero constant. We now show that $\mathcal{D}$ is given by a commutator.

▼

After putting this result into (2.3), one can obtain

$$\begin{aligned}(\mathcal{D}(B))^d_c = B^i_j \left(A^k_i E^j_k - A^j_l E^l_i\right)^d_c = B^i_j \left(A^k_i (E^j_k)^d_c - A^j_l (E^l_i)^d_c\right) = \\ = B^i_j \left(A^k_i \delta^j_c \delta^d_k - A^j_l \delta^l_c \delta^d_i\right) = B^i_c A^d_i - B^d_j A^j_c = [A, B]^d_c.\end{aligned} \tag{2.12}$$

[2]Except for the multiplication by 0 which leads to an identity $0 = 0$ that will not be discussed here, because it is out of the scope of this work and we offer opportunity to solve it to someone else with a better funding necessary to compute it; but it seems likely that a solution to the initial problem, if found, is a solution to this problem, too, even though the equations are not equivalent.

▲

This means that all derivations have the form

$$\mathcal{D}(\cdot) = [A, \cdot], \tag{2.13}$$

where A is a matrix characterizing the derivation. Note that $[A+\lambda\mathbb{1}, \cdot] = [A, \cdot]$ because the unit matrix commutes with all matrices.

We would like to point out that the commutator of two derivations characterized by different matrices is a derivation again, namely

$$[\mathcal{D}_{A_1}, \mathcal{D}_{A_2}]B = \mathcal{D}_{[A_1,A_2]}B. \tag{2.14}$$

▼

$$\begin{aligned}
[\mathcal{D}_{A_1}, \mathcal{D}_{A_2}]B &= \mathcal{D}_{A_1}\mathcal{D}_{A_2}B - \mathcal{D}_{A_2}\mathcal{D}_{A_1}B = \mathcal{D}_{A_1}[A_2, B] - \mathcal{D}_{A_2}[A_1, B] = \\
&= \mathcal{D}_{A_1}(A_2B) - \mathcal{D}_{A_1}(BA_2) - \mathcal{D}_{A_2}(A_1B) + \mathcal{D}_{A_2}(BA_1) = \\
&= [A_1, A_2B] - [A_1, BA_2] - [A_2, A_1B] + [A_2, BA_1] = \\
&= A_1A_2B - A_2BA_1 - A_1BA_2 + BA_2A_1 - \\
&\quad - A_2A_1B + A_1BA_2 + A_2BA_1 - BA_1A_2 = \\
&= A_1A_2B - A_2A_1B - BA_1A_2 + BA_2A_1 = \\
&= [A_1A_2 - A_2A_1, B] = [[A_1, A_2], B] = \mathcal{D}_{[A_1,A_2]}B
\end{aligned} \tag{2.15}$$

▲

Chapter 3

Automorphisms of $M_n(\mathbb{C})$

An automorphism f of the algebra $M_n(\mathbb{C})$ is in general an endomorphism that preserves all the structure. Here it is a linear bijective map which respects the product, so for any $\lambda \in \mathbb{C}$, $A, B \in M_n(\mathbb{C})$:

$$f(AB) = f(A)f(B), \tag{3.1}$$

$$f(A + \lambda B) = f(A) + \lambda f(B). \tag{3.2}$$

Why should we study automorphisms of an algebra? In differential geometry, we can take a smooth manifold M and a compatible Lie group G.[1] Then for $g \in G$ there can be defined a right action $R_g : M \to M$, which is a diffeomorphism (a smooth bijective mapping with smooth inverse) with properties:

$$R_{gh} = R_h \circ R_g, \qquad R_e = \mathrm{id}_M =: \mathrm{id} \qquad \forall g, h \in G \tag{3.3}$$

[3].

Because each group is isomorphic to some group of transformations (Cayley's theorem), we can say that an action of a group describes how a group moves points of M. Left actions can be defined analogously, but the order in the composition will be the opposite. [11]

First of all, $\mathrm{id}_M(x) = R_e(x) = x$ where e is a unit element of the group.
$(R_g \circ R_h)(x) = R_g(R_h(x)) = R_h g(x)$, because it is a right action. For a fixed g: $R_g \circ R_{g^{-1}} = R_{g^{-1}} \circ R_g = R_e$.

Now consider $\mathcal{F}(M) \ni f : M \to \mathbb{R}$. It is a function that takes a point from the manifold and returns a number. Nothing can prevent us from applying this function to a point already moved by a diffeomorphism, so we may define $(R_g^* f)(x) = f(R_g(x))$. This is well defined because the function f works on the whole manifold and the action just moves points to (possibly) other places. Moreover, the new function called the pull-back of f by R_g is smooth. Since the group action $M \times G \to M$ is smooth and

[1] We choose a manifold and a group of symmetries of it, which does not have to exist in general.

for any fixed g the map $R_g: M \to M$ is smooth, the composition of smooth functions is smooth, too.

We have defined a new mapping $R_g^*: \mathcal{F}(M) \to \mathcal{F}(M)$. It is linear because

$$R_g^*(f+\lambda h)(x) = (f+\lambda h)(R_g(x)) = f(R_g(x))+\lambda h(R_g(x)) = R_g^* f(x)+\lambda R_g^* h(x). \quad (3.4)$$

Moreover, the product is preserved:

$$R_g^*(fh)(x) = (fh)(R_g(x)) = f(R_g(x))h(R_g(x)) = (R_g^* f R_g^* h)(x). \quad (3.5)$$

This shows that $R_g^* \in \operatorname{Aut} \mathcal{F}(M)$ is an automorphism of $\mathcal{F}(M)$. The map $g \mapsto R_g^*$ is a representation of G on $\mathcal{F}(M)$.

$$(R_g^* \circ R_h^*)f(x) = R_g^*(R_h^* f)(x) = f(R_h(R_g(x))) = f(R_{gh}(x)) = R_{gh}^* f(x), \quad (3.6)$$

so it is a left action.[2] More details can be found in Chapter 13 of [3].

We see that actions of groups on a manifold induce, via pull-back, actions on the algebra of smooth functions on the manifold.

In our case, we replace the algebra of smooth functions by matrices where e.g. the group $SO(3)$ does not act by rotations as in the Euclidean space, but by conjugations by unitary matrices.

3.1 Group of all automorphisms of $M_n(\mathbb{C})$

All automorphisms form a group $\operatorname{Aut} M_n(\mathbb{C})$ and it would be nice to have another interesting group. It can be seen from the fact that automorphisms are maps that preserve all structures, so their composition has to preserve them, too, and it means that the composition is in the group. An inverse element exists because of the bijectivity and the neutral element is the identity map.

We state without proof that any automorphism of the algebra $M_n(\mathbb{C})$ must be a conjugation by a fixed matrix.[3] A hint on how to prove it can be found in an exercise in [2], in the chapter about the Skolem-Noether theorem (which treats a more general case).

$$\operatorname{Aut} M_n(\mathbb{C}) = \left\{ I_G;\ G \in GL(n,\mathbb{C}),\ I_G(M) := GMG^{-1} \quad \forall M \in M_n(\mathbb{C}) \right\} \quad (3.7)$$

[2] A very important observation that protects the reader from reading about contravariant functors.

[3] Note that matrices which differ just in a scalar multiple produce the same result of the conjugation.

▼

We can check that conjugations are automorphisms. Multiplication of a square matrix by two square matrices results in a square matrix, so the result is from the same set.

$$I_G(AB) = GABG^{-1} = GAG^{-1}GBG^{-1} = I_G(A)I_G(B) \tag{3.8}$$

$$I_G(A+\lambda B) = G(A+\lambda B)G^{-1} = GAG^{-1} + G\lambda BG^{-1} = I_G(A) + \lambda I_G(B) \tag{3.9}$$

$$\mathrm{id}(A) = I_{G^{-1}}\left(I_G(A)\right) = G^{-1}GAG^{-1}G = A = GG^{-1}AGG^{-1} = I_G\left(I_{G^{-1}}(A)\right) \tag{3.10}$$

This shows that conjugation preserves multiplication and addition and enables us to simply define an inverse element while the identity map is formally a conjugation by a unit matrix. Scalars are fixed because they commute with every matrix: $I_G(\lambda) = \lambda$.

▲

Now we can construct the algebra of derivations $\mathrm{Der}(M_n(\mathbb{C}))$ just by considering a group action generated by elements near the identity, i.e. we will construct a one-parameter subgroup. Let

$$I(t) \in \mathrm{Aut}\, M_n(\mathbb{C}) \quad \forall t \in \mathbb{R} \tag{3.11}$$

and

$$I(0) = \mathrm{id}\,. \tag{3.12}$$

Let $G(t) = e^{tC},\ C \in M_n(\mathbb{C})$ be a one-parameter subgroup, and $I(t)(M) = G(t)MG^{-1}(t)$. Then for $0 < \varepsilon \ll 1$:

$$I(\varepsilon)(M) = (\mathbb{1} + \varepsilon C)\, M\, (\mathbb{1} - \varepsilon C) + \mathcal{O}\big(\varepsilon^2\big) = M + \varepsilon[C, M] + \mathcal{O}\big(\varepsilon^2\big). \tag{3.13}$$

As expected from (2.13), we have obtained the commutator with a fixed matrix which is exactly the derivation.

We have implicitly used the fact that a representation of G on a vector space V induces, by conjugation, a representation on the space of linear operators $\mathrm{End}\, V$.

Let $V = \mathbb{C}^n$, $W \ni A :\ V \to V$ is a linear operator (in our case $W = \mathrm{End}\, V = M_n(\mathbb{C})$) and $\rho(g)$ is a representation of G on V, $g \in G$. We define

$$\tilde{\rho}(g)A := \rho(g)A\rho^{-1}(g). \tag{3.14}$$

$\tilde{\rho}(g)A$ is linear, because it is a conjugation by a fixed matrix. It is a representation, because

$$\tilde{\rho}(gh)A = \rho(gh)A\rho^{-1}(gh) = \rho(g)\rho(h)A\rho^{-1}(h)\rho^{-1}(g) = \left(\tilde{\rho}(g) \circ \tilde{\rho}(h)\right) A, \tag{3.15}$$

so

$$\tilde{\rho}(gh) = \tilde{\rho}(g) \circ \tilde{\rho}(h). \tag{3.16}$$

What is more, it is an automorphism of the algebra $\operatorname{End} V$.

$$\tilde{\rho}(g)(AB) = \rho(g)AB\rho^{-1}(g) = \rho(g)A\rho^{-1}(g)\rho(g)B\rho^{-1}(g) = (\tilde{\rho}(g)A)\,(\tilde{\rho}(g)B) \tag{3.17}$$

Returning to our motivation, our algebra is $M_n(\mathbb{C})$ which has a known group of automorphisms $\operatorname{Aut} M_n(\mathbb{C})$. Finding fundamental vector fields is equivalent to finding the corresponding derivation, which has been done both from the definition and from the properties of one-parameter subgroups. Our next aim (Chapter 5) is to take the action of a Lie group and to calculate the generators, i.e., fundamental vector fields for $X = E_i$ (basis element of the Lie algebra[4]).

Just to remind, in differential geometry the fundamental field of the action R_g on M is defined as

$$\xi_X(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 R_{\exp tX}x, \qquad x \in M \tag{3.18}$$

or

$$\xi_X(x) = \dot{\gamma}(0), \qquad \gamma(t) := xe^{tX} \tag{3.19}$$

[3].

Subsequently, we can find analogues of vector fields on $M_n(\mathbb{C})$, but we already know that right actions of one-parameter subgroups are in our case conjugations; therefore the fundamental fields are commutators with the matrix X that we choose to be a basis matrix of the corresponding Lie algebra, namely $[E_i, N]$, $N \in M_n(\mathbb{C})$ because

$$\xi_{E_i}(N) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 e^{tE_i}Ne^{-tE_i} = [E_i, N]. \tag{3.20}$$

[4] Or, as it is done in Chapter 5, i-multiples of them.

Chapter 4

Matrix norm and an invariant scalar product

We want to observe how a group acts on the algebra $M_n(\mathbb{C})$ and we will see that the latter decomposes into subspaces. We want to check whether basis vectors of these subspaces form an orthonormal basis, but to define norm and orthogonality one has to define a scalar product.

First, we want to define the norm of a matrix. We define the scalar product, so that the norm of a matrix is the square root of the sum of squares of the absolute values of its entries.[1]

$$\langle A, A\rangle = A_{ji}A^*_{ji} = A^\dagger_{ij}A_{ji} = \operatorname{tr}\big(A^\dagger A\big) \tag{4.1}$$

In the last term, the indices j arise from the matrix multiplication and indices i from the trace. The trace happens to be linear,

$$\operatorname{tr}(A + \lambda B) = (A + \lambda B)_{ii} = A_{ii} + \lambda B_{ii} = \operatorname{tr}(A) + \lambda \operatorname{tr}(B). \tag{4.2}$$

There is another useful property of the trace:

$$\operatorname{tr}(AB \cdots YZ) = A_{ib}B_{bc} \cdots Y_{yz}Z_{zi} = Z_{zi}A_{ib}B_{bc} \cdots Y_{yz} = \operatorname{tr}(ZAB \cdots Y). \tag{4.3}$$

In line with the original motivation, we define the scalar product as

$$\langle A, B\rangle := \operatorname{tr}\big(A^\dagger B\big). \tag{4.4}$$

The first question is whether this is indeed a scalar product. It preserves addition and is linear in the second slot and anti-linear in the first (so sesquilinear)[2], conjugate-symmetric and positive-definite,

$$\langle A, B + \lambda C\rangle = \langle A, B\rangle + \lambda \langle A, C\rangle\,, \tag{4.5}$$

[1] One may ask why we define it in this way. The inspiration is taken from vectors where it is the same. There are other possible norms of matrices (one can be found in [5]), but we will not use them.

[2] Just to note, [11] uses an opposite convention in Chapter 17.3.

$$\langle A, B\rangle^* = \langle B, A\rangle \,. \tag{4.6}$$

Positive definiteness comes directly from the construction as we sum squares of norms (so squares of non-negative numbers).

Now we can observe what happens if we multiply the second argument by another matrix from the left.

$$\langle M, AN\rangle = \mathrm{tr}\big(M^\dagger AN\big) = \mathrm{tr}\Big(\big(A^\dagger M\big)^\dagger N\Big) = \big\langle A^\dagger M, N\big\rangle \tag{4.7}$$

A natural question can arise: what if we act with a derivation $\hat{A} = [A, \cdot]$ on N? It can be shown that

$$\Big\langle M, \hat{A}N\Big\rangle = \big\langle \big[A^\dagger, M\big]\,, N\big\rangle =: \Big\langle \hat{A}^\dagger M, N\Big\rangle \,, \tag{4.8}$$

which means that if the matrix A is Hermitian the entire operator is Hermitian.

▼

$$\begin{aligned}\Big\langle M, \hat{A}N\Big\rangle &= \mathrm{tr}\big(M^\dagger AN\big) - \mathrm{tr}\big(M^\dagger NA\big) = \mathrm{tr}\Big(\big(A^\dagger M\big)^\dagger N\Big) - \mathrm{tr}\big(AM^\dagger N\big) = \\ &= \mathrm{tr}\Big(\big(A^\dagger M\big)^\dagger N\Big) - \mathrm{tr}\Big(\big(MA^\dagger\big)^\dagger N\Big) = \big\langle A^\dagger M, N\big\rangle - \big\langle MA^\dagger, N\big\rangle = \\ &= \big\langle \big[A^\dagger, M\big]\,, N\big\rangle\end{aligned} \tag{4.9}$$

▲

4.1 Invariant scalar product

Let $A, B \in M_n(\mathbb{C})$ be vectors from a vector space, G is a compact Lie group and ρ its representation (conjugation by group element). It is known ([3] 12.1.12) that for a compact Lie group G (or a finite group ([3] 12.1.11)) there exists a ρ-invariant scalar product. It means that a scalar product remains unchanged under the action of any $g \in G$ on both vectors via ρ.

Let us check whether our scalar product is ρ-invariant. The groups we may be interested in are $SO(3)$ and $SU(2)$ (both Lie groups and compact). Elements g (represented as matrices A) of the first one have a property $A^\mathsf{T} A = \mathbb{1}$ and of the second one $A^\dagger A = \mathbb{1}$ while in both of them $\det(A) = 1$. These properties hold for the matrix representations we consider. Because elements of $SO(3)$ are represented as real matrices, Hermitian conjugation is equivalent to the transposition (we want them to be unitary, but for real matrices unitarity reduces to orthogonality). We will study the representation by conjugations. The result

$$\big\langle AMA^{-1}, ANA^{-1}\big\rangle = \langle M, N\rangle \tag{4.10}$$

shows that our scalar product is ρ-invariant.

▼

$$\begin{aligned}\left\langle AMA^{-1}, ANA^{-1}\right\rangle &= \operatorname{tr}\Big(\left(AMA^{-1}\right)^{\dagger} ANA^{-1}\Big) = \\ &= \operatorname{tr}\Big(\left(A^{-1}\right)^{\dagger} M^{\dagger}A^{\dagger}ANA^{-1}\Big) = \\ &= \operatorname{tr}\left(AM^{\dagger}NA^{-1}\right) = \operatorname{tr}\left(A^{-1}AM^{\dagger}N\right) = \\ &= \operatorname{tr}\left(M^{\dagger}N\right) = \langle M, N\rangle\end{aligned} \tag{4.11}$$

▲

4.2 Matrix norm and a dual space

The norm of a matrix M can be defined as

$$\|M\| = \sqrt{\langle M, M\rangle}. \tag{4.12}$$

We do this to obtain an orthonormal basis.

Now one can recall a basic property of orthonormal vectors [7]:

$$\langle j, m|j', m'\rangle = \delta_{jj'}\delta_{mm'}. \tag{4.13}$$

This symbolism may be a little bit confusing, because $\langle j, m|$ is a covector. We can define

$$\langle j, m| := \langle |j, m\rangle\, , \cdot\rangle\, . \tag{4.14}$$

Since it belongs to the dual space, it should obey certain transformation conditions. First of all,

$$\langle \langle j, m|\, ; |j', m'\rangle\rangle = \delta_{jj'}\delta_{mm'}, \tag{4.15}$$

where $\langle\cdot;\cdot\rangle$ is bilinear pairing. This is obvious from our definition of the dual basis and the condition of orthogonality (this equation is just a combination of the previous ones). Next, if we change the basis of the linear space $M_n(\mathbb{C})$ by a transformation by a matrix A, $|j, m\rangle \mapsto \widetilde{|j, m\rangle} = A\,|j, m\rangle$ we can observe how the (proposed) dual basis changes.

$$\langle j, m| \mapsto \widetilde{\langle j, m|} = \left\langle \widetilde{|j, m\rangle}, \cdot\right\rangle = \langle A\,|j, m\rangle\, , \cdot\rangle = \left\langle |j, m\rangle\, , A^{\dagger}\cdot\right\rangle = \left\langle |j, m\rangle\, , A^{-1}\cdot\right\rangle \tag{4.16}$$

The proposed dual basis is really a basis of $M_n^{\star}(\mathbb{C})$.

Chapter 5

Representations of $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$

In this chapter, we explicitly calculate what the generators look like. We will be interested in the group $SO(3)$ ($A \in SO(3): \; A^{\mathsf{T}}A = \mathbb{1}$, $\det(A) = 1$).

The first observation is that the Lie algebra $\mathfrak{su}(2)$ is isomorphic to the Lie algebra $\mathfrak{so}(3)$ and they are the Lie algebras of $SU(2)$ and $SO(3)$ (note that $SO(3) = SU(2)/\mathbb{Z}_2$, so we will be effectively interested only in $SO(3)$). [3]

It is known from quantum mechanics [7] that all irreducible complex representations are labeled by j and the dimension of a representation labeled by j is $2j+1$. For a fixed j, m can have values $-j, -j+1, \ldots, j-1, j$, so the allowed j values are half-integers (even though only integers will appear). Our aim is to find $|j, m\rangle$ as matrices.

5.1 Representations of $\mathfrak{su}(2)$ by derivations of $M_2(\mathbb{C})$

The algebra $\mathfrak{su}(2)$ can be represented by 2×2 anti-Hermitian traceless matrices. After multiplication by i, these matrices become Hermitian.[1] Because a basis of $\mathfrak{su}(2)$ is given by $-i/2$ multiples of Pauli matrices, when using Hermitian matrices, we will take $1/2$ multiples.

We will use matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.1}$$

and

$$s_i = \frac{1}{2}\sigma_i. \tag{5.2}$$

The following commutation formulae are useful:

$$[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c, \qquad [s_a, s_b] = i\varepsilon_{abc}s_c, \tag{5.3}$$

[1] To be more precise, i multiplies matrices that represent the algebra, not the abstract generators themselves. In the real life, we do not want imaginary spectra and this step is exactly what is needed to obtain the real ones which seem to be a little bit more realistic. Furthermore, thanks to this property, derivations as commutators with these matrices are Hermitian, too. See (4.8).

where the last one satisfies the condition for structure constants $c_{ab}^{c} = i\varepsilon_{abc}$.

There is an identity:

$$\sigma_a \sigma_b = \delta_{ab} \mathbb{1} + i\varepsilon_{abc}\sigma_c. \tag{5.4}$$

Consider the representation given by derivations of $M_2(\mathbb{C})$:

$$J_i := [s_i, \cdot]. \tag{5.5}$$

From the general theory, if E_a are basis elements of an algebra, structure constants can be defined as

$$[E_a, E_b] = c_{ab}^{c} E_c. \tag{5.6}$$

This has to hold even after taking the corresponding representations $J_i \equiv \rho'(E_i)$ (Appendix B: Commutation relations preserved by a representation).

As is known from quantum mechanics, we can choose two commuting operators of angular momentum, in our case those satisfying the following equations:

$$J^2 \left|j, m\right\rangle = j(j+1) \left|j, m\right\rangle \tag{5.7}$$

and

$$J_3 \left|j, m\right\rangle = m \left|j, m\right\rangle, \tag{5.8}$$

where $\left|j, m\right\rangle$ is a common eigenvector and $J^2 = J_a J_a$.

It will be helpful to compute $s_a s_a$ using (5.4).

$$s_a s_a = \frac{1}{4}\sigma_a\sigma_a = \frac{1}{4}\delta_{aa}\mathbb{1} + \frac{1}{4} i\varepsilon_{aac}\sigma_c = \frac{3}{4}\mathbb{1} \tag{5.9}$$

From equation (5.7) can be obtained:

$$J_a J_a M = [s_a, [s_a, M]] \stackrel{!}{=} j(j+1)M \tag{5.10}$$

which leads to

$$\left[\frac{3}{2} - j(j+1)\right] M \stackrel{!}{=} 2 s_a M s_a. \tag{5.11}$$

▼

Indeed,

$$\begin{aligned} J_a J_a M &= J_a[s_a, M] = [s_a, [s_a, M]] = s_a[s_a, M] - [s_a, M]s_a = \\ &= s_a s_a M - s_a M s_a - s_a M s_a + M s_a s_a = \frac{3}{4}M - 2 s_a M s_a + \frac{3}{4}M = \\ &= \frac{3}{2}M - 2 s_a M s_a \stackrel{!}{=} j(j+1)M. \end{aligned} \tag{5.12}$$

▲

5.1.1 Finding all possible j

Let

$$M = \begin{pmatrix} n & o \\ p & q \end{pmatrix} \in M_2(\mathbb{C}). \tag{5.13}$$

Then

$$\sigma_a M \sigma_a = \begin{pmatrix} q & p \\ o & n \end{pmatrix} + \begin{pmatrix} q & -p \\ -o & n \end{pmatrix} + \begin{pmatrix} n & -o \\ -p & q \end{pmatrix} = \begin{pmatrix} 2q+n & -o \\ -p & 2n+q \end{pmatrix}. \tag{5.14}$$

After rewriting equation (5.11), we obtain

$$\begin{pmatrix} (1-j(j+1))n - q & (2-j(j+1))o \\ (2-j(j+1))p & (1-j(j+1))q - n \end{pmatrix} \overset{!}{=} \mathbb{0}. \tag{5.15}$$

If $o \neq 0$ or $p \neq 0$, $2 - j(j+1) = 0$, so $j \in \{-2, 1\}$, which means $n + q = 0$. If both $p = o = 0$, $j(j+1) = 0$, so $j \in \{-1, 0\}$, but we cannot distinguish between these two different j, so they are totally equivalent (we are interested only in the eigenvalue and it is conventional to take the non-negative j). Similar equivalence is in the previous case, so there are only two values of j with unique solutions.

5.1.2 Case $j = 0$

From equation (5.11):

$$\frac{3}{2}M \overset{!}{=} 2 s_a M s_a \qquad \Longrightarrow \qquad 3M = \sigma_a M \sigma_a. \tag{5.16}$$

Equation (5.16) can be rewritten as

$$3\begin{pmatrix} n & o \\ p & q \end{pmatrix} \overset{!}{=} \begin{pmatrix} 2q+n & -o \\ -p & 2n+q \end{pmatrix}, \tag{5.17}$$

whose solutions are $M = \lambda \mathbb{1}$, $\lambda \in \mathbb{C}$.

Due to the commutation with all matrices, we will not offend the reader by computing $[s_3, \lambda\mathbb{1}]$, which is obviously $\mathbb{0}$.

5.1.3 Case $j = 1$

It will be much easier to work directly with the fact that any 2×2 matrix can be expressed as a linear combination of the unit matrix and the Pauli matrices

$$M = \alpha\mathbb{1} + \vec{\beta} \cdot \vec{\sigma}, \tag{5.18}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^\mathsf{T}$.

From equation (5.11),

$$-M \overset{!}{=} \sigma_a M \sigma_a, \tag{5.19}$$

so

$$-\alpha \mathbb{1} - \beta_d \sigma_d \overset{!}{=} 3\alpha \mathbb{1} - \beta_d \sigma_d. \tag{5.20}$$

Fortunately, the second terms on both the left- and right-hand sides are the same, so M has to satisfy

$$\alpha = 0 \qquad \Longrightarrow \qquad M = \vec{\beta} \cdot \vec{\sigma}. \tag{5.21}$$

Next, we have to check whether there are any conditions $\vec{\beta}$ must obey by inserting the solution to equation (5.8).

$$J_3 \left(\vec{\beta} \cdot \vec{\sigma} \right) = i \left(\beta_1 \sigma_2 - \beta_2 \sigma_1 \right) \overset{!}{=} m \vec{\beta} \cdot \vec{\sigma} \tag{5.22}$$

Let $m = 0$. It is possible only if $\beta_1 = \beta_2 = 0$, so

$$M = \beta_3 \sigma_3. \tag{5.23}$$

Let $m = 1$.

$$i\beta_1 \sigma_2 - i\beta_2 \sigma_1 \overset{!}{=} \beta_1 \sigma_1 + \beta_2 \sigma_2 + \beta_3 \sigma_3 \tag{5.24}$$

Comparing the coefficients of the Pauli matrices one can obtain a system of equations

$$\beta_3 = 0 \quad \wedge \quad i\beta_1 = \beta_2, \tag{5.25}$$

which leads to

$$M = \beta_1 \sigma_1 + i\beta_1 \sigma_2. \tag{5.26}$$

Let $m = -1$. Compared to equation (5.24), every term of the right-hand side acquires an extra $-$ sign, so the system of equations will change to

$$\beta_3 = 0 \quad \wedge \quad i\beta_1 = -\beta_2, \tag{5.27}$$

and

$$M = \beta_1 \sigma_1 - i\beta_1 \sigma_2. \tag{5.28}$$

A valuable observation is that our four results are linearly independent; therefore, they are a basis of $M_2(\mathbb{C})$. Moreover, they form an orthogonal system. Orthonormality can be achieved just by multiplying each basis matrix by the inverse of its norm.

Table 5.1: Normalized basis matrices for $M_2(\mathbb{C})$ as eigenvectors of J_3 and J^2.

j	m	Basis matrix
0	0	$\frac{1}{\sqrt{2}}\mathbb{1}$
1	1	$\frac{1}{2}(\sigma_1 + i\sigma_2) = E_1^2$
1	0	$\frac{1}{\sqrt{2}}\sigma_3$
1	-1	$\frac{1}{2}(\sigma_1 - i\sigma_2) = E_2^1$

5.2 Representations of $\mathfrak{so}(3)$ by derivations of $M_3(\mathbb{C})$

Let

$$M = \begin{pmatrix} n & o & p \\ q & r & s \\ t & u & v \end{pmatrix} \in M_3(\mathbb{C}) \tag{5.29}$$

and

$$(A_a)_{bc} = \varepsilon_{abc} \tag{5.30}$$

or more precisely

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.31}$$

Then we define $l_a = -iA_a$, which leads to $[l_a, l_b] = i\varepsilon_{abc}l_c$. Next, we take a representation given by derivations of $M_3(\mathbb{C})$:

$$J_i = [l_i, \cdot]. \tag{5.32}$$

It can be shown that for a fixed a we obtain $l_a l_a = \mathbb{1} - E_a^a$. If we consider the summation,

$$l_a l_a = \left(\mathbb{1} - E_1^1\right) + \left(\mathbb{1} - E_2^2\right) + \left(\mathbb{1} - E_3^3\right) = 2\mathbb{1}. \tag{5.33}$$

Analogously to before, equation (5.7) gives:

$$M\left(4 - j(j+1)\right) + 2A_a M A_a \stackrel{!}{=} \mathbb{0}. \tag{5.34}$$

We will not do any abstract theory about the multiplication by A_a from both sides (even though the parametrization is very simple), we will just write what will be obtained by a direct multiplication.

$$A_1 M A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -v & u \\ 0 & s & -r \end{pmatrix},\ A_2 M A_2 = \begin{pmatrix} -v & 0 & t \\ 0 & 0 & 0 \\ p & 0 & -n \end{pmatrix},$$
$$A_3 M A_3 = \begin{pmatrix} -r & q & 0 \\ o & -n & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.35}$$

After the summation:

$$(4 - j(j+1))\, M + 2\mathcal{M} \stackrel{!}{=} \mathbb{0}, \tag{5.36}$$

where

$$\mathcal{M} = \begin{pmatrix} -v-r & q & t \\ o & -v-n & u \\ p & s & -r-n \end{pmatrix}. \tag{5.37}$$

$$J_3 M = [l_3, M] = -i[A_3, M] \tag{5.38}$$

5.2.1 Case $j = 0$

It is obvious that $M = \alpha\mathbb{1}$, because

$$2\begin{pmatrix} n & o & p \\ q & r & s \\ t & u & v \end{pmatrix} = \begin{pmatrix} v+r & -q & -t \\ -o & v+n & -u \\ -p & -s & r+n \end{pmatrix} \tag{5.39}$$

has no other solution, and a reader is free to check the commutation with l_3.

5.2.2 Case $j = 1$

$$M \stackrel{!}{=} -A_a M A_a, \tag{5.40}$$

so

$$\begin{pmatrix} n & o & p \\ q & r & s \\ t & u & v \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} v+r & -q & -t \\ -o & v+n & -u \\ -p & -s & r+n \end{pmatrix}, \tag{5.41}$$

which leads to a system of equations

$$o = -q \qquad \wedge \qquad t = -p \qquad \wedge \qquad u = -s. \tag{5.42}$$

All solutions can be parametrized using (5.30) $\vec{A} = (A_1, A_2, A_3)^\mathsf{T}$ and $\vec{\beta} \in \mathbb{C}^3$.

$$(J_3 M)_{ef} = i\beta_d(\delta_{df}\delta_{e3}) \tag{5.43}$$

▼

$$
\begin{aligned}
(J_3 M)_{ef} &= \left(-i\left[A_3, \vec{\beta}\cdot\vec{A}\right]\right)_{ef} = \\
&= (-i[A_3, \beta_d A_d])_{ef} = \\
&= i\left(\beta_d (A_d A_3)_{ef} - \beta_d (A_3 A_d)_{ef}\right) = \\
&= i\beta_d\left((A_d)_{ec}(A_3)_{cf} - (A_3)_{ec}(A_d)_{cf}\right) = \\
&= i\beta_d\left(\varepsilon_{dec}\varepsilon_{3cf} - \varepsilon_{3ec}\varepsilon_{dcf}\right) = \\
&= i\beta_d\left[(\delta_{3d}\delta_{ef} - \delta_{3f}\delta_{ed}) - (\delta_{d3}\delta_{ef} - \delta_{df}\delta_{e3})\right] = \\
&= i\beta_d(\delta_{df}\delta_{e3}).
\end{aligned}
\tag{5.44}
$$

▲

It will be cleaner to work with entire matrices rather than their elements.

For $m = 0$: $J_3 M \overset{!}{=} \mathbb{0}$.

$$
i\begin{pmatrix} 0 & 0 & -\beta_1 \\ 0 & 0 & -\beta_2 \\ \beta_1 & \beta_2 & 0 \end{pmatrix} \overset{!}{=} \mathbb{0}
\tag{5.45}
$$

$\beta_1 = \beta_2 = 0$, β_3 is arbitrary, so $M = A_3\beta_3$.

For $m = 1$:

$$
i\begin{pmatrix} 0 & 0 & -\beta_1 \\ 0 & 0 & -\beta_2 \\ \beta_1 & \beta_2 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix},
\tag{5.46}
$$

so $\beta_3 = 0$ and $i\beta_1 = \beta_2$, i.e., $M = A_2 i\beta_1 + A_1\beta_1$.

For $m = -1$:

$$
i\begin{pmatrix} 0 & 0 & -\beta_1 \\ 0 & 0 & -\beta_2 \\ \beta_1 & \beta_2 & 0 \end{pmatrix} \overset{!}{=} -\begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix},
\tag{5.47}
$$

so $\beta_3 = 0$ and $\beta_1 = i\beta_2$, i.e., $M = A_2\beta_2 + iA_1\beta_2$.

5.2.3 Case $j = 2$

$$
o = p \qquad \wedge \qquad p = t \qquad \wedge \qquad u = s \qquad \wedge \qquad v + n + r = 0,
\tag{5.48}
$$

which leads to five basis matrices:

$$
\begin{aligned}
&S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ S_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&S_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ S_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{5.49}
$$

Let $\vec{S} = (S_1, \dots, S_5)^\mathsf{T}$ and $\vec{\gamma} \in \mathbb{C}^5$. Now we can write M as

$$M = \begin{pmatrix} \gamma_4 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_5 & \gamma_1 \\ \gamma_2 & \gamma_1 & -\gamma_4 - \gamma_5 \end{pmatrix}. \tag{5.50}$$

From (5.8):

$$\begin{aligned} J_3 M &\overset{1}{=} [l_3, M] = l_3 M - M l_3 = -iA_3 M + iMA_3 = -i \begin{pmatrix} o+q & r-n & s \\ r-n & -o-q & -p \\ u & -t & 0 \end{pmatrix} \\ &\overset{2}{=} mM, \\ &\overset{3}{=} -i \begin{pmatrix} 2\gamma_3 & \gamma_5 - \gamma_4 & \gamma_1 \\ \gamma_5 - \gamma_4 & -2\gamma_3 & -\gamma_2 \\ \gamma_1 & -\gamma_2 & 0 \end{pmatrix} \end{aligned} \tag{5.51}$$

For $m = 0$, $\gamma_4 = \gamma_5$, the others are 0, so

$$M = \operatorname{diag}(1, 1, -2)\gamma_4. \tag{5.52}$$

For $m = 1$, $i\gamma_2 = \gamma_1$ and the others are 0, so

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix} \gamma_2. \tag{5.53}$$

For $m = 2$, $\gamma_3 = i\gamma_4$, $-\gamma_4 = \gamma_5$ and the others are 0, so

$$M = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \gamma_4. \tag{5.54}$$

For $m = -1$, $i\gamma_1 = \gamma_2$, the others are 0, so

$$M = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix} \gamma_1. \tag{5.55}$$

For $m = -2$, $i\gamma_3 = \gamma_4$, $-i\gamma_3 = \gamma_5$ and the others are 0, so

$$M = \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \gamma_3. \tag{5.56}$$

The coefficients gamma next to the matrices are arbitrary and there is no connection between them for different m. They are chosen just to easily see where these matrices come from.

As can be seen from (5.36) after a computation or directly from the sum of the dimension of the found subspaces, there are no other j with unique solutions.

To sum up, the space with $j = 0$ consists of matrices with non-zero trace (except the trivial case $\mathbb{0}$), $j = 1$ consists of antisymmetric traceless matrices, and $j = 2$ of symmetric traceless matrices.

Table 5.2: Normalized basis matrices for $M_3(\mathbb{C})$ as eigenvectors of J_3 and J^2.

j	m	Basis matrix
0	0	$\frac{1}{\sqrt{3}}\mathbb{1}$
1	1	$\frac{1}{2}(A_1 + iA_2)$
1	0	$\frac{1}{\sqrt{2}}A_3$
1	-1	$\frac{1}{2}(A_1 - iA_2)$
2	2	$\frac{1}{2}(S_4 - S_5 + iS_3)$
2	1	$\frac{1}{2}(S_2 + iS_1)$
2	0	$\frac{1}{\sqrt{6}}(S_4 + S_5)$
2	-1	$\frac{1}{2}(iS_2 + S_1)$
2	-2	$\frac{1}{2}(iS_4 - iS_5 + S_3)$

Conclusion

Our aim was to determine what the vector fields look like on a fuzzy manifold, observe the action of a Lie group and calculate the corresponding generators. We have found that to describe a fuzzy manifold it is better to work with an analogue of the algebra of functions on it, i.e., matrices. This enables us to say that analogues of vector fields in terms of the algebra are its derivations. This is in accordance with the fact that all automorphisms are conjugations. Then we have observed the action of $SO(3)$ by conjugations and found that the algebra of "functions" decomposes into irreducible representations, namely $M_2(\mathbb{C}) = \mathbf{0} \oplus \mathbf{1}$ and $M_3(\mathbb{C}) = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ (numbers mean values of j in the subspaces with the dimension $2j+1$). This shows that in comparison with spherical harmonics, we need only finitely many basis vectors, which have been found for $n = 2$ and $n = 3$. A straightforward computation shows that the bases are orthonormal.

Table C.1: Analogies between commutative geometry and non-commutative geometry.

Commutative geometry	**Non-commutative geometry**
Algebra of smooth functions $\mathcal{F}(M)$	Algebra of matrices $M_n(\mathbb{C})$
Point $x \in M$	No points in the usual sense
Vector field V (derivation of $\mathcal{F}(M)$)	Derivation $\mathcal{D}(B) = [A, B]$
Isometry $\varphi : M \to M$ (preserves metric)	Automorphism $I_G(M) = GMG^{-1}$ (conjugation by unitary matrix)
Killing vectors (infinitesimal isometries)	Commutator with a fixed matrix $[A, \cdot]$
Angular momentum operators L_i (differential operators on sphere)	Generators $J_i = [l_i, \cdot]$ or $[s_i, \cdot]$ acting on matrices
Spherical harmonics $Y_{lm}(\theta, \phi)$ – eigenfunctions of L^2, L_3	Basis matrices $\|j, m\rangle$ – eigenvectors of J^2, J_3
Decomposition $\mathcal{F}(S^2) = \bigoplus_{l=0}^{\infty}(2l+1)$	Decomposition $M_n(\mathbb{C}) = \bigoplus_j (2j+1)$

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Appendix A: Algebra

Let $\mathbb{K}$ be a field. An algebra $\mathcal{A}$ over $\mathbb{K}$ is an algebraic structure that can be regarded as a linear space with a vector multiplication as a bilinear operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying, for all $a, b, c \in \mathcal{A}, k \in \mathbb{K}$,

$$a(b + c) = ab + ac, \qquad (b + c)a = ba + ca \tag{A.1}$$

and

$$(ka)b = k(ab) = a(kb). \tag{A.2}$$

If additionally,

$$a(bc) = (ab)c = abc, \tag{A.3}$$

it is called an associative algebra. [3], [11]

There is another way to define it. The algebra $\mathcal{A}$ over a field $\mathbb{K}$ is a ring $\mathcal{A} = (\mathcal{A}, +, \cdot)$ in which multiplication by $k \in \mathbb{K}$ satisfies property (A.2).

Appendix B: Commutation relations preserved by a representation

Let

$$[C_a, C_b] = t_{abc} C_c \tag{B.1}$$

be a commutator defining a Lie algebra. Structure constants are replaced by a set of constants with only lower indices $t_{ijk} := c^k_{ij}$. It is important that these numbers are the same regardless of the position (upper or lower) of an index.

Now we define an operator $\widehat{C_a} := [C_a, \cdot]$. Let us check how the commutation relation changes.

$$\begin{aligned}
\left[\widehat{C_a}, \widehat{C_b}\right] M = \left(\widehat{C_a}\widehat{C_b} - \widehat{C_b}\widehat{C_a}\right) M &= [C_a, [C_b, M]] - [C_b, [C_a, M]] = \\
&= [C_a, (C_b M - M C_b)] - [C_b, (C_a M - M C_a)] = \\
&= C_a C_b M - C_a M C_b - C_b M C_a + M C_b C_a - \\
&\quad - C_b C_a M + C_b M C_a + C_a M C_b - M C_a C_b = \\
&= [C_a, C_b] M - M [C_a, C_b] = [[C_a, C_b], M] = [t_{abc} C_c, M] = \\
&= t_{abc} \widehat{C_c} M
\end{aligned} \tag{B.2}$$

This shows that when we represent the algebra by commutators acting on $M_n(\mathbb{C})$ as derivations, the commutation relation is preserved.