15

Parallel transport and linear connection on M

• In Chapter 4 we already encountered the possibility of transporting vectors, namely Lie transport and the related Lie derivative. Here we introduce another type of transport, the so-called *parallel* transport. The corresponding derivative is called the *covariant derivative*. Although the two transports share some common features, in many respects their geometrical meaning differs and one should understand which one is appropriate for use in a concrete application.

15.1 Acceleration and parallel transport

• Recall (see Section 2.2) that it makes no direct sense to perform a linear combination of a vector at point x with a vector at point y since the tangent spaces at different points $x, y \in M$ are vector spaces which are not at all related (except for the dimension).

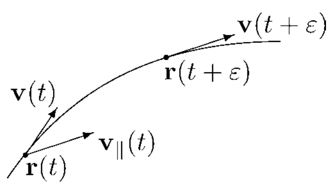
15.1.1 Let $B : W \to V$ be an isomorphism of linear spaces V and W. Check that the rule

$$v + \lambda w := v + \lambda B(w) \qquad v \in V, \quad w \in W$$

gives a sense to the linear combination of two vectors from *different* spaces.

Hint: B enables one to identify V with W. □

• So if a *distinguished* (canonical, independent of arbitrary choices) isomorphism $B : T_y M \to T_x M$ existed, we might define the combination by the trick $v + \lambda w := v + \lambda B(w)$, $v \in T_x M$, $w \in T_y M$. However, already in Section 2.2 we warned that although on a "bare" manifold the spaces $T_x M$ and $T_y M$ are isomorphic, the isomorphism is *not* canonical.



After these general considerations let us have a look at how it is related to the definition of the concept of *acceleration* of a point mass in elementary mechanics. Given $\mathbf{r}(t)$, the trajectory of the point, its (instantaneous) velocity is $\mathbf{v}(t) := \dot{\mathbf{r}}(t)$ and the (instantaneous) acceleration is $\mathbf{a}(t) := \dot{\mathbf{v}}(t)$. The conception of the velocity of the point mass helped us to introduce the key concept of a vector on a manifold as early as in Chapter 2. The acceleration turns out to be equally inspiring (although it had to wait patiently for its chance until Chapter 15) – it brings us to the concept of the parallel transport of the vector as well as the covariant derivative.

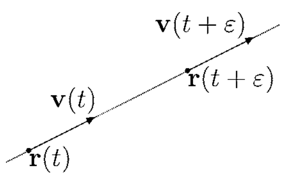
In order to compute $\mathbf{a}(t)$ at the point $\mathbf{r}(t)$ one has by definition to perform the *difference* (i.e. a linear combination) of the vectors $\mathbf{v}(t+\varepsilon)$ and $\mathbf{v}(t)$, i.e. just the procedure which makes no sense officially. It is clear, however, what is meant by this (and often even explicitly stated) in mechanics: the "obvious" fact is used that in E^3 the vectors may be shifted (not altering either their length or direction) around and then used at the point where we need them to sit.[278] In the case of acceleration one is namely to shift the vector $\mathbf{v}(t+\varepsilon)$ from the point $\mathbf{r}(t+\varepsilon)$ back to the point $\mathbf{r}(t)$ (thus obtaining $\mathbf{v}_{\parallel}(t)$) and only this vector may be compared with the vector $\mathbf{v}(t)$. So it is nothing but the trick from (15.1.1), the role of the isomorphism B being played by an appropriate shift. Everything is so clear *here* that one might even be abashed at why an issue should be made of all this.[279]

Nevertheless, there is a tiny cloudlet in the blue sky, namely a slightly conspicuous significance of the Cartesian coordinates in the technical realization of the shifts (on a general manifold all the local coordinates should be equivalent; the fact that this is *not the case* here indicates that E^3 is exceptional from this point of view).

15.1.2 Verify that the operation of the shift of a vector in the Euclidean space E^3 (or, even simpler, in E^2) happens to be technically trivial in Cartesian coordinates *alone* (if we accept as Cartesian coordinates also those with a shifted origin and a modified direction of axes, so that they are related through an *affine* transformation to some "true" Cartesian coordinates).

Hint: in Cartesian coordinates the components of the shifted vector remain *the same*; transform this explicitly to polar, spherical polar, cylindrical, etc. coordinates and check that it becomes complicated. □

• The particular trajectories with *vanishing* acceleration are closely related to the concept of acceleration. They correspond to *uniform straight-line* motion of the point mass. By definition, vanishing of the acceleration means that on this trajectory the velocity $\mathbf{v}(t+\varepsilon)$ is equal to the velocity $\mathbf{v}(t)$. Or, more precisely (since they sit at different points), the velocity vector $\mathbf{v}(t+\varepsilon)$ arises by a *shift* alone (not changing either its length or direction) of the vector $\mathbf{v}(t)$ from the point $\mathbf{r}(t)$ to the point $\mathbf{r}(t+\varepsilon)$ (so that we set $\mathbf{v}(t+\epsilon) = \mathbf{v}_{\parallel}(t+\epsilon)$).



By an iteration of such infinitesimal shifts of the velocity vector the resulting straight line arises, being the trajectory of the point mass (the uniform straight-line motion).

[278] One speaks about *free* and *bounded* vectors there.

[279] Mathematical physics is sometimes blamed for "making an issue" of quite "simple" things. There is a perfect consensus in that this blame is indeed legitimate in p percent of concrete cases, a bit less concord takes place in the numerical value of the number p. Extensive research (based on elaborate questionnaires) revealed that the distribution of p over the world population is actually uniform, bounded by the values $p = 0$ and $p = 100$.

Now, let us try to repeat the same procedure on a different manifold, for example on the sphere S^2. Imagine that an ambitious technocratic ideal was accomplished at last – throughout the Earth, first all the irregularities were straightened out by bulldozers (they were, one should admit, both impractical and unaesthetic) and then the whole surface of the Earth was nicely covered by a neat asphalt. If we now roll a ball along such a smooth surface,[280] it has to roll, according to the laws of mechanics, uniformly along a straight line, since the only force available is the gravitational force, directed everywhere downwards. This force constrains the ball to remain on the two-dimensional *surface* of the Earth (it keeps the ball from flying away along a "truly" straight-line trajectory and escaping into space); the ball gets accustomed to this status quo and it does not regard it as a restriction.[281] It considers pragmatically the sphere S^2 to be its living space and it does not care whether the sphere actually is or is not a subset of any larger ambient space. Since the projection of the gravitational force on to the plane which is tangent to the sphere always vanishes, the ball feels[282] *no force* acting on it and it thus has no reason to change its velocity (neither length nor direction); it therefore moves with vanishing acceleration along a straight line. Note, however, that from the point of view of the ambient space E^3 this is by no means an ordinary straight line, but rather it is a circle (with maximum possible radius), which encircles the whole Earth. The uniform motion along this circle which arises by the iteration of the (infinitesimal) shifts of the velocity vector is the straightest possible motion on the sphere. The shift of the velocity vector keeping its length as well as direction unaltered *in the sense of the sphere* is, as we see from the resulting trajectory, something considerably different from the same procedure performed *in the sense of* E^3 – from the point of view of E^3, in the course of the shifts the vector also continually *rotates* a bit in order to remain tangent to the sphere.

The lesson from this as well as numerous similar particular cases resulted in the following picture: the definition of the concept of *acceleration* as well as *uniform straight-line motion* (i.e. motion with zero acceleration) which is based on it requires the ability to *transport* the velocity *vector* (at least by infinitesimal distances) along a given trajectory. In the space E^3 there is a natural rule of transport and this rule is indeed used in elementary mechanics in E^3. However, in general the matter may not be so simple. It turns out that the most fruitful point of view is to regard the rule of transporting vectors on a manifold as an *independent structure*, which is *a priori not available* on a general manifold, although in particular cases (like in E^3) there may exist most natural realizations.

If such a rule (satisfying some requirements) is introduced on a manifold, we say that a *parallel transport* (or an associated concept – *linear connection*) is defined on M, denoted by (M, ∇). For example, the natural parallel transport of vectors in E^3 is realized as an ordinary shift, but if we introduced another connection into E^3 (which *is* perfectly possible), the parallel transport would be performed in a different way. The straight lines which result

[280] We also ensure zero air resistance and a couple of similar technical details.
[281] This is confidential information from one such ball; for reasons of protection of privacy it has no wish to make either its center or radius public.
[282] See the previous footnote.

from the iteration of the infinitesimal parallel transport of the velocity vector (the trajectories with zero acceleration) are called the (affinely parametrized) *geodesics* on (M, ∇).

The concept of a linear connection is very important in physics, although its presence is fairly obscure in many applications (like in acceleration in elementary mechanics).

15.1.3 Estimate (or evaluate exactly) the fraction

$$f = lc/a$$

where a denotes the number of people on Earth who understand what the acceleration is (including the formula which enables one to compute it) and lc denotes the number of people on Earth who are aware that the linear connection is used in this formula.

Hint: ask all of them and then divide the two numbers; (1.1.1)–(22.5.12). □

• However, there are also disciplines like the general theory of relativity, in which the linear connection lies at the very heart of the mathematical formulation, being explicitly present in the fundamental equations of the theory.

15.1.4 Estimate (or evaluate exactly) the fraction

$$f = lc/gr$$

where gr denotes the number of people on Earth who understand elements of general relativity (including the basic formulas) and lc denotes the number of people on Earth who are aware that the linear connection is used in these formulas.

Hint: see the hint in (15.1.3). □

• The far-reaching generalization of the linear connection, to be explained in more detail in Chapter 20 and beyond, is the basis of the formalism of modern gauge field theories.

15.2 Parallel transport and covariant derivative

• We convinced ourselves that the introduction of the concept of acceleration requires the ability to transport velocity vectors along curves (the trajectories of a point mass). A similar requirement also occurs in numerous other contexts. We say that a *rule of parallel transport* is given on a manifold M, if, for an arbitrary curve γ on M and two points x, y on the curve, there is a prescription which assigns uniquely to vectors in x vectors in y, i.e. a map

$$\tau^{\gamma}_{y,x} : T_x M \to T_y M \qquad v \mapsto \tau^{\gamma}_{y,x} v$$

Clearly, one can think out lots of such rules, but if they are to be useful in the contexts from which the motivation for their introduction came, they should satisfy some restrictive conditions. For the moment we mention the two most important of them.

First, it is natural to ask that the transport of a sum of vectors or a multiple of a vector by a constant should yield the sum of the results of the transport of the individual vectors or the multiple of the transported vector, i.e. to ask for the *linearity* of the map $\tau^{\gamma}_{y,x}$. Secondly, if there are *three* points x, y, z on a curve γ, we expect the parallel transport from x to y followed by the transport (of the vector just brought to y) from y to z to yield the same result as the direct transport (without a moment's rest in y) from x to z would yield. This may be written as the composition property of the maps $\tau^{\gamma}_{y,x}$

$$\tau^{\gamma}_{z,y} \circ \tau^{\gamma}_{y,x} = \tau^{\gamma}_{z,x} \qquad x, y, z \text{ on the curve } \gamma \text{ (otherwise arbitrary)}$$

and, in particular,

$$\tau^{\gamma}_{x,x} = \text{ identity} \qquad \left(\tau^{\gamma}_{y,x}\right)^{-1} = \tau^{\gamma}_{x,y}$$

Note that the rule of parallel transport needs as an input not only the edge points x, y, but also a *path* connecting them.[283] So if we are given at the point x a vector v and a path from x to y, we are able to transport v uniquely to the point y; given another path, the transport is unique as well, but the resulting transported vector may be different in general. We will see that the path dependence of the parallel transport is an important and typical phenomenon in the situations where the connection is applied and it enables one to speak about the *curvature* of the manifold (M, ∇).[284]

Suppose we have some particular fixed rule of a parallel transport of vectors. This rule then enables one to introduce a derivative, which is based on it. Namely, let $\gamma(t)$ be a curve and let $V(t) \equiv V_{\gamma(t)}$ be a vector field defined on the curve.[285] If we intend to differentiate the vector field V along the curve γ, in order to find out whether (and how much) it varies in this direction), we are to compare the vectors $V(t+\varepsilon)$ and $V(t)$. However, these two vectors sit at different points and it means that their difference has no *direct* meaning. Still, the difference of the vectors may be legalized by making use of the rule of parallel transport. Namely, we first *transport* the vector $V(t+\varepsilon)$ along the curve γ from the point $\gamma(t+\varepsilon)$ backwards to the point $\gamma(t)$ and then we compare (subtract) the vector transported back with the initial vector $V(t)$.

Denote the vector transported backwards by $V^{\parallel}_{\varepsilon}(t)$. Then the corresponding derivative, which is called the *absolute derivative* of the vector field V along the curve γ, is defined as

$$\frac{DV(t)}{Dt} := \lim_{\varepsilon \to 0} \frac{V^{\parallel}_{\varepsilon}(t) - V(t)}{\varepsilon}$$

Let us contemplate some immediate consequences of the definition. First, it is clear that the derivative depends on the particular rule of parallel transport.

Next, note that it uses only the behavior of the objects on the curve γ – the field V may (but need not) be defined also outside the curve, but nothing from outside the curve has any

[283] We mentioned a *curve* a minute ago, here we speak about (only) a *path*, i.e. a non-parametrized curve. The transport to be studied here actually depends only on the path alone (see (15.2.6) and (15.2.12)).

[284] This does not mean that the parallel transport always *indeed* depends on a path, but rather that *in general it may* depend on it. For example, the ordinary shifts of vectors in E^3 are evidently path-independent, whereas the transport of the vectors on the sphere, which we discussed in Section 15.1 really depends on a path (15.3.9).

[285] The vector $V(t)$ is an element of the tangent space $T_{\gamma(t)}M$ and it may not be directed along the curve; i.e. we contemplate n-dimensional vectors which need not exist at each point of an n-dimensional domain, as is the case when we treat vector fields on a domain, but they are instead defined only on a one-dimensional domain, at the points of the curve γ.

influence on the value of the derivative along the curve. This differs essentially from the *Lie* derivative. If the curve γ were the integral curve of a vector field W and if both the fields, W and V, were defined in some neighborhood of the curve γ, then the Lie derivative of the field V along W (which corresponds to the derivative of V along the curve γ) would also depend on the values of the field W *outside* the curve γ (15.2.4), so that the transport of the field V along the curve γ actually depends on the structure of *additional curves* aside from the curve γ (the neighboring integral curves of the field W; note that they are indeed necessary since the Lie derivative may be performed only on the fields defined (at least) in a domain).

Realize finally that the *vanishing* of the absolute derivative on some (part of a) curve means that the field $V(t)$ may then be regarded as that its values everywhere on γ arose by (only) a parallel transport of its value at a single fixed point into all the points of the (above mentioned part of the) curve. Such a field on a curve (one might say that it is *constant* on the curve) is called an *autoparallel field*. Thus the absolute derivative informs us exactly about the *deviation from being autoparallel*.

The relation between the absolute derivative of a vector field and the rule of the parallel transport may be used to reverse the roles of what is a "primary" concept and what is a "secondary" one: if we were technically able to perform the derivative, it would allow us in turn to reconstruct the rule of parallel transport. Namely, the rule says: do the transport so *as to make* the derivative *vanish*. This is exactly the way one usually introduces the concept of the linear connection on a manifold. Instead of specifying in detail the requirements which the parallel transport should satisfy, one postulates, on the contrary, the properties of the derivative and the parallel transport is then in turn defined by the simple equation "the derivative should vanish." The corresponding defining properties of the derivative are to be chosen so as to be clear and brief and so as not to contradict any particular case of the transport, which served as the source of inspiration for the general theory (like E^3, the sphere, etc.; i.e. so as to guarantee that all the useful known cases might be regarded as "particular cases of a general theory").

Before we write down the resulting requirements regarding the derivative, we realize that we also have to contemplate the issue of the parallel transport (as well as the derivative) of general *tensors* (just like we did for the Lie derivative).[286] For the Lie stuff, where the "primary" concept was the (Lie) transport (being realized technically as the pull-back Φ_t^* of the flow $\Phi_t \leftrightarrow W$), this issue was simply computed and it *turned out* that the transport preserves the degree and commutes with the tensor product and the contraction, so that the (Lie) derivative *turned out* to be the derivation of the tensor algebra, which preserves the degree and commutes with contraction.

Here it is necessary to *postulate* the properties either at the level of the (parallel) transport, or at the level of the (covariant) derivative. The standard definition says that in this respect we simply copy the properties in the Lie case: one postulates that the (parallel) transport

[286] Note that there are the same problems with the linear combinations of tensors of type $\binom{p}{q}$ at different points x, y, as with vectors (being the special case $p = 1, q = 0$), *except for the case of scalars* ($p = q = 0$): the numbers in x and in y are "canonically" combined without any problems.

preserves the degree and commutes with the tensor product and contraction.[287] The (covariant) derivative, which corresponds to the (parallel) transport, is then necessarily a derivation of the tensor algebra, which preserves the degree and commutes with contractions.

So at last we now state the official definition of the concept of *linear connection* on a manifold M. It says that with each vector field W on M one may associate an operator ∇_W, the *covariant derivative* along the field W, enjoying the following properties:

1. it is a linear operator on the tensor algebra, which preserves the degree

$$\nabla_W : \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M)$$
$$\nabla_W(A + \lambda B) = \nabla_W A + \lambda \nabla_W B \qquad A, B \in \mathcal{T}^p_q(M), \quad \lambda \in \mathbb{R}$$

2. on a tensor product it behaves according to the Leibniz rule

$$\nabla_W(A \otimes B) = (\nabla_W A) \otimes B + A \otimes (\nabla_W B) \qquad A \in \mathcal{T}^p_q(M), B \in \mathcal{T}^{p'}_{q'}(M)$$

3. in degree $\binom{0}{0}$ (i.e. the functions) it gives

$$\nabla_W \psi = W\psi \equiv \mathcal{L}_W \psi \qquad \psi \in \mathcal{F}(M) \equiv \mathcal{T}^0_0(M)$$

4. it commutes with contractions

$$\nabla_W \circ C = C \circ \nabla_W \qquad C = \text{ (any) contraction}$$

5. it is $\mathcal{F}$*-linear* with respect to W, i.e.[288]

$$\nabla_{V+fW} = \nabla_V + f\nabla_W \qquad V, W \in \mathfrak{X}(M), \quad f \in \mathcal{F}(M)$$

Now, let us have a look at how such a connection may be technically determined.

15.2.1 Show that the covariant derivative is uniquely specified by the *coefficients of linear connection* $\Gamma^a_{bc}(x)$ with respect to a frame field e_a, which are the functions defined by

$$\nabla_a e_b =: \Gamma^c_{ba} e_c \qquad \nabla_a := \nabla_{e_a}$$

Solution:[289] we have (the numbers indicate the property used)

$$\nabla_W\big(A^{a\ldots b}_{c\ldots d} e^c \otimes \cdots \otimes e_b\big) = \text{making use of } 1, 2, 3$$
$$= \big(W A^{a\ldots b}_{c\ldots d}\big) e^c \otimes \cdots \otimes e_b + A^{a\ldots b}_{c\ldots d}(\nabla_W e^c) \otimes \cdots \otimes e_b + \cdots + A^{a\ldots b}_{c\ldots d} e^c \otimes \cdots \otimes (\nabla_W e_b)$$

Thus, one needs to be able to compute $\nabla_W e_a$ and $\nabla_W e^b$. If $W = W^b e_b$ then

$$\nabla_W e_a \equiv \nabla_{(W^b e_b)} e_a \overset{5}{=} W^b \nabla_b e_a \equiv \big(\Gamma^c_{ab} W^b\big) e_c$$

[287] Preserving of a degree is clear, commuting with the contraction in plain English says that the transported tensor yields the same number on the transported arguments as the original tensor did on the original arguments.
[288] This is the only property in which the operator of the covariant derivative ∇_W *differs* from the operator of the Lie derivative $\mathcal{L}_W$ ($\mathcal{L}_W$ happens to be only $\mathbb{R}$-linear with respect to W); it turns out that it reflects the requirement mentioned above, so as the parallel transport does not depend (in contrast with the Lie transport) on objects outside the curve.
[289] It may be briefly summarized as follows: ∇_W is a derivation of the tensor algebra $\Rightarrow$ one comes to $\binom{0}{0}$ and the bases of $\binom{1}{0}$ and $\binom{0}{1}$. The case of $\binom{0}{0}$ is handled by property 3, the commuting with contractions enables one to reduce $\binom{0}{1}$ to $\binom{1}{0}$.

and since

$$\begin{aligned} 0 &\overset{3}{=} \nabla_W \delta^a_b = \nabla_W \langle e^a, e_b\rangle = \nabla_W (C(e^a \otimes e_b)) \\ &\overset{4,2}{=} C((\nabla_W e^a) \otimes e_b + e^a \otimes (\nabla_W e_b)) = \langle \nabla_W e^a, e_b\rangle + \langle e^a, \nabla_W e_b\rangle \\ &= \langle \nabla_W e^a, e_b\rangle + \left\langle e^a, \Gamma^d_{bc} W^c e_d\right\rangle = (\nabla_W e^a)_b + \Gamma^a_{bc} W^c \end{aligned}$$

we obtain

$$\nabla_W e^a = -\left(\Gamma^a_{bc} W^c\right) e^b \qquad \text{and in particular} \quad \nabla_b e^a = -\Gamma^a_{cb} e^c$$

The knowledge of the coefficients of linear connection $\Gamma^c_{ab}(x)$ with respect to a frame field $e_a(x)$ thus indeed enables one to compute $\nabla_W A$ for arbitrary W and A, i.e. there is complete information about the connection in them. □

• The coefficients of the linear connection have one upper index and two lower indices. One might anticipate from this that they form the components of a tensor field of type $\binom{1}{2}$. A computation yields something different, however.

15.2.2 Let $e_a \mapsto e'_a = A^b_a(x) e_b$ be a change of a frame field. Check that the primed coefficients of linear connection (given by the prescription $\nabla_{e'_a} e'_b =: \Gamma'^c_{ba} e'_c$) are related to the unprimed coefficients by

$$\Gamma'^c_{ab} = \Gamma^d_{ef} (A^{-1})^c_d A^e_a A^f_b + (A^{-1})^c_d A^f_b \left(e_f A^d_a\right)$$

so that in addition to the first term, corresponding (if it were alone) to a tensor of type $\binom{1}{2}$, there is also the "non-tensorial" second term (which does not contain the unprimed coefficients Γ at all; one speaks about an *inhomogeneous* transformation rule).

Hint: $\nabla_{e'_a} e'_b = A^c_a \nabla_{e_c} (A^d_b e_d) = \dots$; use the properties of the covariant derivative and the definition of the initial coefficients themselves. □

15.2.3 The coefficients of linear connection $\Gamma^k_{ij}(x)$ with respect to the *coordinate* frame field $e_i = \partial_i$ are called the *Christoffel symbols* of the *second kind*. Thus, they are defined by

$$\nabla_j \partial_i =: \Gamma^k_{ij} \partial_k \qquad \nabla_i := \nabla_{\partial_i}$$

Check that under the change of coordinates $x^i \mapsto x'^i(x)$ the following transformation rule holds:

$$\Gamma'^i_{jk} = \frac{\partial x'^i}{\partial x^r} \frac{\partial x^s}{\partial x'^j} \frac{\partial x^m}{\partial x'^k} \Gamma^r_{sm} + \frac{\partial x'^i}{\partial x^r} \frac{\partial^2 x^r}{\partial x'^j \partial x'^k}$$

Hint: (15.2.2), $A^i_j = (J^{-1})^i_j = \partial x^i / \partial x'^j$, $A^f_b (e_f A^d_a) = (e'_b A^d_a)$. □

• Since the connection is a global structure on a manifold, this fairly complicated transformation rule for the Christoffel symbols should necessarily have the correct composition properties on a triple overlap of charts (see Section 2.5). This may be verified "by brute force" here, but one can do this more easily after learning how to encode the coefficients of linear connection into so-called connection 1-forms (15.6.2).

Making use of the covariant derivative we are now in a position to realize the program outlined at the beginning of the section: to express the absolute derivative and then to describe the parallel transport.

15.2.4 Let (M, ∇) be a manifold endowed with a connection, ∇_W the corresponding covariant derivative operator, $\gamma(t)$ a curve and V a vector field. The *absolute derivative* of the field V along γ is defined as[290]

$$\frac{DV(t)}{Dt} := \nabla_{\dot\gamma} V$$

Assume that the curve γ happens to be the integral curve of the field W (i.e. $\dot\gamma = W$ on γ) and that both fields V, W as well as the curve γ are given in a coordinate patch. Check that

(i) in local coordinates we get on the curve γ

$$\nabla_W V = \left(\dot V^i + \Gamma^i_{jk}\dot x^k V^j\right)\partial_i$$
$$\mathcal{L}_W V = \left(\dot V^i - V^j W^i_{\ ,j}\right)\partial_i$$

where $V^i(t) := V^i(\gamma(t))$ are the components of the field V, regarded as the functions on the curve alone

(ii) from the expression of the covariant derivative we can infer that no knowledge of the field W outside the curve is necessary for its computation and thus the formula $\nabla_{\dot\gamma} := \nabla_W$ is indeed correct (recall that the connection officially defines only the notion of the covariant derivative along the vector *field* W) and, consequently, also the definition of the absolute derivative is all right; the expression of the Lie derivative, on the contrary, shows that the behavior of W in a neighborhood of the curve has an influence on this object.

Hint: (i) $\nabla_W V = (\nabla_W V^i)\partial_i + V^i W^j \nabla_j \partial_i = (\dot\gamma V^i)\partial_i + \Gamma^k_{ij}\dot x^j V^i \partial_k$; here $\Gamma^k_{ij}\dot x^j \equiv \Gamma^k_{ij}(\gamma(t))\dot x^j(t)$ is a known function of t on the curve; (ii) $\mathcal{L}_W V = (WV^i - VW^i)\partial_i \equiv (\dot V^i - V^j W^i_{\ ,j})\partial_i$; to compute $VW^i \equiv V^j W^i_{\ ,j}$ we also need to know W in a neighborhood of γ. □

15.2.5 A vector field V on γ will be called *autoparallel* if its absolute derivative along γ vanishes, i.e. if

$$\frac{DV(t)}{Dt} \equiv \nabla_{\dot\gamma} V = 0$$

Check that

(i) the components $V^i(t) := V^i(\gamma(t))$ then satisfy the equations

$$\dot V^i + \Gamma^i_{jk}\dot x^k V^j = 0$$

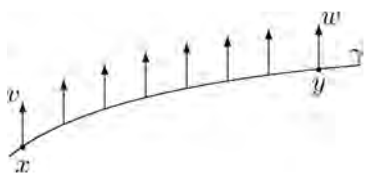
[290] The absolute derivative was defined before in terms of the parallel transport (assumed to be known), here it is defined from the opposite point of view, namely in terms of the (known) covariant derivative.

(ii) these equations may be written in the form

$$\dot{V}^i(t) = S^i_j(t)V^j(t) \qquad S^i_j(t) \text{ known functions of } t$$

so that they form an autonomous system of $n = \dim M$ ordinary first-order linear differential equations with *non*-constant coefficients.

Hint: (i) (15.2.4); (ii) $S^i_j(t) = -\Gamma^i_{jk}(\gamma(t))\dot{x}^k(t)$, which is a known function, provided that the connection (represented by $\Gamma^i_{jk}(x)$) and the curve (in the form of $x^i(t)$) are given. □



15.2.6 In terms of the covariant derivative we may now introduce the operation of the *parallel transport* of a vector along a curve as follows: if there is a vector v at a point x on a curve γ and we want to transport the vector to a point y, then we first construct the autoparallel field generated by the vector v and then take its value at the point y; this value w will be regarded as the result of the parallel transport of the vector, $w = \tau^\gamma_{y,x} v$. Show that

(i) if v has the components (with respect to a coordinate basis in x) $v^i \in \mathbb{R}$, then the components $w^i \in \mathbb{R}$ of the transported vector w (with respect to the coordinate basis in y) are obtained by solving the *equations of parallel transport*

$$\dot{V}^i + \Gamma^i_{jk}\dot{x}^k V^j = 0 \qquad \text{i.e.} \quad \dot{V}^i(t) = S^i_j(t)V^j(t)$$

(for $S^i_j(t) = -\Gamma^i_{jk}(\gamma(t))\dot{x}^k(t)$) with the initial condition $V^i(t_1) = v^i$ (if $x = \gamma(t_1)$); then w^i are obtained as the value of the solution for $t = t_2$ (if $y = \gamma(t_2)$); so in brief

$$\dot{V}^i(t) = S^i_j(t)V^j(t) \qquad V^i(t_1) = v^i \quad w^i := V^i(t_2)$$

(ii) if the assignment $v \mapsto w$ is interpreted as a map $\tau^\gamma_{y,x} : T_x M \to T_y M$, then the map (the *operator of the parallel transport*) is *linear* and it satisfies the requirement

$$\tau^\gamma_{z,y} \circ \tau^\gamma_{y,x} = \tau^\gamma_{z,x} \qquad x, y, z \text{ on the curve } \gamma \text{ (otherwise arbitrary)}$$

(iii) the parallel transport does not feel the parametrization of the curve, i.e. it depends on the *path* rather than on the *curve*.

Hint: (i) according to (15.2.5) the equations $\dot{V}^i + \Gamma^i_{jk}\dot{x}^j V^k = 0$ along with the initial condition $V^i(t_1) = v^i$ yield the unique autoparallel field $V(t)$ generated by the vector v; (ii) the composition property of the operator $\tau^\gamma_{y,x}$ results immediately from the fact that the solution of the equations is *unique* – its linearity stems from the linearity of the equations (a solution linearly depends on the initial conditions); (iii) from the form of the equations of parallel transport

$$\frac{dV^i}{dt} + \Gamma^i_{jk}\frac{dx^k}{dt}V^j = 0$$

we see the reparametrization invariance of the latter (there is the relation $\delta V^i = -\Gamma^i_{jk} V^k \delta x^j$ between the infinitesimal increments irrespective of a parametrization of $x^i(t)$, or alternatively if $V^i(t)$ is a solution for $x^i(t)$, then the solution for $x^i(\sigma(t))$ reads $V^i(\sigma(t))$).

□

• Before we embark on developing a technique of transporting general tensor fields, we need to derive the coordinate component formulas for the computation of the covariant derivatives of an arbitrary tensor field, i.e. to finalize the computation from (15.2.1).

15.2.7 Check that

(i) for the covariant derivative of the coordinate frame and coframe fields there holds

$$\begin{aligned} \nabla_j \partial_i &= +\Gamma^k_{ij} \partial_k & \qquad \nabla_W \partial_i &= +\Gamma^k_{ij} W^j \partial_k \\ \nabla_j dx^i &= -\Gamma^i_{kj} dx^k & \qquad \nabla_W dx^i &= -\Gamma^i_{kj} W^j dx^k \end{aligned}$$

(ii) the component formula for the covariant derivative of a general tensor field reads

$$(\nabla_W A)^{i\dots j}_{k\dots l} = W^m A^{i\dots j}_{k\dots l,m} - \Gamma^n_{km} W^m A^{i\dots j}_{n\dots l} - \dots + \Gamma^j_{nm} W^m A^{i\dots n}_{k\dots l}$$

(iii) this result may be concisely summarized in the form of a table – a recipe for cooking the house speciality $(\nabla_W A)^{i\dots j}_{k\dots l}$ (the recipe for the Lie derivative is also repeated for the convenience of gourmets)

	for preparation of $\nabla_W A$	for $\mathcal{L}_W A$
put on the bottom of a pan	$W A^{\cdots}_{\cdots} \equiv W^m A^{\cdots}_{\dots,m}$	$W A^{\cdots}_{\cdots} \equiv W^m A^{\cdots}_{\dots,m}$
plus for each $A^{\dots i \dots}$ add	$+W^m \Gamma^i_{nm} A^{\dots n \dots}$	$-W^i_{\ ,m} A^{\dots m \dots}$
plus for each $A_{\dots i \dots}$ add	$-W^m \Gamma^n_{im} A_{\dots n \dots}$	$+W^m_{\ ,i} A_{\dots m \dots}$

Hint: (15.2.1) and (15.2.3); compare with (4.3.4). □

15.2.8 Compute the components of the tensor $\nabla_W g$, the covariant derivative of the *metric* tensor along the vector field W.

Hint: the table yields (there is the lump part plus two terms for two lower indices) $(\nabla_W g)_{ij} = W^m g_{ij,m} - W^m \Gamma^n_{im} g_{nj} - W^m \Gamma^n_{jm} g_{in} \equiv W^m (g_{ij,m} - \Gamma^n_{im} g_{nj} - \Gamma^n_{jm} g_{in})$. □

15.2.9 The $\mathcal{F}$-linearity of the operator ∇_W with respect to W enables one to introduce the operation of the *covariant gradient* by

$$\nabla : \mathcal{T}^p_q(M) \to \mathcal{T}^p_{q+1}(M) \qquad (\nabla A)(V, \dots, W; \alpha, \dots) := (\nabla_W A)(V, \dots; \alpha \dots)$$

Check that

(i) ∇A is indeed a tensor field of the type stated above, so that ∇ is a *tensor* operation
(ii) in components (with respect to the coordinate basis) it gives

$$(\nabla A)^{i\dots j}_{k\dots lm} = (\nabla_m A)^{i\dots j}_{k\dots l} =: A^{i\dots j}_{k\dots l;m}$$

where

$$A^{i\ldots j}_{k\ldots l;m} := A^{i\ldots j}_{k\ldots l,m} - \Gamma^n_{km} A^{i\ldots j}_{n\ldots l} - \cdots + \Gamma^j_{nm} A^{i\ldots n}_{k\ldots l}$$

(iii) the computation of $A^{i\ldots j}_{k\ldots l;m}$ is performed according to the recipe

	for preparation of $A^{\cdots}_{\ldots;m}$
first put on the bottom of a pan	$\partial_m A^{\cdots}_{\cdots} \equiv A^{\cdots}_{\ldots,m}$
plus for each $A^{\cdots i \cdots}$ add	$+\Gamma^i_{nm} A^{\cdots n \cdots}$
plus for each $A_{\ldots i \ldots}$ add	$-\Gamma^n_{im} A_{\ldots n \ldots}$

(iv) for a general (possibly non-coordinate) frame field we have

$$(\nabla A)^{a\ldots b}_{c\ldots df} \equiv (\nabla_f A)^{a\ldots b}_{c\ldots d} =: A^{a\ldots b}_{c\ldots d;f}$$

where

$$A^{a\ldots b}_{c\ldots d;f} := e_f A^{a\ldots b}_{c\ldots d} - \Gamma^n_{cf} A^{a\ldots b}_{n\ldots d} - \cdots + \Gamma^b_{nf} A^{a\ldots n}_{c\ldots d}$$

(v) for $p = q = 0$ (on functions) the covariant gradient coincides with the "ordinary" gradient (regarded as a *co*vector field)

$$\nabla f = df \qquad f \in \mathcal{F}(M)$$

(vi) the covariant derivative along W may be written in terms of the covariant gradient as

$$(\nabla_W A)^{i\ldots j}_{k\ldots l} = W^m A^{i\ldots j}_{k\ldots l;m}$$

Hint: (15.2.7). □

• So it holds that "semicolon" = comma *plus* a term containing the Christoffel symbols added for each index; the expression $A^{i\ldots j}_{k\ldots l;m}$ is usually called in short the "*covariant derivative* of $A^{i\ldots j}_{k\ldots l}$ *by m*" and it consists of the partial derivative by m plus the terms with Christoffel symbols.

The covariant gradient may be regarded as a "derivative in an unspecified direction"; if then one intends to compute the (covariant) derivative along a particular vector W, a "scalar product" of the vector is to be performed with the "semi-finished product" ∇A (this immediately results from the $\mathcal{F}$-linearity: $\nabla_W A = W^m \nabla_m A$). If the covariant gradient of a tensor field happens to vanish in some domain, the field is said to be *covariantly constant*. Then the covariant derivative of the field along *any direction* vanishes and so the field may be regarded as being transported into all the points within the domain from its value at a single point (just like the value of a constant *function* in an arbitrary point is known as long as its value at a single point is known).

15.2.10 Evaluate the components of the tensor ∇g, the covariant gradient of the *metric* tensor.

Hint: the table yields (there is the lump part plus two terms for two lower indices) $(\nabla g)_{ijk} \equiv g_{ij;k} = g_{ij,k} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{il}$. □

• Now we may return to the parallel transport of tensors. The concepts of the absolute derivative, the autoparallel field and parallel transport may be extended in a straightforward way from vector fields to arbitrary tensor fields.

15.2.11 The *absolute derivative* of a tensor field A along γ is defined as

$$\frac{DA(t)}{Dt} := \nabla_{\dot{\gamma}} A$$

and the field A on γ is called *autoparallel* if its absolute derivative along γ vanishes. Check that

(i) the concept of the absolute derivative is well defined (if on γ there holds $\dot{\gamma} = W$, then the derivative does not depend on W outside γ)
(ii) in local coordinates the condition for $A^{i\dots j}_{k\dots l}(t) := A^{i\dots j}_{k\dots l}(\gamma(t))$ being autoparallel reads

$$\dot{A}^{i\dots j}_{k\dots l} + \dot{x}^m \left(\Gamma^i_{nm} A^{n\dots j}_{k\dots l} + \dots - \Gamma^n_{lm} A^{i\dots j}_{k\dots n} \right) = 0$$

(iii) this equation may also be written as

$$\dot{A}^{i\dots j}_{k\dots l}(t) = S^{i\dots jc\dots d}_{k\dots la\dots b}(t) A^{a\dots b}_{c\dots d}(t) \qquad S^{i\dots jc\dots d}_{k\dots la\dots b}(t) \text{ known functions of } t$$

so that (if $n = \dim M$) they form an autonomous system of n^{p+q} ordinary first-order linear differential equations with *non*-constant coefficients.

Hint: (15.2.4), (15.2.5) and (15.2.7). □

15.2.12 The operation of *parallel transport* of a tensor field A along γ is introduced as follows: if there is a tensor $\hat{a}$ at a point x on a curve γ and we want to transport the tensor to a point y, then we first construct the autoparallel field generated by the tensor $\hat{a}$ and then take its value at the point y; this value $\hat{b}$ will be regarded as the result of the parallel transport of the tensor, $\hat{b} = \tau^{\gamma}_{y,x} \hat{a}$. Show that

(i) the components $\hat{b}^{i\dots j}_{k\dots l} \in \mathbb{R}$ of the transported tensor $\hat{b}$ (with respect to the coordinate basis in y) are obtained by solving the *equations of parallel transport*

$$\dot{A}^{i\dots j}_{k\dots l} + \dot{x}^m \left(\Gamma^i_{nm} A^{n\dots j}_{k\dots l} + \dots - \Gamma^n_{lm} A^{i\dots j}_{k\dots n} \right) = 0$$

with the initial condition $A^{i\dots j}_{k\dots l}(t_1) = \hat{a}^{i\dots j}_{k\dots l}$ and $\hat{b}^{i\dots j}_{k\dots l}$ are obtained as the value of the solution for $t = t_2$; so in brief

$$\dot{A}^{i\dots j}_{k\dots l}(t) = S^{i\dots jc\dots d}_{k\dots la\dots b}(t) A^{a\dots b}_{c\dots d}(t) \qquad A^{i\dots j}_{k\dots l}(t_1) = \hat{a}^{i\dots j}_{k\dots l} \qquad \hat{b}^{i\dots j}_{k\dots l} := A^{i\dots j}_{k\dots l}(t_2)$$

(ii) if the assignment $\hat{a} \mapsto \hat{b}$ is interpreted as a map $\tau^{\gamma}_{y,x} : T^p_{qx} M \to T^p_{qy} M$, then the map (the *operator of parallel transport*) is *linear* and it satisfies the requirement

$$\tau^{\gamma}_{z,y} \circ \tau^{\gamma}_{y,x} = \tau^{\gamma}_{z,x} \qquad x, y, z \text{ on the curve } \gamma \text{ (otherwise arbitrary)}$$

(iii) the parallel transport depends on the *path* rather than on the *curve*.

Hint: just like in (15.2.6). □

• In Section 4.4 we learned that the operator of Lie transport Φ_t^* may be expressed in the form of the exponent of the Lie derivative, $\Phi_t^* = e^{t\mathcal{L}_W}$. There is a similar possibility also for the parallel transport and the covariant derivative, since the formula stems from the composition property of the transport, being valid in *both* cases under consideration.

15.2.13 Let $\gamma(t)$ be the integral curve of a field W. Denote by τ_t^γ the operator of parallel transport *backwards* along γ by the parametric distance t, i.e.

$$\tau_t^\gamma := \tau_{\gamma(s),\gamma(s+t)}^\gamma$$

for any s. Show that

(i) τ_t^γ has the composition property

$$\tau_{t+s}^\gamma = \tau_t^\gamma \circ \tau_s^\gamma$$

(ii) the covariant derivative may be expressed as

$$\nabla_W A = \left.\frac{d}{ds}\right|_0 \tau_s^\gamma A$$

(iii) for the derivative of τ_t^γ with respect to t there holds

$$\frac{d}{dt}\tau_t^\gamma = \tau_t^\gamma \circ \nabla_W$$

(iv) for C^ω tensor fields we may write

$$\tau_t^\gamma = e^{t\nabla_W} \equiv 1 + t\nabla_W + \frac{t^2}{2!}\nabla_W\nabla_W + \cdots$$

(v) the ordinary Taylor expansion of a function

$$\psi(x+t) = \psi(x) + t\psi'(x) + \frac{t^2}{2!}\psi''(x) + \cdots$$

may be regarded as a special case for $(M, \nabla) = (\mathbb{R}[x],$ *arbitrary* connection on $\mathbb{R})$, $W = \partial_x$.

Hint: (i) (15.2.6) and (15.2.12); (iii) $\frac{d}{dt}\tau_t^\gamma = \frac{d}{ds}\big|_{s=0}\tau_{t+s}^\gamma$; (iv) $(\frac{d}{dt})^n\tau_t^\gamma = \cdots = \tau_t^\gamma(\nabla_W)^n$, (4.4.2); (v) (4.4.1). □

• This expression enables one, just like in the case of the Lie derivative, to perform a systematic expansion of the operator of *infinitesimal* parallel transport τ_ε^γ in terms of powers of ε; for example, to within second-order accuracy in ε we have $\tau_\varepsilon^\gamma = e^{\varepsilon\nabla_W} \equiv 1 + \varepsilon\nabla_W + \frac{\varepsilon^2}{2!}\nabla_W\nabla_W$. This will be used for the study of the relation between the *curvature* and the dependence of the parallel transport on a path in Section 15.5.

15.3 Compatibility with metric, RLC connection

• All the particular examples of parallel transport which we mentioned in Section 15.1, namely in E^2 and E^3 as well as on the sphere S^2, shared a common property: the vectors preserve the *length* under the transport. This means, however, that we actually treat the manifolds (M, g, ∇) endowed with a *pair* of structures, the metric tensor g, which enables us to measure the lengths of the vectors and the linear connection ∇, which enables us to transport the vectors along paths. The invariance of the length of vectors under parallel transport means that the connection is *compatible* with the metric, or in short that we treat the *metric connection*. In the component language this may also be stated as that for some particular Christoffel symbols $\Gamma^i_{jk}(x)$, being dependent on given $g_{ij}(x)$, a computation of the change of the length of an arbitrary vector under parallel transport yields zero. Let us focus our attention on this fact in more detail, now.

15.3.1 Consider a vector v at a point x on a curve γ. Starting from v, generate an autoparallel field V ($\nabla_{\dot\gamma} V = 0$). Check that

(i) the requirement of preservation of the length of v by parallel transport may be stated as

$$\nabla_{\dot\gamma}(g(V, V)) = 0 \qquad \text{if} \quad \nabla_{\dot\gamma} V = 0$$

(ii) if this is to be true for an *arbitrary* curve γ and an *arbitrary* initial vector v, then for any *two* vector fields W, V one should demand

$$\nabla_W(g(V, V)) = 0 \qquad \text{if} \quad \nabla_W V = 0$$

(iii) if this is to be true for any two *equal* arguments V, V, it should also be true for *any* two (possibly different) arguments,[291] i.e. for any *three* vector fields W, V, U the covariant derivative should obey

$$\nabla_W(g(V, U)) = 0 \qquad \text{if} \quad \nabla_W V = 0 = \nabla_W U$$

(iv) this condition is equivalent to the requirement

$$\nabla g = 0 \qquad \text{or in local coordinates} \qquad g_{ij;k} = 0$$

A connection ∇ which satisfies this equation is called the *metric connection*.

Hint: (i) the expression $f(t) := g(V(t), V(t))$ is a function on the curve and $\dot f = \dot\gamma f = \nabla_{\dot\gamma} f$; (iii) $g(U+V, U+V) = g(U,U) + g(V,V) + 2g(U,V)$; (iv) $(\nabla g)(V, U, W) = (\nabla_W g)(V, U)$ and $g(V, U) = CC(g \otimes V \otimes U) \;\Rightarrow\; \nabla_W(g(V,U)) = (\nabla_W g)(V, U) + g(\nabla_W V, U) + g(V, \nabla_W U)$. □

15.3.2 Check that the requirement

$$g_{ij;k} = 0$$

represents $n^2(n+1)/2$ constraints imposed on n^3 functions (the Christoffel symbols $\Gamma^i_{jk}(x)$), so that it is very promising; it even seems that one could satisfy an *additional* $n^2(n-1)/2$ constraints.

Hint: $g_{ij} = g_{ji}$. □

[291] Preserving of all lengths under the parallel transport thus also automatically leads to the preserving of all angles between the vectors.

• It turns out that this naive counting of the "degrees of freedom" indeed leads to a true conclusion. A metric indeed induces a lot of compatible linear connections. Moreover, if one *adds one more* condition, the vanishing *torsion* of the connection, the solution is even unique. So let us first explain what exactly the torsion is and then prove the main result indicated above.

15.3.3 Let (M, ∇) be a manifold with a linear connection. Check that

(i) the map

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad T(U, V) := \nabla_U V - \nabla_V U - [U, V]$$

is $\mathcal{F}(M)$-linear in both arguments, so that actually a tensor field of type $\binom{1}{2}$, which is associated with the connection ∇, is defined by this rule; it is called the *torsion tensor*[292] (or briefly the *torsion*) connection ∇

(ii) the tensor is antisymmetric (in the lower indices)

$$T(U, V) = -T(V, U) \qquad \text{i.e.} \quad T^i_{jk} = -T^i_{kj}$$

and so it has $n^2(n-1)/2$ independent components

(iii) for its (coordinate) components one obtains the expression

$$\langle dx^i, T(\partial_j, \partial_k)\rangle \equiv T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk} \equiv -2\Gamma^i_{[jk]} \qquad \text{i.e.} \quad \Gamma^i_{jk} = \Gamma^i_{(jk)} - \frac{1}{2}T^i_{jk}$$

(iv) if the torsion of the connection *vanishes*, i.e. if

$$\nabla_U V - \nabla_V U = [U, V]$$

then the Christoffel symbols are *symmetric* in the lower indices

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

this is the motivation to call it the *symmetric connection*

(v) the coefficients of a symmetric connection Γ^a_{bc} with respect to a *non-holonomic* basis e_a *are not* symmetric in the lower indices.[293]

Hint: (v) (non-vanishing) coefficients of *anholonomy* (see (9.2.10)) enter the formula. □

• Each linear connection is thus characterized (also) by its torsion and, in particular, the torsion of the symmetric connection (by definition) vanishes (the connection is then said to be *torsion-free*). If the connection is required to be at the same time metric and symmetric, it imposes altogether n^3 constraints on n^3 functions Γ^i_{jk}. This "rule of thumb" calculation indicates that the connection with this property might be unique.

15.3.4 Show that there is a unique connection which is simultaneously metric and symmetric. In order to do this check step by step that

[292] Geometrical meaning of the torsion is studied in (15.8.1).

[293] Since the coefficients of a connection *do not* constitute the components of a tensor, it *may* happen that they are symmetric with respect to one basis, but they are not symmetric in another one. It may even happen, just as is true for the anholonomy coefficients, that they *vanish* in one basis, but they do not vanish in another one; see, for example, (15.3.5).

(i) the Christoffel symbols of the *first kind*[294]

$$\Gamma_{ijk} := g_{il}\Gamma^l_{jk}$$

of the connection which is metric and symmetric satisfy

$$\begin{aligned} \Gamma_{ijk} + \Gamma_{jik} &= g_{ij,k} && \text{since it is metric} \\ \Gamma_{ijk} - \Gamma_{ikj} &= 0 && \text{since it is symmetric} \end{aligned}$$

(ii) the two relations result in

$$g_{ij,k} + g_{ik,j} - g_{jk,i} = 2\Gamma_{ijk}$$

and eventually

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})$$

so that the requirement of being metric and symmetric indeed leads to the unique result for the Christoffel symbols of the connection.[295] This distinguished linear connection on a Riemannian manifold is usually called the *Riemann connection* or the *Levi-Civita connection*; we will therefore use the abbreviation *RLC connection*[296]

(iii) the non-coordinate definition of the RLC connection reads

$$\begin{aligned} g(\nabla_U V, W) := \frac{1}{2}\{&Ug(V, W) + Vg(U, W) - Wg(U, V) \\ &+ g([U, V], W) - g([U, W], V) - g(U, [V, W])\} \end{aligned}$$

which is to be regarded as a definition of the expression $\nabla_U V$ in terms of the right-hand side, where no covariant derivative occurs.

Hint: (i) (15.3.1), (15.2.10) and (15.3.3); (iii) check the $\mathcal{F}$-linearity of the right-hand side and set the coordinate basis as U, V, W. □

• Since we already created a stockpile of the manifolds endowed with the metric, the formulas obtained in this problem enable us to examine everything concerning connections on real examples.

15.3.5 Compute the Christoffel symbols of the RLC connection in E^n directly from the formula in (15.3.4) and check that

(i) for arbitrary n we obtain in Cartesian coordinates $\Gamma^i_{jk} = 0$
(ii) for $n = 2$ in polar coordinates the only non-vanishing gammas read[297]

$$\Gamma^r_{\varphi\varphi} = -r \qquad \Gamma^\varphi_{r\varphi} = 1/r$$

(iii) the same result for polar coordinates may be also obtained by transforming the Cartesian Christoffel symbols (which are zero according to item (i)) to the polar coordinates by means of (15.2.3)

294 Since $\Gamma^i_{jk}(x)$ *do not* constitute the components of a tensor, this *is not* the operation of the lowering of the index; it is indeed a *definition*.
295 One may check that the transformational properties of $g_{ij}(x)$ under the change of coordinates yield the proper (15.2.3) transformational properties of $\Gamma^i_{jk}(x)$.
296 Its role in the analysis of RLC circuits in electronics still remains obscure.
297 Due to the symmetry we do not list Γ^i_{kj} explicitly, if Γ^i_{jk} is already there.

(iv) for $n = 3$ in spherical polar coordinates the only non-vanishing gammas are

$$\Gamma^r_{\vartheta\vartheta} = -r \qquad \Gamma^\vartheta_{r\vartheta} = 1/r \qquad \Gamma^\varphi_{r\varphi} = 1/r$$
$$\Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta$$

and in cylindrical coordinates one gets (just like in polar coordinates in the plane)

$$\Gamma^r_{\varphi\varphi} = -r \qquad \Gamma^\varphi_{r\varphi} = 1/r.$$

□

15.3.6 Check that these expressions for Christoffel symbols in E^n result in a common concept of the parallel transport rule of vectors in the Euclidean space E^n (the vectors are just shifted with no change either of length or direction).

Hint: according to (15.2.6) and (15.3.5) the equations of parallel transport along *any* curve read in *Cartesian* coordinates $\dot{V}^i = 0$, whence $V^i(t) = \text{constant}$. □

15.3.7 Use the formula obtained in (15.3.4) to compute the Christoffel symbols of the RLC connection on the sphere S^2 with the standard metric; check that in coordinates ϑ, φ the only non-vanishing symbols are

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \qquad \Gamma^\varphi_{\vartheta\varphi} = \frac{\cos\vartheta}{\sin\vartheta}$$

Hint: see (3.2.4). □

15.3.8 Check that on the sphere S^2 with the standard metric the equations of parallel transport of a vector V read as follows:

(i) along a general curve $\vartheta(t), \varphi(t)$

$$\dot{V}^\vartheta - \dot{\varphi}\sin\vartheta\cos\vartheta\, V^\varphi = 0 \qquad \dot{V}^\varphi + \frac{\cos\vartheta}{\sin\vartheta}(\dot{\vartheta}V^\varphi + \dot{\varphi}V^\vartheta) = 0$$

(ii) along a parallel (of latitude) parametrized as $\vartheta(t) = \vartheta_0, \varphi(t) = t$

$$\dot{V}^\vartheta - \sin\vartheta_0\cos\vartheta_0\, V^\varphi = 0 \qquad \dot{V}^\varphi + \frac{\cos\vartheta_0}{\sin\vartheta_0}V^\vartheta = 0$$

and, in particular, along the equator

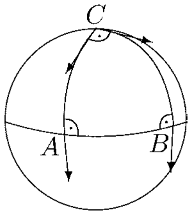
$$\dot{V}^\vartheta = 0 \qquad \dot{V}^\varphi = 0$$

(iii) along a meridian parametrized as $\vartheta(t) = t, \varphi(t) = \varphi_0$

$$\dot{V}^\vartheta = 0 \qquad \dot{V}^\varphi + \frac{\cos t}{\sin t}V^\varphi = 0$$

Hint: (i) see (15.2.6). □

15.3.9 Let us test the equations from (15.3.8) and check the result, which is well known from the pictures in popular books trying to illustrate the subtleties of parallel transport in "curved spaces." Namely, the parallel transport of a vector around a *right spherical* triangle.



So let ABC be a right triangle on the sphere S^2 with the vertices being the north pole ($= C$) and two points at the equator, the point B lying a quarter of the equator's perimeter eastwards from A. Check that if we transport (in the sense of the RLC connection on the sphere) along the route $C \to A \to B \to C$ a vector which is directed to the point A at the beginning, the transported vector is rotated by $\pi/2$ counterclockwise with respect to the initial one (so that it has the same length and is directed to the point B) and if we traversed the same route in the opposite direction (along $C \to B \to A \to C$), then it is rotated by the same angle clockwise.

Hint: the result itself is clear immediately *without any computation* from the *metric compatibility* of the RLC connection: the transported vector must not change its length as well as the angle it makes with the line along which it is transported. Concerning the equations, the "singularity" in a neighborhood of the point C (the coordinates happen to be defective there) may be "healed" by a substitution of the edge of the triangle by an infinitesimal quarter-circle $\vartheta = \epsilon$. The three "long" parts are computed trivially, on this short one the equations linearize (due to $\epsilon \ll 1$) to $\dot{V}^\vartheta = -\epsilon V^\varphi$, $\dot{V}^\varphi = (1/\epsilon)V^\vartheta$ and they are easily solved; altogether one obtains that $k\partial_\vartheta \mapsto \frac{k}{\epsilon}\partial_\varphi$, which is just what is needed. □

15.3.10 Check that if a vector is transported around the parallel line $\vartheta(t) = \vartheta_0$ on the sphere[298] S^2, it ends up rotated by the angle $\beta_\parallel = 2\pi \cos\vartheta_0$ with respect to its initial direction. Around this parallel of the *fictitious non-rotating* globe an arbitrary object passes in just one day, which is at rest on the *real rotating* globe. In particular, this also holds for a *Foucault pendulum* that is observed somewhere on the Earth. Check that the angle β_{Fouc} of the rotation of the plane in which it swings *coincides* with $\beta_\parallel$

$$\beta_\parallel = \beta_{\text{Fouc}} = 2\pi \cos\vartheta_0$$

(and also in detail the angle of the rotation of the Foucault pendulum due to the shift along the parallel line by a small angle $\delta\varphi$ is $\delta\varphi \cos\vartheta_0$, coinciding with the angle of the rotation of a vector due to the parallel transport along the same trajectory). What does this result say about the Foucault pendulum?

Hint: see (15.3.8); (a vector in the direction of the swinging undergoes parallel[299] transport).

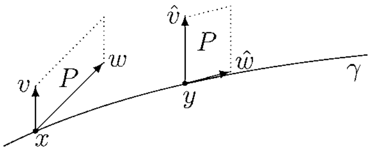
□

15.3.11 Let (M, ω, ∇) be an n-dimensional manifold endowed with a volume form and a linear connection. Given n vectors $v, \dots, w$ at a point, the volume of the parallelepiped spanned by them is $\omega(v, \dots, w)$. A parallel transport of the vectors results in n new vectors $\hat{v}, \dots, \hat{w}$ (at a different point) and the corresponding new volume $\omega(\hat{v}, \dots, \hat{w})$. The two

[298] The angle is measured *from the z-axis*, the standard *latitude* α is measured *from the equator*; therefore $\sin\alpha = \cos\vartheta_0$.

[299] "The vector of swinging" tries to remain parallel in the "ambient" space E^3, but the situation continually forces it to "project" into the tangent plane to the sphere ≡ the Earth; one can prove that this is exactly the way in which the RLC connection works with respect to the *induced* metric.

structures are said to be compatible if an arbitrary parallel transport preserves the volume of each such parallelepiped. Show that



(i) the condition of compatibility of the structures may be expressed in the form

$$\nabla\omega = 0 \qquad \text{and in local coordinates} \qquad \omega_{i\dots j;k} = 0$$

(ii) if in local coordinates $\omega = f dx^1 \wedge \dots \wedge dx^n$ (i.e. $\omega_{i\dots j} = f\epsilon_{i\dots j}$), then the condition from item (i) relates f and Γ^i_{jk} by

$$f_{,k} = f\Gamma^i_{ik} \qquad \text{i.e.} \quad (\ln f)_{,k} = \Gamma^i_{ik}$$

(iii) the RLC connection *is* compatible with the metric volume form ω_g, i.e. the RLC Christoffel symbols obey[300]

$$(\ln\sqrt{|g|})_{,k} = \Gamma^i_{ik}$$

Hint: (i) like in (15.3.1); (ii) write down explicitly $\omega_{i\dots j;k} = 0$ and use (5.6.4); (iii) according to (5.6.7) we have $\partial g/\partial g_{ij} = g g^{ji}$ ($g \equiv \det g$); it is convenient to use the machinery of connection forms (see Section 15.6) and to write in an orthonormal basis $\nabla_V \omega_g = \nabla_V(e^1 \wedge \dots \wedge e^n) = (\nabla_V e^1) \wedge \dots \wedge e^n + \dots + e^1 \wedge \dots \wedge (\nabla_V e^n) = \dots = -\omega^a_a(V)(e^1 \wedge \dots \wedge e^n) = 0$ due to $\omega_{ab} = -\omega_{ba}$ (15.6.6). □

• Let us have a look at some practical manipulations with the coordinate expressions containing covariant derivatives.[301]

15.3.12 Check that

(i) the "semicolon" operation (just like the "colon" operation, the ordinary partial derivative) is linear and on a product it behaves according to the Leibniz rule; so, for example,

$$\left(A^i_{jk} + \lambda B^i_{jk}\right)_{;l} = A^i_{jk;l} + \lambda B^i_{jk;l} \qquad \left(A^i_j B^k_{lm}\right)_{;n} = A^i_{j;n} B^k_{lm} + A^i_j B^k_{lm;n}$$

(ii) also the inverse metric tensor g^{ij} is covariantly constant with respect to the metric connection, i.e.

$$g_{ij;k} = 0 \quad \Rightarrow \quad g^{ij}{}_{;k} = 0$$

(iii) the semicolon operation in the sense of the *metric* connection (in particular, RLC) commutes with the raising and lowering of indices; e.g.

$$(g^{ij} A_{jl})_{;k} = g^{ij} A_{jl;k}$$

Hint: (i) this is the behavior of ∇_i, see (15.2.9); (ii) $\nabla\hat{1} = 0$ (commuting with contractions), i.e. $\delta^i_{j;k} = 0$; (iii) both $\sharp_g$ and $\flat_g$ are combinations of the tensor product with the (covariantly

[300] It is clear intuitively that if parallel transport preserves the scalar products (consequently, also the unit cube) then it also preserves the volume, since the volume form ω_g is just "tuned" to the unit cube (see Section 5.7). The compatibility of the pairs metric ↔ volume form and metric ↔ connection thus results automatically in the compatibility of the pair volume form ↔ connection.

[301] This may be regarded as a continuation of the exercises of the *index gymnastics* from (2.4.14) and (5.2.6) (see footnote 50), which is made possible by the addition of a further popular gymnastic apparatus, the semicolon.

constant) tensors g or g^{-1} and contractions; e.g. $\nabla_W(\flat_g V) = \nabla_W(C(g \otimes V)) = C((\nabla_W g) \otimes V) + C(g \otimes \nabla_W V) = \flat_g(\nabla_W V)$. □

• For the computations of the covariant derivatives of *forms* it is sometimes fairly useful to realize how the operator ∇_V behaves with respect to the Hodge star $*_g$. For the *metric* connection the simplest possible behavior takes place.

15.3.13 Check that the operator of the covariant derivative ∇_V with respect to the metric connection ∇ *commutes* with the operator of dualization $*_g$

$$[\nabla_V, *_g] = 0$$

Hint: realize that $*_g$ is composed from the operations of the raising of indices, contractions and the tensor product with the (covariantly constant, $\nabla_V \omega_g = 0$) volume form: $\nabla_V *_g \alpha \sim \nabla_V\{C \dots C((\sharp_g \dots \sharp_g \alpha) \otimes \omega_g)\} = C \dots C((\sharp_g \dots \sharp_g \nabla_V \alpha) \otimes \omega_g) \sim *_g \nabla_V \alpha$. □

15.3.14* Consider a connection which is metric, yet not necessarily symmetric. Generalize the results of problem (15.3.4) for this case. In particular, check that

(i) the Christoffel symbols of the first kind of the connection with a given torsion satisfy

$$\begin{aligned} \Gamma_{ijk} + \Gamma_{jik} &= g_{ij,k} && \text{since it is metric} \\ \Gamma_{ijk} - \Gamma_{ikj} &= -T_{ijk} && \text{from the definition of the torsion} \end{aligned}$$

(ii) the two relations yield

$$g_{ij,k} + g_{ik,j} - g_{jk,i} = 2\Gamma_{i(jk)} + (T_{jki} + T_{kji})$$

from where

$$\Gamma_{i(jk)} = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) - \frac{1}{2}(T_{jki} + T_{kji})$$

and eventually

$$\Gamma^i_{jk} = \Gamma^i_{(jk)} - \frac{1}{2}T^i{}_{jk} = \frac{1}{2}g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}) - \frac{1}{2}\left(T_{jk}{}^i + T_{kj}{}^i + T^i{}_{jk}\right)$$

The metricity plus the prescribed torsion thus result in a unique expression for the Christoffel symbols of the sought connection.[302] (The torsion being *zero*, we return to the RLC connection.)

Hint: (i) (15.3.1), (15.2.10) and (15.3.3); (iii) set the coordinate basis for U, V, W. □

15.4 Geodesics

• Now, having been equipped with the machinery of the linear connection, we may return to the concept which opened the chapter, the concept of acceleration. If we realize what is actually performed with the velocity field defined on a curve in order to compute the acceleration, we can immediately conclude that the acceleration at a given point on the

[302] Note that the *symmetric* part of the Christoffel symbols is not given by the expression for the RLC connection *alone*, but rather it contains in addition a part composed of (the tensor) $-T_{(jk)}{}^i$.

curve is nothing but the *absolute* derivative of the velocity field $\dot{\gamma}$ along the curve, or in terms of (15.2.4) the *covariant* derivative of the velocity along the velocity itself[303]

$$a = \nabla_{\dot{\gamma}}\dot{\gamma} = \nabla_v v \qquad v := \dot{\gamma} = \text{the velocity vector}$$

A case of particular interest arises when the acceleration *vanishes*. This is good old *uniform straight-line* motion. The corresponding curve on (M, ∇) is thus characterized by the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$; it is called an (affinely parametrized) *geodesic* and it represents the most reasonable realization of the concept of a "straight line" (together with a particular "speed" of the motion along the line) on a general manifold with a linear connection.

15.4.1 Let γ be an affinely parametrized geodesic on (M, ∇). Show that

(i) in local coordinates we have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \quad \leftrightarrow \quad \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \quad \text{the } \textit{geodesic equation}$$

so that we get a system of n ordinary quasi-linear second-order differential equations for the unknown functions $x^i(t)$

(ii) the geodesic feels only the symmetric part of the Christoffel symbols[304]

(iii) the geodesics in E^n, when expressed in Cartesian coordinates, are just the curves

$$x^i(t) = x^i_0 + v^i t \qquad v^i \equiv \dot{x}^i(0), x^i_0 \equiv x^i(0)$$

(iv) in the general case the first two terms of the expansion in t of the coordinate presentation of a geodesic read

$$x^i(t) = x^i_0 + v^i t - \frac{1}{2}\Gamma^i_{jk}v^j v^k t^2 + \cdots$$
$$v^i \equiv \dot{x}^i(0), \quad x^i_0 \equiv x^i(0), \quad \Gamma^i_{jk} \equiv \Gamma^i_{jk}\left(x^i_0\right)$$

Hint: (i) see (15.2.5) for $V = \dot{\gamma}$; (iii) see (15.3.5); (iv) $x^i(t) = x^i(0) + \dot{x}^i(0)t + \frac{1}{2}\ddot{x}^i(0)t^2 + \cdots$. □

15.4.2 Let $(M, \nabla) = (S^2, \nabla_{\text{RLC}})$. Check that

(i) the acceleration corresponding to the uniform motion along a meridian is zero

(ii) the acceleration corresponding to the uniform motion along a parallel is *not* zero (even $a \not\parallel v$), except for the longest parallel = *the equator* (and trivially also for the opposite extreme "parallel," staying still at any pole)

(iii) all the meridians are geodesics, the only parallel which happens to be a geodesic is the equator; in general the only geodesics on the sphere are the *great circles* (the circles with the maximum possible radius; trivially also the curve which represents standing still at any point).

Hint: according to the results from (15.3.7) we get: (i) $a \sim \nabla_\vartheta \partial_\vartheta = 0$; (ii) $a \sim \nabla_\varphi \partial_\varphi = -\sin\vartheta\cos\vartheta\,\partial_\vartheta$. □

[303] So *the dot* in the expression $\mathbf{a} = \dot{\mathbf{v}}$, corresponding to the rate of change of the vector $\mathbf{v}$, implicitly contains its parallel transport; thus it is actually the *covariant* derivative.

[304] If the torsion of the connection does not vanish, it contributes to the symmetric part (15.3.14). Then if there are two connections, both of them being metric (with respect to the same g), differing, however, in the torsion, they will generate in general *different* families of geodesics – see an example in (15.8.3).

• Now let us concentrate on the issue of the parametrization. One may also traverse the path which corresponds to the uniform straight-line motion non-uniformly. Although the acceleration does not vanish in this case, it is rather specific, being at each point tangent to the curve, i.e. proportional to the velocity $\nabla_{\dot{\gamma}}\dot{\gamma} \sim \dot{\gamma}$. From the opposite point of view, a curve which satisfies the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = f(t)\dot{\gamma}$ is also straight if regarded as a path, only the motion along this path fails to be uniform. Therefore such curves are called geodesics as well, being, however, not affinely parametrized.

15.4.3 Let γ be an affinely parametrized geodesic and let $\hat{\gamma} := \gamma \circ \sigma$ be a reparametrized curve, $\hat{\gamma}(t) := \gamma(\sigma(t))$, $\sigma'(t) > 0$. Check that

(i)

$$\nabla_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \sigma''\dot{\hat{\gamma}}$$

(ii) the affine reparametrization $\sigma(t) = at + b$ (and no other one) does not spoil the affine parametrization of a geodesic
(iii) by means of a unique affine reparametrization one can "tune" a given geodesic to two given points P, Q on it (not too far from each other) so as to satisfy $P = \gamma(0)$ and $Q = \gamma(1)$; now the points P, Q are at a *parametric* distance $= 1$ from each other and the point P is its "origin"
(iv) if a geodesic $\gamma(t)$ turns out to be "badly" (non-affinely) parametrized, one can always make a reparametrization such that the new geodesic $\bar{\gamma}$ is already affinely parametrized.

Hint: (i) (2.3.5) $\Rightarrow \nabla_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \sigma'(\sigma''\dot{\gamma} + \sigma'\nabla_{\dot{\gamma}}\dot{\gamma})$; (ii) we need $\sigma'' = 0$; (iii) we need to map $(t_1, t_2) \mapsto (0, 1)$ by means of $t \mapsto at + b$; (iv) if $\nabla_{\dot{\gamma}}\dot{\gamma} = f(t)\dot{\gamma}$, then for $\gamma := \bar{\gamma} \circ s$ ($s = s(t) \Rightarrow \dot{\gamma} = s'\dot{\bar{\gamma}}$) we get $\nabla_{\dot{\gamma}}\dot{\gamma} = s''\dot{\bar{\gamma}} + (s')^2\nabla_{\dot{\bar{\gamma}}}\dot{\bar{\gamma}} = fs'\dot{\bar{\gamma}}$ so that to reach $\nabla_{\dot{\bar{\gamma}}}\dot{\bar{\gamma}} = 0$ it is enough to solve $s'' = s'f$ (to find $s(t)$ for given $f(t)$), which is easy. □

• So we see that the parametrization which is optimal from the point of view of the simplicity of the equations (the affine one) can always be achieved. Therefore we will automatically understand an *affinely parametrized* geodesic when speaking about a geodesic from now on and we will specially point out if this will not be the case.

The procedure of finding geodesics of the *RLC connection* as we learned up to now is fairly lengthy and laborious. Fortunately, there is a convenient alternative way, which is based on the Lagrange equations from analytical mechanics. The steps performed in both approaches may be summarized as follows:

$$g \mapsto \Gamma \mapsto \ddot{x} + \Gamma\dot{x}\dot{x} = 0 \mapsto x(t) \qquad \text{straightforward approach}$$
$$g \mapsto L \mapsto \mathcal{E} \mapsto x(t) \qquad \text{Lagrangian approach}$$

(see the problem). For a given g, the Lagrangian approach actually turns out to be the easiest and quickest way for

1. finding the explicit form of the equations for geodesics
2. finding the solutions of the equations
3. finding the explicit expressions for the Christoffel symbols themselves

(the main source of the power of the new approach concerning item 2 lies in making use of well-known tricks from the Lagrangian machinery – a relation between the cyclic coordinates and the conservation laws, see (18.4.2)).

15.4.4 Let (M, g) be a Riemannian manifold, γ a curve on M and $\nabla = \nabla_{\text{RLC}}$ the RLC connection corresponding to g. Then the functional

$$S[\gamma] := \frac{1}{2}\int g(\dot\gamma, \dot\gamma)\, dt \equiv \int L\, dt \qquad L(x, \dot x) = \frac{1}{2} g_{ij}(x)\dot x^i \dot x^j$$

may be regarded as an *action integral* for the *free* motion $\gamma(t)$, since the Lagrangian L contains the kinetic energy alone ($L = T$). Check that

(i) the *Euler–Lagrange expression* $\mathcal{E}_i$ corresponding to this Lagrangian is[305]

$$\mathcal{E}_i(x, \dot x) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x^i} = -g_{ij}\left(\ddot x^j + \Gamma^j_{kl}\dot x^k \dot x^l\right) \equiv -g_{ij}(\nabla_{\dot\gamma}\dot\gamma)^j$$

(ii) the Lagrange equations are equivalent to the geodesic equations

$$\mathcal{E}_i(x, \dot x) = 0 \quad \Leftrightarrow \quad \nabla_{\dot\gamma}\dot\gamma = 0$$

(iii) the explicit form of the Lagrange equations for this Lagrangian enables one to immediately read off the Christoffel symbols Γ^i_{jk}.

Hint: (i) see (15.3.4); (ii) g_{ij} is non-singular; (iii) the antisymmetric part of Γ^i_{jk} vanishes for the RLC connection and the symmetric part may be read off from $\Gamma^j_{kl}\dot x^k \dot x^l$. □

15.4.5 Let (T^2, g) be the torus in E^3 with the induced metric. Check by plugging into the equations that the following curves happen to be geodesics and draw the corresponding pictures:

(i) $\psi(t) = kt$, $\varphi(t) = \varphi_0$
(ii) $\psi(t) = \psi_0$, $\varphi(t) = kt$ for particular values of ψ_0 (which ones?).

Hint: see (3.2.2), $L = \frac{1}{2}[(a + b\sin\psi)^2\dot\varphi^2 + b^2\dot\psi^2]$. □

• Consider now a more general Lagrangian, also containing the potential energy, $L = T - U$. The motion deviates from a "straight line" due to the force corresponding to U.

15.4.6 Let the action integral be

$$S[\gamma] := \int L\, dt \qquad L(x, \dot x) = \frac{1}{2} g_{ij}(x)\dot x^i \dot x^j - U(x) \equiv T(x, \dot x) - U(x)$$

Check that

(i) the Euler–Lagrange expression, corresponding to this Lagrangian, comes out as

$$\mathcal{E}_i(x, \dot x) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x^i} = -g_{ij}(\nabla_{\dot\gamma}\dot\gamma)^j - U_{,i}$$

[305] For a coordinate-free derivation see (15.4.16).

(ii) the Lagrange equations are equivalent to

$$\mathcal{E}_i(x, \dot{x}) = 0 \quad \Leftrightarrow \quad \nabla_{\dot{\gamma}} \dot{\gamma} = -\sharp_g dU$$

so that the motion is no longer along a geodesic in general, but rather it has an acceleration of $-\text{grad}\, U$.

Hint: see (15.4.4). □

15.4.7 We know from analytical mechanics that the *Lagrange equations* (of the second kind) in general (even if there does not exist any potential energy) read

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} - \frac{\partial T}{\partial x^i} = Q_i \qquad T = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j$$

where Q_i is the ith *generalized force* and T is the kinetic energy of a system.

(i) Check that their coordinate-free version is

$$a \equiv \nabla_{\dot{\gamma}} \dot{\gamma} = Q$$

with Q being a force (vector) field $Q = Q^i \partial_i$, $Q^i = g^{ij} Q_j$, so that actually they represent the "Newton" equation[306]

"acceleration = force"

on a configuration manifold $(M, g, \nabla_{\text{RLC}})$

(ii) according to textbooks of analytical mechanics the generalized force is computed by the formula

$$Q_i := \sum_{k=1}^{N} \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial x^i}$$

where $\mathbf{r}_k(x^1, \dots x^n)$ represent the parametrization of the positions of individual point masses in terms of the generalized coordinates x^i and $\mathbf{F}_k$ is the force acting on the kth point mass. Check that if this parametrization is regarded as a map $f : M \to \mathbb{R}^{3N}$, then the expression for Q_i is nothing but a component expression of the *pull-back* of the force (as a covector field) from $\mathbb{R}^{3N}$ to the configuration space M (see also (3.2.9)). □

• Our lifelong experience results in a clear feeling that the *shortest* path connecting two points is the *straight* path. This experience stems from the particular spaces E^3 or E^2. Now we are in a position, however, to investigate the relation between the straight and the shortest lines on an arbitrary Riemannian manifold. Since Section 2.6 we can compute the lengths of curves and now we have learned that the straight lines are the geodesics. So the question is whether geodesics (regarded as straight lines) happen to also be at the same time the shortest paths.

Right at the beginning we should realize that the length of a curve does not depend on the parametrization (2.6.9), so that the shortest path is indeed only a path (without

[306] With unit mass; recall that if a system of particles with various masses is under consideration, it is formally described as the motion of *a single* particle in a many-dimensional Riemannian (configuration) space, the masses being hidden in the *metric tensor* g; see (2.6.7) and (3.2.9).

parametrization). This means that even if we find that the shortest path turns out to be a geodesic, the result certainly may not come out as the *affinely parametrized* geodesic.

15.4.8 The *functional of the length* of a curve (see (2.6.9) and (4.6.1)) may be regarded as an action integral with the Lagrangian[307]

$$L(x,\dot{x}) := \sqrt{g_{ij}(x)\,\dot{x}^i\dot{x}^j}$$

Check that

(i) the Lagrange equations for this Lagrangian are

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \frac{\dot{L}}{L}\dot{x}^i \qquad \text{i.e. in a coordinate-free form} \qquad \nabla_{\dot{\gamma}}\dot{\gamma} = \chi\dot{\gamma}$$

where $\dot{L}$ is the "total" time derivative[308] of the Lagrangian L, Γ^i_{jk} correspond to the RLC connection and $\chi \equiv \dot{L}/L$

(ii) the shortest path is a geodesic

(iii) the affine parametrization of this geodesic is achieved in the *natural parameter* s, being a parameter such that its increment coincides with the increment of the actual length of the curve (i.e. the length of the path between the points $\gamma(s_1)$ and $\gamma(s_2)$ is simply $s_2 - s_1$)

(iv) if ψ is a non-zero function, then

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{\dot{\psi}}{\psi}\dot{\gamma} \quad \Leftrightarrow \quad \nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{\psi}\right) = 0$$

so that the equation for the shortest path may also be written in the form

$$\nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{L}\right) \equiv \nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{\sqrt{g_{ij}(x)\,\dot{x}^i\dot{x}^j}}\right) \equiv \nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{||\dot{\gamma}||}\right) = 0$$

(in the natural parameter we have $L \equiv \sqrt{g_{ij}(x)\,\dot{x}^i\dot{x}^j} \equiv ||\dot{\gamma}|| = 1$, so that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ then).

Hint: (i) see (15.3.4); (ii) see (15.4.3); (iii) if $\nabla_{\dot{\gamma}}\dot{\gamma} = (\dot{L}/L)\dot{\gamma}$, then the improving procedure from (15.4.3) yields the equation $s'' = s'L'/L$ with a solution $L = s'$, or $ds = L\,dt$; hence $s_2 - s_1 = \int_{t_1}^{t_2} L\,dt =$ *the length* of the curve. □

• By variation of the functional of the length of the curve we investigate in principle only its *stationary points* (*local* extrema or saddle points). In numerous simple particular cases it is intuitively clear what the situation looks like "globally." For example, consider two points A, B on a sphere, which do not happen to be opposite one another. If we join them by the shorter part of a great circle, we get the path with *minimal* length, whereas the complementary longer part of the great circle turns out to be only a saddle point of the functional of length, since any warp evidently results in its prolongation and its "rotation" along the sphere (with a view to deforming it step by step to the shorter part) makes it shorter

307 Note that this (reparametrization invariant) Lagrangian is (up to a factor of 2) a *square root* of the Lagrangian from problem (15.4.4), which *was not* reparametrization invariant and therefore it *could* yield as extremals the curves with a particular parametrization; see also (15.4.16).

308 The derivative of the function $L(x(t),\dot{x}(t))$ with respect to time, which takes into account the fact that the time enters through both $x(t)$ and $\dot{x}(t)$; it may be written in detail as $\dot{L} = \frac{\partial L}{\partial x^i}\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i$, but here it is more convenient to leave it as it is.

(as our intuition signals and an elementary computation confirms). In general, however, the issue of the global properties of the critical points of the length functional may be fairly complicated and they are beyond the scope of this book.

Additional complication refers to certain particular pairs of points (they are called *conjugate* points). If we were to choose, for example, the north and south poles as A, B on the sphere, there would be an infinite number of (globally) shortest paths (each meridian). This occurs only for certain exceptional pairs of points, being always "very far" to each other. It turns out that for each point $x \in M$ there is a neighborhood $\mathcal{O}$ (called the *geodesic neighborhood*), in which there is just one[309] shortest path leading to each $y \in \mathcal{O}$ from x (it is clearly a geodesic).

In this neighborhood one can introduce extremely useful coordinates, tailored to the linear connection; they are known as the *normal* coordinates. For their construction recall that a geodesic may be uniquely fixed either by a point where it is in some "time" and a tangent vector at that point (the position and the velocity at a *single time*) or by the positions at *two* instants of time.[310]

15.4.9 Denote by $\gamma_v(t)$ the geodesic starting at time zero from the point P with velocity v,

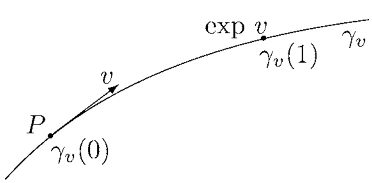
$$\gamma_v(0) = P \in M \qquad \dot{\gamma}_v(0) = v \in T_P M$$

Check that a simple relation holds

$$\gamma_v(bt) = \gamma_{bv}(t) \qquad b \in \mathbb{R}$$

or, put another way, a b-fold increase of the *initial* velocity results in a b-fold increase of the velocity along the *whole* trajectory (the motion takes place along the same path).

Hint: the curve $\hat{\gamma}(t) := \gamma_v(bt)$ is a geodesic ($\dot{\hat{\gamma}} = b\dot{\gamma}_v \Rightarrow \nabla_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \cdots = 0$) which satisfies $\hat{\gamma}(0) = P$, $\dot{\hat{\gamma}}(0) = bv \Rightarrow$ due to the uniqueness it necessarily coincides with $\gamma_{bv}(t)$; or alternatively: the operator of the parallel transport (of the velocity vector along the geodesic) is linear. □



15.4.10 On a manifold with connection (M, ∇) define the *exponential map* (centered at $P \in M$)

$$\exp : T_P M \to M \qquad v \mapsto \exp v := \gamma_v(1)$$

where $\gamma_v(t)$ is the geodesic from problem (15.4.9). So one assigns to a vector v the point from M to which we arrive at $t = 1$, if at time $t = 0$ we start from the point P with initial velocity v and all the while the motion is uniform and straight-line (i.e. along a geodesic). Check that

[309] This may be obtained as a result of an analysis of differential equations governing a geodesic. It is a second-order equation and a contemplation of additional conditions leading to a unique solution yields the conclusion mentioned above.

[310] Each of these input data evidently fixes the uniform straight-line motion; from a formal point of view a second-order system $\ddot{x} + \Gamma\dot{x}\dot{x} = 0$ needs either $x(t_0)$ and $\dot{x}(t_0)$, or $x(t_0)$ and $x(t_1)$, the second possibility being trouble-free only in the geodesic neighborhood of the point $\gamma(t_0)$.

(i)

$$\exp(v = 0) = P$$

(ii) the coordinate presentation of the exponential map reads

$$\exp : v^i \mapsto x^i(v^1, \dots, v^n) \equiv x^i(P) + v^i - \frac{1}{2}\Gamma^i_{jk}(P)v^j v^k + \cdots$$

(iii) $\exp_{*0}$ is a *non-degenerate* (i.e. its kernel vanishes) linear map, so that exp maps bijectively (*diffeomorphically*) some neighborhood of zero in $T_P M$ to some neighborhood of the point P

(iv) the uniform straight-line motion in the *tangent* space is mapped to the uniform straight-line motion *on a manifold* (i.e. along a geodesic)

$$\exp(vt) = \gamma_v(t)$$

Hint: (ii) $t = 1$ in the expression of a geodesic (15.4.1); (iii) the Jacobian matrix at zero is $J^i_j(0) = \delta^i_j$ (for small v^i it reduces to a *translation* $v^i \mapsto x^i_0 + v^i$); (iv) see (15.4.9). □

• The fact that a neighborhood of a point P may be diffeomorphically mapped on a neighborhood of the zero in a *linear* space $T_P M$ means in practice that we obtain *local coordinates* in the neighborhood of the point P. The most important property of the coordinates constructed in this particular way is the vanishing of all Christoffel symbols *in the point* P. This fact greatly simplifies numerous computations and proofs (actually all that is needed in doing so is only to be aware of their existence, there is no need to construct them explicitly).[311]

15.4.11 Let exp be the exponential map centered at $P \in M$. If in $T_P M$ an (arbitrary) basis e_i is fixed, we may introduce in a neighborhood of the point P *Riemann normal coordinates* by the prescription

$$x^i \leftrightarrow Q \quad \Leftrightarrow \quad Q = \exp(v) \equiv \exp(x^i e_i)$$

So a geodesic is constructed starting ($t = 0$) in P and passing at $t = 1$ through *the* point Q, which is to be assigned coordinates. The geodesic has the unique initial velocity v and this velocity in turn has components with respect to e_i; *these components* are declared (by definition) as the coordinates x^i. Check that in these coordinates

(i) the geodesic $\gamma_v(t)$ reads

$$x^i(t) = v^i t \qquad \text{if} \quad v = v^i e_i$$

(ii) for any *symmetric* connection (in particular, also RLC)

$$\Gamma^i_{jk}(P) = 0$$

(iii) in these coordinates there holds

$$g_{ij,k}(P) = 0 \qquad \text{so that in a neighborhood of } P \qquad g_{ij}(x) \doteq g_{ij}(P) + \frac{1}{2} g_{ij,kl}(P) x^k x^l$$

[311] Let us mention also that in general relativity these coordinates have a direct physical meaning as the coordinates with respect to a frame of reference which freely falls in a gravitational field (locally inertial frame), so that the action of the force due to the gravitational field (locally) vanishes.

(i.e. a linear term is missing in the expansion) and, in particular, for an *orthonormal* basis e_i the coordinate expression of the metric tensor in a neighborhood of the point P is

$$g_{ij}(x) = \eta_{ij} + K_{ijkl}x^k x^l + \cdots \qquad K_{ijkl} = K_{(ij)(kl)} = \text{ constant}$$

Hint: (i) according to (15.4.10) we have $\exp((v^i t)e_i) = \gamma_v(t)$; (ii) the general equation for a geodesic $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$ and the fact that here $x^i(t) = v^i t$ yields $\Gamma^i_{jk}(x(t))v^j v^k = 0$ on the whole geodesic $x(t)$; for $t = 0$ this gives $\Gamma^i_{jk}(P)v^j v^k = 0$ for *all* $v^i \Rightarrow \Gamma^i_{(jk)}(P) = 0$ (for $t \neq 0$ $\Gamma^i_{jk}(x(t))$ depends on v^i via $x(t)$ so that on different v^i actually different quadratic forms vanish, which *does not* allow one to deduce the vanishing of the forms); (iii) for an RLC connection it yields item (ii) and (15.3.4):

$$\begin{aligned} \Gamma_{ijk} + \Gamma_{jik} &= g_{ij,k} && \text{metric connection (holds in arbitrary coordinates)} \\ \Gamma_{ijk} - \Gamma_{ikj} &= 0 && \text{symmetric connection (holds in arbitrary coordinates)} \\ \Gamma_{ijk}(P) + \Gamma_{ikj}(P) &= 0 && \text{holds in normal coordinates centered at } P \end{aligned}$$

so that all $\Gamma_{jik}(P) = 0$ and then also $g_{ij,k}(P) = 0$. □

• As a simple illustration (see also (15.5.8)) of the use of these coordinates let us mention the following useful technicality, which holds for the coordinate computation of Lie and exterior derivatives.

15.4.12 Let α be a 1-form and V a vector field. Then in (arbitrary) coordinates we have

$$(\mathcal{L}_V\alpha)_i = \alpha_{i,j}V^j + V^j_{\ ,i}\alpha_j \qquad (d\alpha)_{ij} = -2\alpha_{[i,j]}$$

Check by a direct computation that

(i) if we substitute in these expressions *each* comma by a semicolon (the partial derivative by the covariant) in the sense of an arbitrary *symmetric* connection (in particular, also RLC), the expression actually does not change (the new terms pairwise cancel)
(ii) the same rule holds in general when the *Lie* derivative of an *arbitrary* tensor field as well as the *exterior* derivative of an *arbitrary* form are computed.

Hint: see (5.2.6), (6.2.5), (4.3.4), (15.2.9) and the symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$ (15.3.3). □

15.4.13 Check the validity of the general statement from (15.4.12) making use of the normal coordinates.

Hint: both expressions to be compared (with commas versus semicolons) are (a priori different) *tensor fields*,[312] the expression with semicolons containing additional terms with Christoffel symbols; in normal coordinates centered at $P \in M$ they coincide at the point P for any *symmetric* connection (15.4.11), so that at *this point* (being arbitrary) and in *these particular* coordinates the two tensors are indeed equal; the equality of two tensors does not, however, depend on the choice of the coordinates. □

[312] Consider, for example, $\alpha_{i,j}V^j + V^j_{\ ,i}\alpha_j$. This *is* a tensor field, since it is $(\mathcal{L}_V\alpha)_i$ (although neither of the two terms by itself is a tensor). After replacing commas by semicolons both terms become tensor fields even by themselves ($\alpha_{i;j}V^j = (\nabla_V\alpha)_i$ and $V^j_{\ ;i}\alpha_j = (\nabla V(\alpha))_i$) so that also their sum is all right.

15.4.14 Show that the *Killing equations* may also be written in terms of covariant derivatives in the sense of RLC connection and then take a form

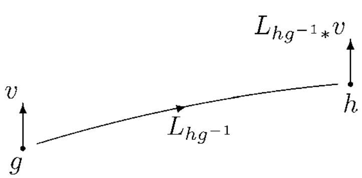
$$\xi_{i;j} + \xi_{j;i} = 0 \qquad \xi_i := g_{ij}\xi^j$$

Hint: see (4.6.6), (15.3.1) and (15.4.13). □

• Recall that we have already encountered the exponential map when speaking about Lie groups; namely in Section 11.4 we studied the map

$$\exp : T_e G \equiv \mathcal{G} \to G \qquad X \mapsto \exp X := \gamma^X(1)$$

We see that this definition coincides with the definition introduced here (15.4.10), provided that $M = G$, $P = e$ (so that it is centered at the unit element of the group), $v = X$ and *if* the one-parameter subgroup $\gamma^X(t)$ *were a geodesic* on the group G in the sense of some linear connection on G. It turns out that such a connection may indeed be easily constructed so that the "group" exponential map actually reduces to be a particular case of the "geodesic" one.



15.4.15* Define on a Lie group the parallel transport of vectors by means of the left translation, i.e. declare the operator

$$\tau_{h,g} := L_{hg^{-1}*} : T_g G \to T_h G \qquad g, h \in G$$

to be the operator of parallel transport (it does not depend on the path between the points). Check that

(i) it is linear and satisfies

$$\tau_{h,k} \circ \tau_{k,g} = \tau_{h,g} \qquad h, k, g \in G$$

(it may indeed serve as a parallel transport operator)

(ii) the covariant derivative corresponding to this parallel transport is defined by

$$\nabla_V W = 0$$

for any *left-invariant* field W (and arbitrary V)

(iii) the coefficients of the connection Γ^a_{bc} with respect to the *left-invariant* frame field e_a vanish

(iv) the tensor of the torsion reads (in the left-invariant basis)

$$T(e_a, e_b) = -[e_a, e_b] \equiv -c^c_{ab} e_c \qquad \text{i.e.} \quad T^a_{bc} = -c^a_{bc}$$

so that this (simple) connection has (for non-Abelian groups) non-vanishing torsion

(v) the geodesics emanating from the unit element of the group happen to coincide with one-parameter subgroups.

Hint: (ii) the covariant derivative measures a *deviation* from the parallel transport and the field W, being left-invariant, satisfies $\tau_{h,g} W(g) = W(h)$, so that it is invariant with respect to the parallel transport along any curve; (iv) $\nabla_a e_b = 0$; (v) see (11.3.3). □

• Now, let us have a look at how the equation of the *geodesics* of the RLC connection may be derived from the functional (15.4.4) in a coordinate-free way.

15.4.16* Let (M, g) be a Riemannian manifold, γ a curve on M and ∇ the RLC connection corresponding to g. Perform an infinitesimal variation of the curve γ by means of the flow of a "deforming" vector field W, i.e. pass to the curve $\gamma_\epsilon(t) \equiv \Phi_\epsilon \circ \gamma(t)$, where Φ_s is the flow generated by the field W (the field W should vanish at the points $\gamma(t_1)$ and $\gamma(t_2)$ since the endpoints of the curve are to be kept fixed in the course of the variation). Check that

(i) the functional $S[\gamma]$ from (15.4.4) responds to the change of argument as follows:

$$S[\gamma] := \frac{1}{2}\int g(\dot\gamma, \dot\gamma)\, dt \;\mapsto\; S[\gamma_\epsilon] = S[\gamma] + \epsilon \int \langle \mathcal{E}, W\rangle\, dt + \cdots$$

where the *Euler–Lagrange 1-form* $\mathcal{E}$ reads

$$\mathcal{E} := -g(\cdot\,, \nabla_{\dot\gamma}\dot\gamma) \equiv -\flat_g \nabla_{\dot\gamma}\dot\gamma$$

so that the critical points of the functional $S[\gamma]$ coincide with the (affinely parametrized) *geodesics* $(\nabla_{\dot\gamma}\dot\gamma = 0)$

(ii) if a potential energy is added to the action (15.4.6), i.e. we add the term $-\int U(\gamma(t))\, dt$, the Euler–Lagrange 1-form undergoes a change to

$$\mathcal{E} = -\flat_g \nabla_{\dot\gamma}\dot\gamma - dU$$

so that the critical points of the functional $S[\gamma]$ turn out to be the solutions of the (actually "Newton") equation (see (15.4.7))

$$\nabla_{\dot\gamma}\dot\gamma = -\sharp_g dU \equiv -\text{grad}\, U$$

$\Rightarrow$ we no longer move along the geodesics, but there is the non-vanishing acceleration $-\text{grad}\, U$

(iii) for the "square root" action $\hat S[\gamma] := \int \sqrt{g(\dot\gamma, \dot\gamma)}\, dt$ (the reparametrization invariant *functional of the length*) we similarly get[313]

$$\hat{\mathcal{E}} := -\flat_g \nabla_{\dot\gamma}\left(\frac{\dot\gamma}{\sqrt{g(\dot\gamma, \dot\gamma)}}\right) \equiv -\flat_g \nabla_{\dot\gamma}\left(\frac{\dot\gamma}{||\dot\gamma||}\right)$$

so that the critical points of the functional $\hat S[\gamma]$ again turn out to be the *geodesics* (this time parametrized arbitrarily, $\nabla_{\dot\gamma}(\dot\gamma/||\dot\gamma||) = 0$).

Hint: (i) $\gamma \mapsto \Phi_\epsilon \circ \gamma \;\Rightarrow\; \dot\gamma \mapsto \Phi_{\epsilon *}\dot\gamma$. Then,

$$S[\Phi_\epsilon \circ \gamma] = \frac{1}{2}\int g(\Phi_{\epsilon *}\dot\gamma, \Phi_{\epsilon *}\dot\gamma)\, dt = \frac{1}{2}\int (\Phi_\epsilon^* g)(\dot\gamma, \dot\gamma)\, dt$$
$$= S[\gamma] + \epsilon \frac{1}{2}\int (\mathcal{L}_W g)(\dot\gamma, \dot\gamma)\, dt + \cdots$$

Disentangling $\mathcal{L}_W\{g(U, V)\} = \nabla_W\{g(U, V)\}$ for the RLC connection gives

$$(\mathcal{L}_W g)(U, V) = g(\nabla_V W, U) + g(\nabla_U W, V)$$

[313] This reduces to $\mathcal{E}$ after the choice of the "natural parameter," in which $g(\dot\gamma, \dot\gamma) = 1$.

(in coordinates it is the identity $(\mathcal{L}_W g)_{ij} = W_{i;j} + W_{j;i}$ from (15.4.14)), from where

$$(\mathcal{L}_W g)(\dot\gamma, \dot\gamma) = 2g(\nabla_{\dot\gamma} W, \dot\gamma) = 2\dot\gamma g(W, \dot\gamma) - 2g(W, \nabla_{\dot\gamma}\dot\gamma)$$

Since $\dot\gamma$ in the first term is actually the derivative with respect to t of the function standing on the right, under the integral sign it may be omitted (we get $g(W, \dot\gamma)$ evaluated at the boundary of the interval $\langle t_1, t_2\rangle$, which is zero since there $W = 0$). One is left with

$$S[\Phi_\epsilon \circ \gamma] = S[\gamma] - \epsilon \int g(W, \nabla_{\dot\gamma}\dot\gamma)\, dt + \cdots \equiv S[\gamma] + \epsilon \int \langle \mathcal{E}, W\rangle\, dt + \cdots$$

from where we immediately get (W is arbitrary, g is non-degenerate) the equation $\nabla_{\dot\gamma}\dot\gamma = 0$. (ii) $\int (U \circ \gamma)\, dt \mapsto \int (U \circ \Phi_\epsilon \circ \gamma)\, dt = \int (\Phi_\epsilon^* U \circ \gamma)\, dt = \int (U \circ \gamma)\, dt + \epsilon \int (WU \circ \gamma)\, dt + \cdots = \int (U \circ \gamma)\, dt + \epsilon \int (\langle dU, W\rangle \circ \gamma)\, dt + \cdots$; (iii) $\delta\sqrt{2u} = (2u)^{-1/2}\delta u$ and (15.4.8). □

15.4.17 Consider two highways, both of them starting from Bratislava (or any other town of your choice) going in a westward direction. The first highway proceeds all the time straight forward (i.e. it is a geodesic) and the second one is always directed westward (i.e. it keeps track along a parallel line). Find out the (approximate) distance between the two highways after 1, 10 and 100 km from the starting point.

Hint: for the geodesic highway there holds $a = \nabla_v v = 0$ (where $v \equiv \dot\gamma$ is the velocity of the motion and a is its acceleration); for the "parallel line" highway choose $v = -e_\varphi$ (e_ϑ, e_φ being the standard orthonormal frame field on the sphere, i.e. the Earth), so that we run like mad at constant unit speed $||v|| = 1$ westwards; the acceleration is

$$a = \nabla_v v = \nabla_{(-e_\varphi)}(-e_\varphi) = \cdots = -\frac{1}{R}\frac{\cos\vartheta}{\sin\vartheta} e_\vartheta \equiv -||a|| e_\vartheta$$

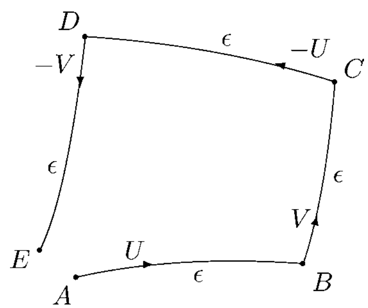
The motion with unit speed along the parallel line highway thus has an acceleration directed to the north (perpendicular to the driving direction) with magnitude $||a||$. The same acceleration corresponds to the motion along a bend with radius $r = 1/||a||$ (recall that for motion on a circle $||a|| = ||v||^2/r$ holds, here we have $||v|| = 1$). The geodesic highway is thus regarded by the driver as being straight and the one running along parallel in turn as a right-hand *bend of radius* $r \equiv R\tan\vartheta$.[314] One can easily check that if we perform a motion by ϵ along the *tangent* to a circle of radius r, we move off the circle by $\Delta l \equiv \epsilon^2/2r$. In our case the distance between the highways is thus $\Delta l \equiv \epsilon^2/2R\tan\vartheta$. Since in Bratislava $\vartheta \sim 42^\circ$ (and the radius of the Earth $R \sim 6378\,\text{km}$), we get approximately $\Delta l = 10^{-4}\,\text{km}^{-1}\epsilon^2$, so that for $\epsilon = 1$, 10 and 100 km we get the distances around 10 cm, 10 m and 1 km respectively. □

[314] This fact (and thus the result of the whole problem as well) may be also seen by an elementary consideration: $\hat r \equiv R\sin\vartheta$ is the radius of the parallel (its center lying on the Earth's axis); the motion along this circle has an acceleration $v^2/\hat r$. From the acceleration, however, the driver feels as the acceleration "due to the bend" only its projection onto the plane of the road (the remaining part raises the car up), producing a factor of $\cos\vartheta$ and this may be reformulated as an effective radius $r = \hat r/\cos\vartheta \equiv R\tan\vartheta$.

15.5 The curvature tensor

• The parallel transport of a vector (as well as an arbitrary tensor) in general depends on the (oriented) path along which it is performed. An alternative formulation of the same fact is that if the tensor is transported along a *closed* path (a loop), the resulting tensor may *differ* from the initial one. We already convinced ourselves that this phenomenon is indeed real in the case of the transport of a vector around a particular spherical triangle (15.3.9), where the change consisted in a *rotation* by $\pi/2$. The fact that the resulting vector had the same length as the initial one (so that the net change consisted *only* in the rotation) is a particular feature of the RLC connection (actually its metricity). In general, only the *linearity* of the operator of the parallel transport along the closed path is guaranteed.

It turns out that an immensely important piece of information about the local dependence of the parallel transport on the path is stored in the further tensor field characterizing the connection, the curvature tensor. In order to motivate its formal definition, let us first compute what the operator of the parallel transport along a particular infinitesimal loop looks like, namely the loop we already encountered in Chapter 4, when studying the geometrical meaning of the commutator of vector fields.



15.5.1 Consider two vector fields U, V. We saw in problem (4.5.3) how an infinitesimal loop

$$A \overset{\Phi^U_\varepsilon}{\to} B \overset{\Phi^V_\varepsilon}{\to} C \overset{\Phi^U_{-\varepsilon}}{\to} D \overset{\Phi^V_{-\varepsilon}}{\to} E \overset{\Phi^{[U,V]}_{-\varepsilon^2}}{\to} A$$

is generated by the fields composed of four pieces of parametric length ε and the fifth "closing" piece of parametric length ε^2 ($E \mapsto A$). Check that the operator $\tau_{A,A} \equiv \tau_{A\mapsto B\mapsto C\mapsto D\mapsto E\mapsto A}$ of the parallel transport of an arbitrary tensor along this loop may be expressed within second-order accuracy in ϵ as

$$\tau_{A,A} = \hat{1} - \varepsilon^2 R(U, V) + \cdots$$

where the *curvature operator* $R(U, V)$ is the expression

$$R(U, V) := \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]} \equiv [\nabla_U, \nabla_V] - \nabla_{[U,V]}$$

Hint: according to (15.2.13) the transport from A to B is performed by the operator $\tau_{B,A} = e^{-\varepsilon\nabla_U} = \hat{1} - \varepsilon\nabla_U + \frac{1}{2}\varepsilon^2\nabla_U^2 + \cdots$; so one should multiply out the product

$$\tau_{A,A} = e^{\varepsilon^2\nabla_{[U,V]}} e^{\varepsilon\nabla_V} e^{\varepsilon\nabla_U} e^{-\varepsilon\nabla_V} e^{-\varepsilon\nabla_U} = \cdots$$

up to order ϵ^2 □

• The curvature operator $R(U, V)$, which we obtained in this way, has some fairly remarkable properties.

15.5.2 Show that for the curvature operator $R(U, V)$ there holds

(i) it is a *derivation* of the tensor algebra $\mathcal{T}(M)$, which commutes with contractions
(ii) it *vanishes* on degree $\binom{0}{0}$, i.e. on $\mathcal{F}(M)$

$$R(U, V)f = 0 \qquad f \in \mathcal{F}(M)$$

(iii) it depends $\mathcal{F}(M)$-linearly on both U and V.

Hint: (i) it has the structure $[D_1, D_2] + D_3$, where D_1, D_2, D_3 *are* such according to the definition of ∇_W and (4.3.7); or alternatively: (each) operator of parallel transport should have certain properties and the operator treated here has the form $\tau_{A,A} = \hat{1} - \varepsilon^2 R(U, V)$, resulting in some properties of $R(U, V)$; (iii) the properties of the covariant derivative. □

• The derivations of the tensor algebra which commute with contractions and vanish on $\mathcal{F}(M)$ turn out to have a fairly simple structure – they are completely given by a certain *tensor* (field) of type $\binom{1}{1}$. Let us investigate this useful fact from a slightly more general perspective.

15.5.3 Show that each derivation D of the tensor algebra $\mathcal{T}(M)$ which preserves degree and commutes with contractions has the form

$$D = \mathcal{L}_V + A \qquad V \in \mathfrak{X}(M), \quad A \in \mathcal{T}^1_1(M)$$

i.e. it is parametrized by a vector field V and a tensor field A of type $\binom{1}{1}$.

Solution: on $\mathcal{F}(M)$, each derivation is given by a vector field (see Section 2.2) so that

$$Df = Vf \equiv \mathcal{L}_V f \qquad \text{for some } V \in \mathfrak{X}(M)$$

Set $\hat{D} := D - \mathcal{L}_V$. According to (4.3.7) it is a derivation of $\mathcal{T}(M)$ commuting with contractions, moreover it is by construction zero on $\mathcal{F}(M)$. Then it is enough (see (4.3.1) and the text after the problem) to specify it on vector fields. There we have $W \mapsto \hat{D}(W) =$ a vector field again (due to preserving of degree) and

$$\hat{D}(fW) \equiv \hat{D}(f \otimes W) = (\hat{D}f)W + f\hat{D}(W) = f\hat{D}(W) \qquad f \in \mathcal{F}(M)$$

so that $W \mapsto \hat{D}(W)$ is $\mathcal{F}(M)$-linear $\Rightarrow$ it is a tensor field $\hat{D} = A$ of type $\binom{1}{1}$, namely $A(W, \alpha) := \langle \alpha, \hat{D}(W) \rangle$. □

15.5.4 Let us see in more detail how the operator A (regarded as a part of a general derivation D from (15.5.3) corresponding to a tensor of type $\binom{1}{1}$) acts on tensors. Check that

(i) the action of the operator A on a type-$\binom{1}{1}$ tensor B, on a volume form ω and on a general[315] tensor T of type $\binom{p}{q}$ looks in components as follows:

$$B^a_b \mapsto B^c_b A^a_c - B^a_c A^c_b \equiv [A, B]^a_b$$
$$\omega \mapsto (-\text{Tr}\, A)\, \omega$$
$$T^{a\ldots b}_{c\ldots d} \mapsto T^{e\ldots b}_{c\ldots d} A^a_e + \cdots - T^{a\ldots b}_{c\ldots e} A^e_d$$

(ii) in particular, for the *covariant* derivative $D = \nabla_V$ we have $A = \nabla V$ so that

$$\nabla_V = \mathcal{L}_V + (\nabla V)$$

Hint: (i) if $Ae_a = A^b_a e_b$, then $Ae^a = -A^a_b e^b$ (since $0 = A\langle e^a, e_b\rangle = A(C(e^a \otimes e_b)) = \cdots$); then

$$A\big(B^b_a e^a \otimes e_b\big) = \big(AB^b_a\big)e^a \otimes e_b + B^b_a(Ae^a) \otimes e_b + B^b_a e^a \otimes (Ae_b) = \cdots$$
$$\begin{aligned} A(fe^1 \wedge \cdots \wedge e^n) &= (Af)e^1 \wedge \cdots \wedge e^n + f(Ae^1) \wedge \cdots \wedge e^n + \cdots + fe^1 \wedge \cdots \wedge (Ae^n) \\ &= f\big(-A^1_1 e^1\big) \wedge \cdots \wedge e^n + \cdots + fe^1 \wedge \cdots \wedge \big(-A^n_n e^n\big) \\ &= \big(-A^a_a\big) fe^1 \wedge \cdots \wedge e^n \equiv (-\text{Tr}\, A)\, \omega \end{aligned}$$

□

• The operator $R(U, V)$ thus has the form (15.5.3) with the *missing* part $\mathcal{L}_W$, so that all the information about the operator is stored in its action (as a tensor field of type $\binom{1}{1}$) on *vector* fields, i.e. in the expression $R(U, V)W$ (a vector field) or, alternatively, in the expression $\langle\alpha, R(U, V)W\rangle$ (a function).

15.5.5 Check that

(i) the expression

$$R(W, U, V; \alpha) := \langle\alpha, R(U, V)W\rangle \equiv \langle\alpha, ([\nabla_U, \nabla_V] - \nabla_{[U,V]})W\rangle$$

is $\mathcal{F}(M)$-linear in all four arguments $U, V, W \in \mathfrak{X}(M)$, $\alpha \in \Omega^1(M)$ so that a tensor field of type $\binom{1}{3}$ is defined by this formula; this important tensor field is called the *curvature tensor*, or also the *Riemann tensor*; in components

$$R^a{}_{bcd} = \langle e^a, R(e_c, e_d)e_b\rangle = \langle e^a, (\nabla_c \nabla_d - \nabla_d \nabla_c - \nabla_{[e_c, e_d]})e_b\rangle$$

(ii) from the definition we have

$$R(U, V)W = \big(R^a{}_{bcd} U^c V^d W^b\big)\, e_a$$

so that the value of the expression at the point $P \in M$ depends *only* on the values of the quantities *at the point* P (in spite of the fact that there are *derivatives* in the detailed expression of $R(U, V)W$ and therefore one could expect that the values of the objects in some infinitesimal *neighborhood* of the point might be necessary)

[315] The first two objects are clearly particular cases of the general one, but on B one sees most easily how it works and ω illustrates specificity of forms (it comes in handy in (15.6.18)).

(iii) the curvature tensor is antisymmetric in the last pair of indices

$$R^a{}_{bcd} = -R^a{}_{bdc}$$

(iv) in the coordinate basis it may be expressed in terms of the Christoffel symbols by the formula

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}$$

Hint: (i) see (15.5.2); (iv) $R^i{}_{jkl} = \langle e^i, R(e_k, e_l)e_j\rangle = \langle dx^i, (\nabla_k\nabla_l - \nabla_l\nabla_k)\partial_j\rangle = \cdots$. □

• If we take the *coordinate* basis vectors ∂_i, ∂_j as the fields U, V in (15.5.1), the loop contains only four steps of parametric length ε along the coordinate curves (the commutator term is not needed for its closure) and it lies completely on the ijth coordinate two-dimensional surface (the remaining coordinates being constant there).

15.5.6 Check that

(i) for $U = \partial_i$, $V = \partial_j$ the curvature operator reduces to the *commutator of the covariant derivatives* in the ith and jth coordinate directions

$$R(\partial_i, \partial_j) = \nabla_i\nabla_j - \nabla_j\nabla_i \equiv [\nabla_i, \nabla_j]$$

so that the components of the Riemann tensor enter the result of the computation of the commutator of the "coordinate" covariant derivatives on the coordinate basis as follows:

$$[\nabla_i, \nabla_j]\partial_k = R^l{}_{kij}\partial_l \qquad [\nabla_i, \nabla_j]\, dx^k = -R^k{}_{lij}\, dx^l$$

(ii) for an arbitrary tensor field there holds

$$A^{k\dots l}_{r\dots s;i;j} - A^{k\dots l}_{r\dots s;j;i} = A^{m\dots l}_{r\dots s}R^k_{mji} + \cdots + A^{k\dots m}_{r\dots s}R^l_{mji} - A^{k\dots l}_{m\dots s}R^m_{rji} - \cdots - A^{k\dots l}_{r\dots m}R^m_{sji}$$

Hint: (i) see (15.5.5) and $0 = R(U, V)\langle\alpha, W\rangle = \langle R(U, V)\alpha, W\rangle + \langle\alpha, R(U, V)W\rangle$; (ii)

$$A^{k\dots l}_{r\dots s;i;j} - A^{k\dots l}_{r\dots s;j;i} = \{[\nabla_j, \nabla_i]A\}^{k\dots l}_{r\dots s} = \{R(\partial_j, \partial_i)A\}^{k\dots l}_{r\dots s}$$

$$R(\partial_j, \partial_i)A = R(\partial_j, \partial_i)\big\{A^{k\dots l}_{r\dots s}dx^r \otimes \cdots \otimes \partial_l\big\} = \cdots$$

□

• The curvature tensor admits (as a $\binom{1}{3}$-type tensor) three contractions

$$R^c{}_{cab} \qquad R^c{}_{acb} \qquad R^c{}_{abc}$$

all the resulting tensors being of type $\binom{0}{2}$. It follows from the antisymmetry in the last pair of indices that the second contraction differs from the third one only in a sign and it turns out that the first one vanishes *for the RLC connection*, so that it is usually ignored. In the case of a *Riemannian* manifold a further contraction is possible (a tool for raising the index on the tensor of type $\binom{0}{2}$ is available) and one can define a further tensor field (being already of type $\binom{0}{0}$, a function). The definitions read

$$R_{ab} := R^c{}_{acb} \qquad \text{Ricci tensor}$$

$$R := R^a{}_a \equiv g^{ab}R_{ba} \equiv R^{ab}{}_{ab} \qquad \text{scalar curvature}$$

Clearly these tensors carry less information in general then the whole Riemann tensor; this may not be true, however, for manifolds with very low dimensions (for example, we will see in (15.6.11) and (15.6.12) that on two-dimensional manifolds the whole Riemann tensor of the RLC connection may be reconstructed from the scalar curvature). A highly effective way of computing the curvature tensor consists in using the machinery of differential forms to be discussed in the next section. We will also mention some of its further properties there.

15.5.7 Consider two Riemannian manifolds. Show that if the scalar curvature of the first manifold vanishes and this is not the case for the second one then the two manifolds cannot be isometric to each other. Infer from this that the sphere S^2 is not (locally) isometric to the Euclidean plane (we have already proved the same result before, referring to different Killing algebras, see (4.6.13)).

Hint: let $f : (M, g) \to (N, h)$ be an isometry; if y^a are the coordinates on N and $h = h_{ab}\, dy^a \otimes dy^b$, then in coordinates $x^a := f^* y^a$ on M we get $g \equiv f^* h = h_{ab}(x)\, dx^a \otimes dx^b$ (with *the same* functions h_{ab}); then according to (15.3.4) also $\Gamma^a_{bc}, \dots, R$, will be the same, i.e. the scalar curvature on M arises by the substitution $y \mapsto x \equiv f^* y$ in the expression of the scalar curvature on N (being a pull-back, $R_M = f^* R_N$). □

15.5.8 In a neighborhood of a point P consider Riemann normal coordinates centered at P, corresponding to an orthonormal basis e_i in P (15.4.11). Check that

(i) the components of the Riemann tensor of the RLC connection *in the point P* in these coordinates read

$$R^{ns}_{ijkl} = -(g_{i[k,l]j} - g_{j[k,l]i}) \qquad \text{at the point } P$$

(ii) the tensor has the symmetries (being already valid *in arbitrary* coordinates)

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}$$

Hint: (i) according to (15.5.5) and (15.4.11) we have $R^i{}_{jkl}(P) = \Gamma^i_{jl,k}(P) - \Gamma^i_{jk,l}(P)$; using (15.3.4) $2\Gamma^i_{jl,k}(P) = \cdots = \eta^{ir}(g_{rj,lk} + g_{rl,jk} - g_{jl,rk})(P)$, so that

$$R^i{}_{jkl}(P) = \Gamma^i_{jl,k}(P) - \Gamma^i_{jk,l}(P) \equiv 2\Gamma^i_{j[l,k]}(P) = -\eta^{ir}(g_{r[k,l]j} - g_{j[k,l]r})(P)$$

(ii) they are explicit in these particular coordinates; the symmetries, however, do not depend on the choice of coordinates. □

• We close the section with a few words about an important concept, which is based on the path dependence of the parallel transport.

On a manifold (M, ∇) with a connection consider a point x and a *loop* c, which starts and ends at the point x. If we take a vector in x and perform the parallel transport along the loop, in general we arrive (according to the result of (15.5.1)) at a *different* vector. However, since the operator of parallel transport is always a *linear isomorphism*, the transported vector may be obtained from the initial one by the action of a certain *linear invertible* operator $T_x M \to T_x M$, i.e. an element of the group $G \equiv GL(T_x M) \cong GL(n, \mathbb{R})$.

Contemplate now all the possible loops emanating from the same point x. To each of them a group element may be assigned, so eventually we get a map "loop $\mapsto$ element of the group." This group element is said to be the *holonomy* (corresponding to the pair (x, c)) and the group itself in which the group elements lie the *holonomy group*. For a linear connection this is not necessarily the *whole* group $GL(n, \mathbb{R})$, but rather it may be only a *subgroup*. This happens when the parallel transport "preserves something"; the preserving of something is thus reflected in a restriction of the resulting group (the automorphism group of a stronger structure is smaller, see the concrete results in Section 10.1). For example, the connection, which is *metric*, assigns to each loop some *rotation*,[316] so that here the holonomy group is at best the rotation group; in reality it may be even smaller sometimes, as the example of the "ordinary" connection in E^3 shows, where the parallel transport does not depend on the path and *each* loop gives the *identity* map (the holonomy group being trivial, containing a single element).[317]

15.6 Connection forms and Cartan structure equations

• The formalism of differential forms is also very efficient in the theory of linear connections. Let us have a look, first, at how information about the connection may be encoded into appropriate 1-forms.

15.6.1 Let e_a be a frame field on $\mathcal{O} \subset (M, \nabla)$. Check that

(i) the relations

$$\nabla_V e_a = \omega^b_a(V) e_b \qquad \omega^b_a \in \Omega^1(\mathcal{O})$$

define a set of 1-forms ω^b_a on $\mathcal{O}$; they are known as *connection forms* with respect to e_a

(ii) these forms are related to the coefficients of the connection via

$$\omega^a_b = \Gamma^a_{bc} e^c$$

and, in particular, for the *coordinate* frame $e_i = \partial_i$ they can be written in terms of Christoffel symbols of the second kind as

$$\omega^i_j = \Gamma^i_{jk}\, dx^k$$

(iii) a different choice of frame field $e_a \mapsto A^b_a(x) e_b$, where $A^b_a(x) \in GL(n, \mathbb{R})$, results in a transformation of connection forms according to the rule

$$\omega'^a_b = (A^{-1})^a_c \omega^c_d A^d_b + (A^{-1})^a_c\, dA^c_b$$

(iv) this general rule contains (as a special case) the correct transformation law for Christoffel symbols

(v) for a *coframe* field one has

$$\nabla_V e^a = -\omega^a_b(V) e^b$$

[316] We learned in problems (15.3.9) and (15.3.10) that to the loop = the spherical right triangle, the group element = the rotation by $\pi/2$ is assigned and similarly to the loop = the parallel line, the group element = the rotation by the *Foucault angle* β_{Fouc} is assigned.

[317] The structure behind this is *complete parallelism* (see Section 15.8).

Hint: (i) according to the axioms of the covariant derivative, $\nabla_V e_a$ is a vector (field) which depends in an $\mathcal{F}(M)$-linear way on V; (ii) see (15.2.1); (iv) see (15.2.3); (v) $0 = \nabla_V \langle e^a, e_b \rangle = \cdots$. □

15.6.2 It is convenient to interpret the 1-forms ω^a_b as well as the functions (0-forms) A^a_b as component forms of *matrix algebra-valued* forms ω and A (in the sense of Section 6.4); if E^b_a is the usual Weyl basis in the matrix algebra $M_n(\mathbb{R})$, then

$$\omega = \omega^a_b E^b_a \in \Omega^1(\mathcal{O}, M_n(\mathbb{R})) \qquad A = A^a_b E^b_a \in \Omega^0(\mathcal{O}, M_n(\mathbb{R}))$$

Show that

(i) the results of (15.6.1) may be written as

$$e' = eA \quad \Rightarrow \quad \omega' = A^{-1}\omega A + A^{-1}\, dA$$

where the operations of multiplication and exterior derivative of forms are to be understood *in the sense of* (6.4.2) and (6.4.4)

(ii) this result is consistent on a *triple* overlap,

$$\begin{aligned} e \mapsto e' = eA \mapsto e'' &= e'B \\ &\equiv e(AB) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \omega \mapsto \omega' = A^{-1}\omega A + A^{-1}\, dA \mapsto \omega'' &= B^{-1}\omega' B + B^{-1}\, dB \\ &= (AB)^{-1}\omega(AB) + (AB)^{-1} d(AB) \end{aligned}$$

which says (cf. the end of Section 2.5) that a *global* structure on a manifold (linear connection ∇) is actually defined by means of *local* quantities (the forms ω on domains $\mathcal{O}$, where the frame fields e_a dwell). □

• If one also encodes tensors related to the connection, namely the curvature and torsion tensor, into suitable forms, their definitions result, after translation into the language of forms, in Cartan structure equations.

15.6.3 On a domain $\mathcal{O}$ with a frame field e_a, let us define *torsion forms* T^a and *curvature forms* Ω^a_b (both of them being 2-forms) with respect to this frame by

$$T^a(U, V) := \langle e^a, T(U, V) \rangle \equiv \langle e^a, \nabla_U V - \nabla_V U - [U, V] \rangle \qquad \text{or} \quad T^a = \frac{1}{2} T^a_{bc} e^b \wedge e^c$$

$$\Omega^a_b(U, V) := \langle e^a, R(U, V) e_b \rangle \equiv \langle e^a, ([\nabla_U, \nabla_V] - \nabla_{[U,V]}) e_b \rangle \qquad \text{or} \quad \Omega^a_b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d$$

where T^a_{bc} and R^a_{bcd} are the components of the *tensors* of torsion and curvature with respect to e_a. Check that

(i) they are indeed 2-forms
(ii) under the transformation $e_a \mapsto e'_a = A^b_a(x) e_b$ of the frame field the forms transform as follows:

$$\Omega^a_b \mapsto \Omega'^a_b = (A^{-1})^a_c \Omega^c_d A^d_b \qquad\qquad T^a \mapsto T'^a = (A^{-1})^a_b T^b$$

(iii) the encoding described above is bijective.

Hint: (ii) $e_a \mapsto A^b_a e_b$, $e^a \mapsto (A^{-1})^a_b e^b$; $R(U, V)$ acts as a derivation which vanishes on functions so that we have $R(U, V)(A^c_b e_c) = A^c_b R(U, V)e_c$. □

15.6.4 One may also interpret the 2-forms Ω^a_b as component forms of *a single* $M_n(\mathbb{R})$-valued 2-form Ω and similarly T^a as component forms of *a single* $\mathbb{R}^n$-valued 2-form T.[318] Check that the transformation rules for Ω and T under the change $e \mapsto eA(x)$ of a frame field then read

$$\Omega \mapsto \Omega' = A^{-1}\Omega A \qquad T \mapsto T' = A^{-1}T$$

Hint: see (15.6.3). □

• Let us now have a look at the geometrical meaning of the connection and curvature forms introduced above. It should not be too surprising to hear that ω^a_b carries information about parallel transport of a frame field e_a in an arbitrary direction and Ω^a_b informs us about the result of such transport along an infinitesimal loop. A more detailed discussion of these topics is in order, however, since it may help the reader to develop some intuition for the work with both forms and, moreover, it paves the way for the theory of general connections and gauge fields, to be developed in Chapter 21.

15.6.5 Let ω^a_b be connection 1-forms on $(\mathcal{O}, e_a)$ and Ω^a_b the corresponding curvature 2-forms. Verify that a parallel transport of the frame e_a by ϵ along V and around the pentagon-shaped ϵ-loop spanned by the vectors V, W (see (15.5.1)), respectively, results in

translation along V	$e_a \mapsto e_a - \epsilon\omega^b_a(V)e_b$
translation around a loop based on V, W	$e_a \mapsto e_a - \epsilon^2\Omega^b_a(V, W)e_b$

or in an index-free version

$$e \mapsto (\hat{1} - \epsilon\omega(V))e \qquad e \mapsto (\hat{1} - \epsilon^2\Omega(V, W))e$$

Hint: according to the definition of the covariant derivative

$$\nabla_V e_a = \frac{e^{\parallel}_a - e_a}{\epsilon} \qquad \text{so that} \qquad e^{\parallel}_a = e_a + \epsilon\nabla_V e_a \equiv \left(\delta^b_a + \epsilon\omega^b_a(V)\right)e_b$$

where $e^{\parallel}_a$ is the vector e_a parallel transported *against* the direction of V by a parameter ϵ; then the translation *along* V (against the vector $-V$) gives an additional minus sign. Curvature: the role of the curvature *operator* $R(U, V)$ within the context of translation along a loop (15.5.1). □

• The situation can be described as follows: if on a domain $\mathcal{O}$ one has a frame field e, one has also, in particular, the frames $e(x)$ and $e(x + \epsilon V)$ at the points x and "$x + \epsilon V$"[319]

[318] $\mathbb{R}^n$ serves as an $M_n(\mathbb{R})$-module here: columns = elements of $\mathbb{R}^n$ can be multiplied by matrices = elements of $M_n(\mathbb{R})$ (the result being again a column, cf. Appendix A.4). According to Section 6.4 we can then introduce the exterior product of two $M_n(\mathbb{R})$-valued forms as well as the product of an $M_n(\mathbb{R})$-valued form with an $\mathbb{R}^n$-valued one.

[319] "$x + \epsilon V$" denotes in a compact way the point at which we arrive when moving by a parameter ϵ from the point x along *any* curve of the equivalence class defining V.

(the values of the field e at the two points). The result of a parallel transport of a frame $e(x)$ by ϵ along the vector V is some particular frame at the point $x + \epsilon V$; each frame there can be, however, obtained by an appropriate "mixing" of the elements of the frame $e(x + \epsilon V)$. We see from the result of (15.6.5) that, as a matter of fact, mixing by the matrix $\hat{1} - \epsilon\omega(V)$ takes place. This matrix is non-singular and infinitesimally close to the identity matrix. Thus, we see that it is natural to treat the matrix $X := \omega(V)$ as an element of the *Lie algebra* $gl(n, \mathbb{R})$ and the frame is then mixed by the element of the *Lie group* $GL(n, \mathbb{R})$ of the form $\hat{1} + \epsilon X \equiv e^{\epsilon X}$. This point of view reveals with no computation at all, as an example, that for the *metric* connection the matrix of the connection 1-forms with respect to an *orthonormal* frame field has to be (pseudo-)antisymmetric. In order for the parallel transport not to spoil orthonormality of a frame, the group element has to belong to a (pseudo-)orthogonal subgroup and, consequently, the element of the Lie algebra $X \equiv \omega(V)$ has to belong to a (pseudo-)orthogonal subalgebra which is, according to (11.7.6), just the algebra of all (pseudo-)antisymmetric matrices. This should be valid for all V, hence the matrix ω itself must be (pseudo-)antisymmetric. The same reasoning is valid for the curvature forms as well; since the matrix $Y \equiv \Omega(V, W)$ has to be (pseudo-)antisymmetric for all V, W, the matrix 2-form Ω must be (pseudo-)antisymmetric, too.

15.6.6 Verify these statements by formal computation; namely check that

(i) for the matrices of connection 1-forms and curvature 2-forms of a *metric* connection with respect to a *general* (that means, including coordinate) frame field there holds

$$dg_{ij} = \omega_{ij} + \omega_{ji} \qquad \omega_{ij} := g_{ik}\omega^k_j$$
$$0 = \Omega_{ij} + \Omega_{ji} \qquad \Omega_{ij} := g_{ik}\Omega^k_j$$

(ii) this results in (anti)symmetry of the Riemann tensor

$$R_{ijkl} = -R_{jikl} \qquad R_{ijkl} := g_{im}R^m{}_{jkl}$$

(iii) for an *orthonormal* frame field the matrices of these forms are (pseudo-)antisymmetric:

$$\omega_{ab} + \omega_{ba} = 0 \qquad \omega_{ab} := \eta_{ac}\omega^c_b$$
$$\Omega_{ab} + \Omega_{ba} = 0 \qquad \Omega_{ab} := \eta_{ac}\Omega^c_b$$

In matrix notation thus $(\eta\omega)^{\mathrm{T}} = -\eta\omega$, which means, according to (11.7.6), that $\omega \in so(r, s)$.

Hint: (i) for a *general* frame field and arbitrary connection there holds

$$Vg_{ij} = \nabla_V(g(e_i, e_j)) = (\nabla_V g)(e_i, e_j) + g(\nabla_V e_i, e_j) + g(e_i, \nabla_V e_j)$$

so that for the *metric* connection $Vg_{ij} = g(\nabla_V e_i, e_j) + g(e_i, \nabla_V e_j)$, which gives just $dg_{ij} = \omega_{ij} + \omega_{ji}$; for the curvature forms, replace $\nabla_V \mapsto R(V, W)$ and use the fact that for the metric connection $R(V, W)g = 0$; (ii) use the relation between Ω^i_j and $R^i{}_{jkl}$ (15.6.3). □

• The forms ω, Ω and T are not independent. Full information about a connection for a given (co)frame field e is encoded in ω. Since the connection determines the torsion and curvature tensors, it determines the forms Ω and T as well. Consequently, there should exist equations relating these forms. This is the way in which the Cartan structure equations are obtained.

15.6.7 Let ω^a_b be connection 1-forms on $(\mathcal{O}, e_a)$, Ω^a_b and T^a the corresponding curvature and torsion 2-forms. Check the validity of the *Cartan structure equations*

$$de^a + \omega^a_b \wedge e^b = T^a$$
$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = \Omega^a_b$$

or in index-free version (i.e. if one regards e, ω, Ω, T as the forms with values in the algebra $M_n(\mathbb{R})$ or the $M_n(\mathbb{R})$-module $\mathbb{R}^n$ respectively)

$$de + \omega \wedge e = T$$
$$d\omega + \omega \wedge \omega = \Omega$$

Hint: a straightforward computation using general arguments, definitions of objects and Cartan formulas (6.2.13) for the exterior derivative, e.g.

$$\begin{aligned}
de^a(U, V) &= U\langle e^a, V\rangle - V\langle e^a, U\rangle - \langle e^a, [U, V]\rangle \\
&= \langle \nabla_U e^a, V\rangle + \langle e^a, \nabla_U V\rangle - \langle \nabla_V e^a, U\rangle - \langle e^a, \nabla_V U\rangle - \langle e^a, [U, V]\rangle \\
&= \dots \text{ (15.6.1)} \\
&= \langle e^a, T(U, V)\rangle - \big(\omega^a_b \wedge e^b\big)(U, V) \\
&= \big(T^a - \omega^a_b \wedge e^b\big)(U, V)
\end{aligned}$$

The second relation in full analogy: start with $d\omega^a_b(U, V) = \cdots$. □

• As we know, an important role among linear connections is played by the RLC connection (see Section 15.3). Let us have a look at the modifications of the Cartan structure equations in this particular case.

15.6.8 Let ω^a_b be RLC connection 1-forms with respect to an *orthonormal* frame field e_a, Ω^a_b and T^a the corresponding curvature and torsion 2-forms. Check that the structure equations in this case read

$$\omega_{ab} + \omega_{ba} = 0 \qquad \omega_{ab} := \eta_{ac}\omega^c_b$$
$$de^a + \omega^a_b \wedge e^b = 0$$
$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = \Omega^a_b$$

or in index-free notation

$$(\eta\omega)^{\mathrm{T}} + \eta\omega = 0$$
$$de + \omega \wedge e = 0$$
$$d\omega + \omega \wedge \omega = \Omega$$

Hint: see (15.6.7). Also consider vanishing of the torsion and ω being a (pseudo-) antisymmetric matrix due to the metricity of the connection (15.6.6). □

• For a given metric tensor g, the application of these equations consists of the following three-step procedure:

1. one finds an orthonormal coframe field e^a (so that $g = \eta_{ab} e^a \otimes e^b$)
2. the *first two* equations from (15.6.8) are written down and solved, i.e. one looks for a set of 1-forms ω^a_b such that they satisfy the second equation and at the same time the matrix $\omega_{ab} \equiv \eta_{ac}\omega^c_b$ is *antisymmetric* (due to this condition there exist only $n(n-1)/2$ unknown 1-forms instead of n^2)
3. if we already do have connection 1-forms ω^a_b, we plug them into the third equation and find curvature 2-forms Ω^a_b and maybe, depending on what we actually need, also the components of the curvature tensor, Ricci tensor and scalar curvature from the relations

$$\Omega^a_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d \qquad R_{ab} = R^c{}_{acb} \qquad R = R^a_a \equiv \eta^{ab} R_{ab}$$

Recall (15.3.4) that the computation of Christoffel symbols (and components of all remaining objects then) for the RLC connection is not a creative procedure, one has simply to plug g_{ij} into the corresponding formulas (and to make no mistake in the course of the computation of all the necessary partial derivatives, the number of which increases rapidly with the dimension of this manifold). That is why the solution of Cartan structure equations should not be a creative procedure either. Step 1 amounts to the diagonalization of the matrix of the metric tensor; in real life situations, however, one often obtains the required frame field without the formal procedure of diagonalization. For step 2, there should exist some formulas expressing ω^a_b in terms of the (already known) 1-forms e^a.

15.6.9 Let e_a be an *orthonormal* frame field, c^a_{bc} its coefficients of anholonomy and let ω^a_b be the RLC connection 1-forms. Check that

(i) the following relations are valid (it is useful to compare them with their coordinate counterparts displayed in (15.3.4)):

$$\omega_{ab}(e_c) + \omega_{ba}(e_c) \equiv \Gamma_{abc} + \Gamma_{bac} = 0 \qquad \text{metric connection}$$
$$\omega_{ab}(e_c) - \omega_{ac}(e_b) \equiv \Gamma_{abc} - \Gamma_{acb} = -c_{abc} \qquad \text{symmetric connection}$$

(ii) the RLC connection 1-forms may be expressed explicitly as

$$\omega^a_b = \eta^{ac}\omega_{cb} = \eta^{ac}\Gamma_{cbd}e^d \equiv \frac{1}{2}\eta^{ac}(c_{dcb} + c_{bcd} - c_{cbd})e^d$$

Hint: (i) see (9.2.10), $0 = T(e_a, e_b) = \nabla_a e_b - \nabla_b e_a - [e_a, e_b]$. □

• One often obtains, however, the solution by plugging an appropriate *ansatz* into the structure equations and solving the rest by "trial and error." This is illustrated most easily on *two-dimensional* manifolds.

15.6.10 From (15.6.6) it follows that if we are given an *orthonormal* frame field on a *two-dimensional* manifold (a surface) (M, g), there is only a *single independent* connection 1-form; we will denote it by $\alpha \equiv \omega_{12} = -\omega_{21}$. By the same reasoning, there is only a single independent curvature form; let us denote it by $\beta \equiv \Omega_{12} = -\Omega_{21}$. This form (as is the case for *any* 2-form) is necessarily a scalar multiple of the metric volume form $e^1 \wedge e^2$; then we can write

$$\omega_{ab} = \epsilon_{ab}\alpha \qquad \Omega_{ab} = \epsilon_{ab}\beta \equiv \epsilon_{ab}Ke^1 \wedge e^2$$

The function $K(x)$ is called the *Gaussian curvature* of the surface. Show that

(i) the symmetries of the Riemann tensor lead to the conclusion that the complete Riemann (curvature) tensor may be reconstructed from the *scalar* curvature R alone (and the same then clearly holds for the Ricci tensor), the latter being simply twice the *Gaussian* curvature; in the case of signature $(+,+)$ (the other ones need minor modifications here and there) we may namely write

$$R_{abcd} = K(x)\epsilon_{ab}\epsilon_{cd} \qquad R_{ab} = K(x)\delta_{ab} \qquad K = R/2$$

(ii) the structure equations for the RLC connection (15.6.8) reduce to the simple system

$$de^1 + \alpha \wedge e^2 = 0$$
$$de^2 - \alpha \wedge e^1 = 0$$
$$d\alpha = \beta \equiv Ke^1 \wedge e^2$$

The computation of all relevant quantities thus consists in the solution of the *first two* (very simple) equations for the unknown 1-form α. Differentiation of α then results in β, the latter being necessarily of the form $Ke^1 \wedge e^2$; eventually K doubled gives R.

Hint: (i) $R_{abcd} = -R_{bacd} = -R_{abdc}$; (ii) $\omega^1_a \wedge \omega^a_2 = 0 \wedge 0 + \alpha \wedge (-\alpha) = 0$. □

15.6.11 Solve the system (15.6.10) for (S^2_ρ, g) = the two-dimensional sphere of radius ρ endowed with the standard "round" metric (3.2.4). Compute the Gaussian curvature (show that it is *constant* $\cdot K(x) = 1/\rho^2$), Ricci tensor and the scalar curvature and check that

$$R_{abcd} = \frac{1}{\rho^2}\epsilon_{ab}\epsilon_{cd} \qquad R_{ab} = \frac{1}{\rho^2}\delta_{ab} \qquad R(x) = \frac{2}{\rho^2} \equiv 2K(x)$$

so that the scalar curvature is constant, inversely proportional to the square of the radius of the sphere (which matches the intuitive notion of the curvature of the sphere: it is everywhere the same and the bigger the sphere the less it is "curved").

Solution: for $e^1 = \rho\, d\vartheta$, $e^2 = \rho \sin\vartheta\, d\varphi$ we have the equations $\alpha \wedge e^2 = 0$, $\rho\cos\vartheta\, d\vartheta \wedge d\varphi - \alpha \wedge e^1 = 0$ from which one easily gets $\alpha = -\cos\vartheta\, d\varphi$, so that $\beta = \sin\vartheta d\vartheta \wedge d\varphi \equiv \rho^{-2}e^1 \wedge e^2$. □

15.6.12 Solve the system (15.6.10) for (T^2, g) = the two-dimensional torus with the metric induced from the embedding into E^3 (3.2.2). Compute its Gaussian curvature, Ricci tensor and the scalar curvature and check that

$$R_{abcd} = \frac{\sin\psi}{b(a + b\sin\psi)}\epsilon_{ab}\epsilon_{cd} \qquad R_{ab} = \frac{\sin\psi}{b(a + b\sin\psi)}\delta_{ab}$$
$$R(x) = \frac{2\sin\psi}{b(a + b\sin\psi)} \equiv 2K(x)$$

so that the scalar curvature is no longer constant, rather it depends on the coordinate ψ (in particular, it *vanishes* on the two circles, where the torus touches the slices of bread when eaten for lunch, it is positive on the part seen by the consumer from outside and negative on the part which is not visible).

Hint: following the lines of (15.6.11) for $e^1 = (a + b\sin\psi)\,d\varphi$, $e^2 = b\,d\psi$ one quickly gets $\alpha = \cos\psi\,d\varphi$ and $\beta = -\sin\psi\,d\psi \wedge d\varphi \equiv (R/2)e^1 \wedge e^2$. □

15.6.13 We mention without proof a quick method of obtaining the scalar curvature of two-dimensional surfaces (like a sphere or a torus). At any given point, imagine two mutually perpendicular circles of appropriate radii such that they match optimally the surface in the neighborhood of the point. Let their radii be r_1, r_2. Then let the Gaussian curvature be the product of the inverse values of the radii, $K = (r_1 r_2)^{-1}$. If the circles lie on the opposite sides of the tangent plane at the given point (so that there is a "saddle" in the neighborhood of this point), the curvature is negative. Verify that this algorithm is consistent with the results we obtained for the sphere and the torus.

Hint: the sphere: the radii of the circles coincide, being $r_1 = r_2 = \rho$; the torus: on the outer perimeter (say) there holds $r_1 = a + b$, $r_2 = b$, on the inner perimeter $r_1 = a - b$, $r_2 = b$ (and with a saddle point there) and on the upper as well as the bottom circle one has $r_1 = \infty$, $r_2 = b$. □

• Important examples[320] of working with the Cartan structure equations on higher than two-dimensional manifolds are provided by ordinary three-dimensional Euclidean space E^3 and four-dimensional Minkowski space $E^{1,3}$, when non-Cartesian frame fields are used; in particular, for orthonormal frame fields generated by the cylindrical and spherical polar coordinates.

15.6.14 Consider the three-dimensional Euclidean space E^3 endowed with the cylindrical and spherical polar orthonormal (co)frame fields (2.6.4),

$$\begin{array}{llll} e^1 = dr & e^2 = r\,d\varphi & e^3 = dz & \text{cylindrical} \\ e^1 = dr & e^2 = r\,d\vartheta & e^3 = r\sin\vartheta\,d\varphi & \text{spherical polar} \end{array}$$

Check that the Cartan structure equations for the RLC connection lead in these two cases

(i) to the following connection forms:[321]

$$\begin{array}{llll} r\omega_{12} = -e^2 & \omega_{13} = 0 & \omega_{23} = 0 & \text{cylindrical} \\ r\omega_{12} = -e^2 & r\omega_{13} = -e^3 & r\omega_{23} = -(\cot\vartheta)\,e^3 & \text{spherical polar} \end{array}$$

and that these forms are consistent with the Christoffel symbols obtained in (15.3.5)

(ii) to vanishing curvature forms (as should be the case in *Euclidean* space).

320 Moreover, their results will turn out to be useful later.
321 The forms, obtained trivially from the antisymmetry $\omega_{ab} = -\omega_{ba}$ are omitted. Recall that $g_{ab} = \delta_{ab}$ so that $\omega_{ab} = \omega^a_{\ b}$.

Hint: (i) for example, for the spherical polar case one obtains in notation $\sigma^a_b \equiv r\omega^a_b$ the equations

$$\begin{aligned}
\sigma^1_2 \wedge e^2 + \sigma^1_3 \wedge e^3 &= 0 \\
\sigma^1_2 \wedge e^1 - \sigma^2_3 \wedge e^3 &= e^1 \wedge e^2 \\
\sigma^1_3 \wedge e^1 + \sigma^2_3 \wedge e^2 &= e^1 \wedge e^3 + (\cot \vartheta)\, e^2 \wedge e^3
\end{aligned}$$

Since we know that the solution is unique, we try to find the *simplest possible* forms satisfying all the equations: e.g. the second equation suggests that (maybe) $\sigma^2_3 \sim e^3$ (e^3 is missing on the right-hand side) and $\sigma^1_2 = -e^2$ (there might be a term $\sim e^1$ there, but we try the simplest ansatz first[322]); the result $\omega^1_2 = -d\vartheta$ gives $\nabla_{e_2} e_2 = \omega^1_2(e_2) e_1 = -(1/r)\partial_r$, at the same time it should be $(1/r^2)\Gamma^i_{\vartheta\vartheta}\partial_i$, from where we get $\Gamma^r_{\vartheta\vartheta} = -r$ and (the remaining) $\Gamma^i_{\vartheta\vartheta} = 0$, which is in agreement with (15.3.5). □

15.6.15 Consider the four-dimensional Minkowski space $E^{1,3}$ endowed with the orthonormal (co)frame fields of the form $e^a \equiv (e^0, e^j)$, where $e^0 = dt$ and e^j, $j = 1, 2, 3$, are the cylindrical and spherical polar frames treated in (15.6.14). Check that the Cartan structure equations for the RLC connection lead in these two cases to

(i) the common result

$$\omega^0_j = 0 \qquad \omega^i_j = \text{ as in } E^3$$

i.e.

$$\omega_{0j} = 0 \qquad \omega_{ij} = -\text{ as in } E^3$$

(ii) vanishing curvature forms (as should be the case in *Minkowski* space).

Hint: (i) in detail the equations take the form

$$\begin{aligned}
de^0 + \omega^0_j \wedge e^j &= 0 \\
de^j + \omega^j_0 \wedge e^0 + \omega^j_k \wedge e^k &= 0
\end{aligned}$$

with the evident solution (recall that it is unique)

$$\omega^0_j = 0 \qquad \omega^j_k = \text{the solutions of the system} \quad de^j + \omega^j_k \wedge e^k = 0$$

This system coincides, however, with the system met in the case of E^3. Since now $g_{ab} = \eta_{ab}$, we have $\omega_{0j} = \eta_{00}\omega^0_j = \omega^0_j$ and (no summation) $\omega_{ij} = \eta_{ii}\omega^i_j = -\omega^i_j$. □

• Let us now return to general manifolds with the connection (M, ∇). The curvature and torsion tensors enter important identities (Bianchi and Ricci), which are most easily derived, and even formulated, in the language of forms.

[322] Recall the "Ockham's razor" (law of parsimony) principle, which advises us: "*Pluralitas non est ponenda sine necessitate*," i.e. plurality should not be posited without necessity. Fortunately, there is no "necessity" for "positing plurality," here.

15.6.16 Let ω^a_b be connection 1-forms on $(\mathcal{O}, e_a)$, Ω^a_b and T^a the corresponding curvature and torsion 2-forms. Check that

(i) the following identities hold[323]

$$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0 \qquad \text{Bianchi identity}$$
$$dT + \omega \wedge T = \Omega \wedge e \qquad \text{Ricci identity}$$

(ii) they are equivalent to

$$\{(\nabla_U R)(V, W) - R(U, T(V, W)\} + \text{ cycl.} = 0 \qquad \text{Bianchi}$$
$$(\nabla_U T)(V, W) + T(T(U, V), W) + \text{ cycl.} = R(U, V)W + \text{ cycl.} \qquad \text{Ricci}$$

(iii) and in components also to

$$R^i{}_{j[kl;m]} + R^i{}_{jr[m} T^r_{kl]} = 0 \qquad \text{Bianchi}$$
$$T^i_{[jk;l]} + T^i_{m[j} T^m_{kl]} = R^i{}_{[jkl]} \qquad \text{Ricci}$$

(iv) in particular, for the RLC connection the identities simplify (in the three different versions mentioned above) to

$$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0 \quad (\nabla_U R)(V, W) + \text{ cycl.} = 0 \quad R^i{}_{j[kl;m]} = 0 \qquad \text{Bianchi identity}$$
$$\Omega \wedge e = 0 \quad R(U, V)W + \text{ cycl.} = 0 \quad R^i{}_{[jkl]} = 0 \qquad \text{Ricci identity}$$

Hint: (i) apply d on the Cartan structure equations (15.6.7); (ii) insert arguments U, V, W and use Cartan formulas (6.2.13) (for $p = 2$ in the form with "+ cycl."); (iii) replace the (general) fields U, V, W by the coordinate basis fields. □

• Let us have a look, next, at how the basic differential operators on forms, the exterior derivative ("differential") d and the codifferential δ, may be expressed in terms of the covariant derivatives.

15.6.17* Let i_a, $j^a \equiv g^{ab} j_b$ be the operators on forms introduced in (5.8.6), (5.8.10) (the fields e_a, e^a are supposed to be dual to each other, but they need not be orthonormal) and T^a the torsion forms. Check that

(i) the exterior derivative of forms may be expressed in terms of covariant derivatives $\nabla_a \equiv \nabla_{e_a}$ as

$$d = j^a \nabla_a + T^a i_a \qquad \text{i.e.} \quad d\alpha = e^a \wedge \nabla_a \alpha + T^a \wedge i_a \alpha$$

(ii) in particular, for the RLC connection this simplifies to

$$d = j^a \nabla_a \qquad \text{i.e.} \quad d\alpha = e^a \wedge \nabla_a \alpha$$

and for the components of the exterior derivative of p-forms in a coordinate basis one obtains

$$(d\alpha)_{i\ldots jk} = (-1)^p (p+1) \alpha_{[i\ldots j;k]}$$

(iii) relate the result of (ii) to (6.2.5)

[323] We will also encounter these identities later in a more general context of connections on principal bundles, see (20.4.4)–(20.4.8). With the help of the exterior *covariant* derivative $\mathcal{D}$ which will be introduced there, they even simplify to $\mathcal{D}\Omega \equiv \mathcal{D}\mathcal{D}\omega = 0$ and $\mathcal{D}T \equiv \mathcal{D}\mathcal{D}e = \Omega \wedge e$ (the Cartan structure equations themselves read $\mathcal{D}e = T$ and $\mathcal{D}\omega = \Omega$, see (21.7.4)).

(iv) for the RLC connection, the *codifferential* may be expressed in terms of covariant derivatives as

$$\delta_g \alpha = -i^a \nabla_a \alpha$$

(v) coordinate expression of the codifferential then reads (compare with (8.3.5))

$$(\delta\alpha)_{i\dots j} = -\alpha_{ki\dots j;}{}^{k} \qquad \text{i.e.} \quad (\delta\alpha)^{i\dots j} = -\alpha^{ki\dots j}{}_{;k}$$

(vi) and, in particular, the *Laplace–Beltrami operator* reads

$$\Delta f \equiv -\delta df = f_{;k}{}^{;k}$$

Hint: (i) according to (6.1.7) and (6.1.8), the right-hand side is a degree +1 derivation of $\Omega(M)$, so that it is enough to check this on $\Omega^0(M)$ (trivial) and on $e^a \in \Omega^1(M)$ (the first Cartan equation); (ii) $(d\alpha)_{i\dots jk} = (dx^r \wedge \nabla_r \alpha)_{i\dots jk} = \cdots$ (iii) see (15.4.12); (iv) using (5.8.9) and (15.3.13) we get $*^{-1} d * \hat{\eta} = *^{-1} j^a \nabla_a * \hat{\eta} = *^{-1} j^a * \hat{\eta} \nabla_a = - *^{-1} * i^a \hat{\eta}\hat{\eta}\nabla_a \equiv -i^a \nabla_a$; (v) $(\delta_g\alpha)_{i\dots j} = -(i^k \nabla_k \alpha)_{i\dots j} = -g^{kr}(\nabla_k\alpha)_{ri\dots j} = -g^{kr}\alpha_{ri\dots j;k} = -\alpha_{ki\dots j;}{}^{k}$. □

15.6.18* Check that for the RLC connection the two apparently different definitions of the divergence (the first one, based on the metric volume form (8.2.1), and the *covariant divergence*, which is defined to be the trace of the covariant gradient of the field V)

$$\mathcal{L}_V \omega_g = (\text{div}_{(1)} V)\omega_g \qquad \text{div}_{(2)} V = \text{Tr}\,(\nabla V) \equiv V^i_{\ ;\, i}$$

actually lead to the same result.

Hint: according to (15.5.4) there holds

$$\mathcal{L}_V \omega_g = \nabla_V \omega_g - (\nabla V) \cdot \omega_g$$

where the action of the tensor $\nabla V \equiv A$ of type $\binom{1}{1}$ on the volume form ω_g is

$$(\nabla V) \cdot \omega_g \equiv A \cdot \omega_g = (-\text{Tr}\, A)\, \omega_g \equiv (-\text{div}_{(2)} V)\omega_g$$

Since according to (15.3.11) for the RLC connection $\nabla_V \omega_g = 0$, we can finally conclude that $\text{div}_{(1)} V = \text{div}_{(2)} V$. □

- In the following exercises we will be concerned with several simple facts one usually encounters when studying *spinor fields* on Riemannian manifolds (M, g) (in "curved spaces"; the spinor fields are treated in more detail in Chapter 22).

15.6.19* Let $\omega^a_b \equiv \omega^a_{b\mu}\, dx^\mu$ be connection 1-forms with respect to a (co)frame field $e^a \equiv e^a_\mu\, dx^\mu$ and let $\Gamma^\mu_{\nu\rho}$ be the Christoffel symbols *of the same* (!) connection ∇ with respect to local coordinates x^μ. Verify that

(i) $\omega^a_{b\mu}$ and $\Gamma^\mu_{\nu\rho}$ are related as follows:

$$\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{b\mu} e^b_\nu = 0$$

(ii) if this is regarded as the prescription for finding $\omega^a_{b\mu}$ in terms of given $\Gamma^\rho_{\mu\nu}$ and e^a_μ, we may rewrite it as

$$\omega^a_{b\mu} = e^a_\rho e^\nu_b \Gamma^\rho_{\mu\nu} + e^a_\rho \partial_\mu e^\rho_b$$

If the (co)frame field happens to be *orthonormal* and the connection ∇ is *metric* (possibly not symmetric, however), then the fields[324] $\omega^a_{b\mu}(x)$ are known as the *spin connection*. (They enter the formulas for the covariant derivative of spinor fields, cf. (22.4.8) and (22.5.1), as well as the explicit expression of the Dirac operator (22.5.4). The formula obtained above may sometimes be found in the literature on spinors in general relativity under the noble name of the *tetrad postulate*. The fields e^a_μ and e^μ_a are usually called *vielbein fields* (in the four-dimensional case *tetrad fields*, see Section 4.5).)

Hint: the particular case of (15.6.1) for the change of a frame field $e_a \mapsto \partial_\mu \equiv e^a_\mu e_a$ (so that $A \leftrightarrow e^a_\mu$, $A^{-1} \leftrightarrow e^\mu_a$, see (4.5.3)). □

15.6.20* Let e_a be an *orthonormal* frame field. In the general theory of relativity the following objects are often introduced when working within the *tetrad formalism*:

$$\gamma_{abc} := e^\mu_a e_{b\mu;\nu} e^\nu_c \qquad \text{Ricci coefficients of rotation}$$

(in particular, for computations with spinors, cf. (22.5.4)). Verify that

(i) they may be expressed as follows:

$$\begin{aligned}\gamma_{abc} &= (\nabla_c g)_{ab} + g(e_a, \nabla_c e_b) \\ &\equiv g_{ab;c} + \Gamma_{abc} \qquad\qquad \Gamma_{abc} := \eta_{ad}\Gamma^d_{bc}\end{aligned}$$

(where Γ^a_{bc} are the coefficients of the connection (15.6.1) with respect to e_a), so that for the *metric* connection (the case notably interesting for the general theory of relativity and spinors) we get that the Ricci coefficients of rotation simply coincide with the coefficients of connection (with respect to an orthonormal frame) with a "lowered index"

$$\gamma_{abc} = \Gamma_{abc} = \langle \omega_{ab}, e_c \rangle \qquad \text{i.e.} \quad \omega_{ab} = \Gamma_{abc} e^c = \gamma_{abc} e^c$$

(ii) they are antisymmetric with respect to the first pair of indices

$$\gamma_{abc} = -\gamma_{bac}$$

(iii) one can also express these coefficients in terms of *coefficients of anholonomy* (9.2.10) of the frame field e_a and this gives (for the metric connection)

$$\gamma_{abc} = \frac{1}{2}(c_{cab} + c_{bac} - c_{abc})$$

Hint: (i)

$$\begin{aligned}e^\mu_a e_{b\mu;\nu} e^\nu_c &= e^\mu_a e^\nu_c \big(g_{\mu\sigma} e^\sigma_b\big)_{;\nu} = e^\mu_a e^\nu_c \big(g_{\mu\sigma;\nu} e^\sigma_b + g_{\mu\sigma} e^\sigma_{b;\nu}\big) \\ &= e^\mu_a e^\nu_c \left\{(\nabla_\nu g)_{\mu\sigma} e^\sigma_b + g_{\mu\sigma}(\nabla_\nu e_b)^\sigma\right\} = (\nabla_c g)_{ab} + g(e_a, \nabla_c e_b)\end{aligned}$$

(ii) $\omega_{ab} = -\omega_{ba}$ because of metricity; (iii) see (15.6.9). □

• Let us have a look at how one can write down the parallel transport equations in terms of connection forms.

[324] That is, the *coordinate* components of the *metric* connection 1-forms with respect to an *orthonormal* frame field.

15.6.21 Let ω^a_b be the connection forms with respect to a frame field e_a, $V = V^a(t)e_a$ a vector field defined on a curve $\gamma(t)$ and $A = A^{a\ldots b}_{c\ldots d}(t)e^c \otimes \cdots \otimes e_b$ a tensor field of type (r, s) at the same curve. Check that

(i) the parallel transport equations of the vector V and the tensor A take the form

$$\dot{V}^a = S^a_b(t)V^b \qquad\qquad S^a_b(t) := -\omega^a_b(\dot{\gamma}(t))$$
$$\dot{A}^{a\ldots b}_{c\ldots d} = S^a_f(t)A^{f\ldots b}_{c\ldots d} + \cdots - \cdots - S^f_d(t)A^{a\ldots b}_{c\ldots f}$$

(ii) the equations from (15.2.6) and (15.2.12) are the special cases for the *coordinate* frame field
(iii) for an *orthonormal* frame field the matrix S^a_b is (pseudo-)antisymmetric
(iv) if S^a_b does not depend on time, the explicit solution (for the vector) may be written in the form

$$V^a(t) = (e^{tS})^a_b V^b(0) \equiv V^a(0) + tS^a_b V^b(0) + \frac{t^2}{2} S^a_c S^c_b V^b(0) + \cdots$$

and the matrix $(e^{tS})^a_b$ is (pseudo-)orthogonal.

Hint: (i) $0 = \nabla_{\dot{\gamma}}(V^a e_a) = \cdots$; (ii) $\omega^i_j = \Gamma^i_{jk}\, dx^k$ (15.6.1). □

15.6.22 Check that for the case of the two-dimensional sphere from problem (15.6.11) the result of (15.6.21) gives

$$S^a_b(t) = \epsilon_{ab}\dot{\varphi}\cos\vartheta$$

and, in particular, for the motion along a parallel $\vartheta = \vartheta_0$, $\varphi = t$ we get

$$S^a_b(t) = S^a_b = \epsilon_{ab}\cos\vartheta_0 \qquad \text{i.e.} \quad e^{tS} \equiv e^{\varphi S} = \begin{pmatrix} \cos(\varphi\cos\vartheta_0) & \sin(\varphi\cos\vartheta_0) \\ -\sin(\varphi\cos\vartheta_0) & \cos(\varphi\cos\vartheta_0) \end{pmatrix}$$

This means that the motion along the parallel with $\vartheta = \vartheta_0$ results in a (clockwise) *uniform rotation* of the vector which is parallel transported. The net effect of the transport by the angle φ (directed toward the east) consists in the rotation of the vector by the angle $\varphi\cos\vartheta_0$; in particular the transport around the entire parallel gives just the *Foucault angle* $2\pi\cos\vartheta_0 \equiv 2\pi\sin\alpha$ from (15.3.10).

Hint: $\omega^a_b = \epsilon_{ab}\alpha = -\epsilon_{ab}\cos\vartheta\, d\varphi$. □

15.7 Geodesic deviation equation (Jacobi's equation)

• Imagine two boats sailing across a lake, their motion being uniform and along a straight line. We may write down their trajectories as

$$\mathbf{r}_1(t) = \mathbf{r}_1(0) + \mathbf{v}_1 t$$
$$\mathbf{r}_2(t) = \mathbf{r}_2(0) + \mathbf{v}_2 t$$

and they represent (affinely parametrized) *geodesics* in E^2. For their *relative* position vector, relative velocity and relative acceleration we get

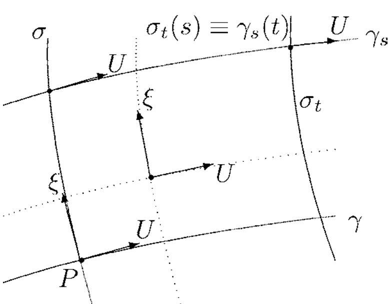
$$\mathbf{r}(t) \equiv \mathbf{r}_2(t) - \mathbf{r}_1(t) = (\mathbf{r}_2(0) - \mathbf{r}_1(0)) + (\mathbf{v}_2 - \mathbf{v}_1)t \equiv \mathbf{r}(0) + \mathbf{v}t$$
$$\dot{\mathbf{r}}(t) = \mathbf{v}$$
$$\ddot{\mathbf{r}}(t) = \mathbf{0}$$

These equations say that also from the point of view of a man sitting in the first boat the motion of the second boat is *uniform* and *along a straight line*. This fact is so evident (we knew it in advance and no computation was needed for it) that the reader might be astonished as to why this trivial stuff should enter Section 15.7 of the chapter devoted to linear connection.

Let's try to have a look at what happens when our freshwater beginners are substituted by fearless sea wolves, moving at the scale of the whole globe. Imagine they start their sails simultaneously, being (say) 100 m from one another (the second one eastwards from the first one). Both of them move *uniformly along a straight line* again (with the same speed) to the south, each one along their meridian. Their trajectories thus also represent (affinely parametrized) *geodesics*, but this time with geodesics *on the sphere* S^2. The behavior of a "relative vector," however, essentially differs in this case: the trajectories of the boats first diverge from one another and then (after passing the equator) they start to converge! From this "oscillation" it is clear that their "relative motion" is no longer "uniform," even though the motion of either of the boats *is* uniform and along a straight line.

Now we will try to discuss all of this in a more general setting, on a *manifold with a connection* (M, ∇). It turns out that the phenomenon already occurs at the *local* level and it is a manifestation of the behavior of *nearby geodesics*.

Contemplate a geodesic $\gamma(t)$. We may construct the whole *one-parameter class* of geodesics from the *single* geodesic $\gamma(t)$ as follows: at the point $P = \gamma(0)$ we consider a vector ξ (which is not directed along $\dot{\gamma}$) and we construct an arbitrary curve $\sigma(s)$ such that it is tangent to ξ at the point P, so that



$$P = \sigma(0) = \gamma(0) \qquad \dot{\sigma}(0) = \xi$$

Now consider any vector field $U(s)$ on (a small piece of) the curve $\sigma(s)$ such that it is smooth and that for $s = 0$, i.e. at P, it coincides with the velocity vector $\dot{\gamma}(0)$ of the initial geodesic. The points of the curve $\sigma(s)$ plus the vectors $U(s)$ at these points induce unique[325] geodesics (see the text before (15.4.9)) $\gamma_s(t)$: there holds

$$\gamma_s(t = 0) = \sigma(s) \qquad\qquad \dot{\gamma}_s(t = 0) = U(s)$$

[325] What this construction (for $s = \epsilon \ll 1$) actually does is a small *variation of the initial conditions* of the original geodesic: we perform the "variation" of the initial point $P \equiv \sigma(0)$ to $\sigma(\epsilon)$ and the variation of the initial velocity $\dot{\gamma}(P) \equiv U(0)$ to $U(\epsilon)$ at the point $\sigma(\epsilon)$. The aim is then to learn what effect the small variation of the initial conditions has on the future course of the geodesic. Put another way, what is the variation of the *rest* of the geodesic for a given variation of its *initial conditions?*

The curve $\sigma(s)$ by construction binds the *initial points* (the points $\gamma_s(0)$) of the *one-parameter class* of the geodesics γ_s. Now define a similar curve $\sigma_t(s)$ *for each* t, i.e. so that the curve $\sigma_t(s) \equiv \sigma(t,s)$ with fixed t binds the points on the geodesics γ_s which share the same value of the parameter t. Thus, there holds

$$\sigma(t,0) = \gamma(t) \equiv \gamma_{s=0}(t) \qquad \sigma(0,s) = \sigma(s) \equiv \sigma_{t=0}(s) \qquad \sigma(t,s) = \gamma_s(t) = \sigma_t(s)$$

The parameter s thus "labels" the geodesics whereas the parameter t "runs in" them. It is intuitively clear that this one-parameter class of geodesics forms a *two-dimensional surface* $\mathcal{S}$ (which is parametrized by $\sigma(t,s)$). There are two natural *vector fields* on the surface (it is enough to consider the fields in an infinitesimal neighborhood of the initial geodesic): the velocity field U of the motion along the individual geodesics (it may be regarded as an extension to the surface $\mathcal{S}$ of its values $U(s)$ on the curve $\sigma(s)$) and the field ξ which "links the individual neighboring geodesics" or, more exactly, whose integral curves are (by definition) the curves which bind the points with the same value of "the time" t, i.e. the curves $\sigma_t(s) \equiv \sigma(t,s)$ with fixed t (for its flow Φ_s^ξ we may write $\Phi_s^\xi \gamma(t) = \gamma_s(t)$).

15.7.1 Check that the vector fields U and ξ on $\mathcal{S}$ *commute*

$$[U, \xi] = 0$$

Hint: they generate the flows $(t,s) \mapsto (t+\lambda, s)$ and $(t,s) \mapsto (t, s+\lambda)$, see (4.5.8). □

• Let us now have a look at how objects in this construction correspond to objects in the situation with the boats. The relative velocity of the boats $\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1$ is the difference of two vectors sitting at two distinct points. In order to make the comparison we need first to (parallel) transport one of the vectors along the line joining the two points – the *relative velocity* of nearby geodesics is thus the covariant derivative of the velocity vector along the joining line, $\nabla_\xi U$. This object is still not the most interesting one since *we can control it* by means of the choice of its value at the time zero.[326] The truly interesting object measures the *rate* of the relative velocity (i.e. the change of the relative velocity along a trajectory), i.e. we are to study the (covariant, *parallel* transport is again implicit) derivative of the relative velocity along the (ordinary) velocity, $\nabla_U(\nabla_\xi U)$. It is natural to call this quantity the *relative acceleration* of "neighboring" geodesics

$$\begin{array}{lcl} \text{relative velocity} & \leftrightarrow & \nabla_\xi U \\ \text{relative acceleration} & \leftrightarrow & \nabla_U(\nabla_\xi U) \end{array}$$

This object, the relative acceleration, is already out of our control (by means of any choices in time $t = 0$) and as Jacobi's equation (to be introduced in a while) shows, the relative acceleration feels the *curvature* of a manifold (M, ∇) where the geodesics are studied.

[326] When we had chosen arbitrarily $U(s)$ on $\sigma(s)$ and also we have already fixed implicitly $\nabla_{\dot\sigma} U \equiv \nabla_\xi U$. In particular, if we had chosen as $U(s)$ on $\sigma(s)$ the *autoparallel* field given by the vector $\dot\gamma(0)$, it should mean that we perform the variation of the *position alone* leaving the initial value of the velocity "the same" or, put another way, that we study a free motion of two objects which start from neighboring points with the same speed, their initial velocities being directed parallel to each other.

15.7.2 Let U and ξ be the fields introduced above and let the connection ∇ be *torsion-free*. Show that

(i) the relative acceleration may also be expressed as $\nabla_U^2 \xi$, since

$$\nabla_U \xi = \nabla_\xi U$$

(ii) there holds the identity

$$\nabla_U^2 \xi = R(U, \xi)U$$

(iii) for the field ξ on the initial geodesic γ this results in *Jacobi's equation* for *geodesic deviation*

$$\frac{D^2\xi}{Dt^2} \equiv \nabla_{\dot\gamma}^2 \xi = R(\dot\gamma, \xi)\dot\gamma \qquad \text{or briefly} \quad \ddot\xi = R(\dot\gamma, \xi)\dot\gamma$$

Notice that in Jacobi's equation the only quantities that occur are (i.e. it is *enough* if they are) defined on the geodesic γ alone (for $\nabla_\xi U$ we need the *field* U also in a neighborhood of γ, whereas for $\nabla_U \xi = \nabla_{\dot\gamma}\xi$ we make do with ξ on the curve γ itself).

Hint: (i) in general, $\nabla_U \xi - \nabla_\xi U - [U, \xi] = T(U, \xi)$; (ii) since U is a "geodesic field," we have $\nabla_U U = 0$; then $\nabla_U \nabla_U \xi = \nabla_U \nabla_\xi U = \nabla_U \nabla_\xi U - \nabla_\xi \nabla_U U - \nabla_{[U,\xi]} U \equiv R(U, \xi)U$.

□

15.7.3 Be sure to understand that on the right-hand side of Jacobi's equation there is a *linear operator* (depending quadratically on $\dot\gamma$) applied on the vector ξ

$$\xi \mapsto B(\xi) \equiv R(\dot\gamma, \xi)\dot\gamma \qquad \xi^i \mapsto B^i_l \xi^l \quad B^i_l := R^i{}_{jkl}\dot x^j \dot x^k$$

so that in components the equation reads

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\xi = \left(R^i{}_{jkl}\dot x^j \dot x^k \xi^l\right)\partial_i \qquad \text{i.e.} \quad (\nabla_{\dot\gamma}\nabla_{\dot\gamma}\xi)^i = R^i{}_{jkl}\dot x^j \dot x^k \xi^l$$

Hint: $R(U, V)W$ is $\mathcal{F}$-linear in *all* arguments (15.5.5). □

15.7.4 Let us examine how this equation works on the ordinary sphere S^2. Here we may consider *meridians* as a one-parameter class of geodesics (15.4.2). Then we take as the fields U and ξ simply the coordinate basis fields ∂_ϑ and ∂_φ respectively. Check that

(i) a direct computation of the left-hand side of Jacobi's equation gives

$$\nabla_{\partial_\vartheta}^2 \partial_\varphi = \cdots = -\partial_\varphi$$

(ii) by comparison with what the right-hand side of the equation should give we obtain

$$-\partial_\varphi \overset{!}{=} R(\partial_\vartheta, \partial_\varphi)\partial_\vartheta \equiv R^\vartheta{}_{\vartheta\vartheta\varphi}\partial_\vartheta + R^\varphi{}_{\vartheta\vartheta\varphi}\partial_\varphi$$

(iii) from there we can read the values of the components of the Riemann curvature tensor

$$R^\vartheta{}_{\vartheta\vartheta\varphi} = 0 \qquad R^\varphi{}_{\vartheta\vartheta\varphi} = -1$$

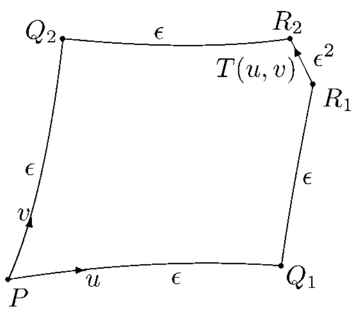
(iv) these results coincide with those obtained by a direct computation from the formula for components of the curvature tensor in terms of Christoffel symbols (15.5.5) or by expressing the result from (15.6.11) ("orthonormal" components obtained from the Cartan structure equations) in coordinate components.

Hint: see (15.3.7). □

15.8* Torsion, complete parallelism and flat connection

• We encountered the concept of (the tensor of) torsion in the section devoted to the RLC connection (15.3.3), where we learned that the requirement of *vanishing* torsion leads in combination with metricity to a unique (i.e. RLC) connection. So in this particular connection the torsion is *by definition* completely "disabled." On the other hand, exactly this particular connection is by far the most frequent linear connection met by most people (say, in general relativity). This results in the torsion mostly remaining hidden in the shadow of its much more popular sibling, the curvature.[327]

The torsion must appreciate then (even be touched to the heart) knowing that we did not forget about it. In this section we will learn in which geometrical situation the (non-vanishing) torsion manifests its presence. Namely it turns out that it causes "disclosure of a geodesic parallelogram."



15.8.1 At a point P consider two vectors u, v. The point P and the vector u define a unique (affinely parametrized) geodesic $\gamma_u(t)$. We parallel transport v to the point $Q_1 \equiv \gamma_u(\epsilon)$ along the geodesic; this results in $v_\parallel$ in Q_1. The point Q_1 and the vector $v_\parallel$ define in turn a geodesic $\gamma_{v_\parallel}(t)$. The point $\gamma_{v_\parallel}(\epsilon)$ will be labeled R_1. Now we perform the same steps with the vectors $u \leftrightarrow v$ being interchanged. In this way we obtain the points Q_2 and R_2. It is clear that in the ordinary plane we should draw a parallelogram with vertices P, Q_1, $R_1 \equiv R_2$, Q_2. It turns out, however, that on a general manifold with connection (M, ∇) there holds $R_1 \neq R_2$, and the step by which we get from R_1 to R_2 up to second-order accuracy in ϵ is realized by the vector $T(u, v)$, where T is the *torsion* of the connection ∇. One may then say that the vector $T(u, v)$ encloses within order ϵ^2 the infinitesimal geodesic parallelogram given by the vectors u and v. Check this statement by a computation.

Hint: for example, in coordinates: according to (15.4.1) the point Q_1 has the coordinates $x^i(Q_1) = x^i(P) + u^i\epsilon - \frac{1}{2}\Gamma^i_{jk}u^j u^k\epsilon^2 + \cdots$. Components of the transported vector v are

[327] As scientists recently discovered (under microscopes, I expect) this spectacular astronomical phenomenon was already pretty well known to Mayan civilization. Mayan astronomers compiled precise tables of positions for the Moon, Venus, Curvature and Torsion and were able to predict with astonishing accuracy torsion eclipses (caused by the curvature; their prediction namely stated that it always happens).

by (15.2.6) $v^i_\parallel = v^i - \epsilon \Gamma^i_{jk} u^k v^j + \cdots$ (since $\dot{x}^k = u^k$). Within order ϵ^2 then the coordinates of the point R_1 are

$$\begin{aligned} x^i(R_1) &= x^i(Q_1) + v^i_\parallel \epsilon - \frac{1}{2}\Gamma^i_{jk} v^j_\parallel v^k_\parallel \epsilon^2 \\ &\equiv x^i(P) + \epsilon(u^i + v^i) + \frac{1}{2}\epsilon^2 \Gamma^i_{jk}(-u^j u^k - v^j v^k - 2v^j u^k) \end{aligned}$$

The corresponding result for $x^i(R_2)$ is obtained by $u \leftrightarrow v$ so that

$$\begin{aligned} x^i(R_2) - x^i(R_1) &= \frac{1}{2}\epsilon^2 \Gamma^i_{jk} 2(v^j u^k - v^k u^j) = \epsilon^2\big(-2\Gamma^i_{[jk]}\big) u^j v^k \equiv \epsilon^2 T^i_{jk} u^j v^k \\ &\equiv (\epsilon^2 T(u, v))^i \end{aligned}$$

□

• These results probably reminded the reader of a similar computation in Chapter 4 where the geometrical meaning of the *commutator* $[U, V]$ of two vector fields U and V was discussed (4.5.3). In what way do these constructions actually differ?

In Chapter 4 we managed without any connection, here we definitely need it. In particular, there we moved along *integral curves* of the vector fields involved, whereas here we move along *geodesics*. There we needed the fields U, V *also in a neighborhood* of the point P, whereas here we make do with the vectors u, v *at the point* P *alone*.

In both cases unclosed parallelograms arose; then due to non-vanishing $[U, V]$, now due to non-vanishing $T(U, V)$. As an enclosing piece (up to order ϵ^2) one had to add then $-\epsilon^2[U, V]$, now it is $+\epsilon^2 T(U, V)$.

There is also an equivalent way of expressing the effect of torsion. Contemplate vectors u, v at the point P. Extend them to vector *fields* U, V in a small *neighborhood* of the point P as follows: if Q is a point in the neighborhood, we construct a geodesic from P to Q and parallel transport the vectors u, v to Q along the geodesic (recall that a parametrization of the geodesic does not matter). All the transported vectors then constitute the vector fields U, V. By construction their covariant derivative in any direction vanishes *at the point* P so that we get for the tensor of torsion at that point $T_P(U, V) = (\nabla_U V)_P - (\nabla_V U)_P - [U, V]_P = -[U, V]_P$. The effect of torsion thus happens to coincide with the effect of (minus) the commutator *of these* vector fields. The latter manifests itself when traveling along their *integral curves*, coinciding here in turn just with geodesics (a geodesic given by a vector v arises by the parallel transport of v "along itself"; that is, however, exactly the way in which the values of the field V arise), so that the above-mentioned "geodesic" construction actually matches the construction in terms of the integral curves used here.

Recall also that Section 15.5, describing curvature, starred yet another important "non-closure phenomenon" which is related to connections (this time, however, not at the level of the points along which we travel, but rather at the level of tensors being transported; a tensor suffers a change due to the parallel transport along a loop). Let us illustrate non-vanishing torsion with the example of a simple connection where the effect of the torsion may be easily grasped visually.

Consider as a manifold the two-dimensional sphere S^2 with both the north and south poles removed, endowed with the common "round" metric tensor. If it is as big as the surface of the Earth, it may easily happen we actually do not recognize it is a sphere (it took some time for mankind, too) and we believe we walk on a Euclidean plane. Then it is natural to perform the parallel transport of vectors as follows.

First, we measure the length of the vector to be transported and arrange the *length* to be *the same* after the transport. Then the only issue which remains is its direction. In order to fix the direction we use a compass and measure the azimuth of the initial vector (i.e. the angle clockwise from due north; this does not work at the poles, but recall they were removed from the manifold at the very beginning with wondrous foresight). We then prescribe the *same azimuth* to the transported vector. If we believe we walk on a Euclidean plane (endowed with a distinguished "north" direction) we have a clear conscience that we did our best to realize parallel transport in the *most common* intuitive sense.[328]

15.8.2 Check that the connection on the sphere with removed poles which was introduced above is metric, it has vanishing curvature and *non-vanishing torsion*.

Hint: by construction it is evident that the scalar product of vectors is preserved (so that it is metric) and that parallel transport does not depend on the path ($\Rightarrow$ vanishing curvature). We also see that the standard orthonormal frame field (e_ϑ, e_φ) on the sphere *is parallel*, i.e. that for *any* V there holds $\nabla_V e_\vartheta = 0 = \nabla_V e_\varphi$ (it is enough to realize how parallel transport of *these* vectors to another point turns out). Then (on the sphere with unit radius),

$$T(e_\vartheta, e_\varphi) \equiv \nabla_{e_\vartheta} e_\varphi - \nabla_{e_\varphi} e_\vartheta - [e_\vartheta, e_\varphi] = -[e_\vartheta, e_\varphi] = -\left[\partial_\vartheta, \frac{1}{\sin\vartheta}\partial_\varphi\right]$$
$$= \frac{\cos\vartheta}{\sin^2\vartheta}\partial_\varphi \equiv \frac{\cos\vartheta}{\sin\vartheta}e_\varphi \neq 0$$

□

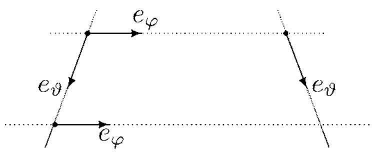
15.8.3 Check that all meridians *as well* (in contrast to RLC) *as parallel lines* (and even in general all *loxodromes*) turn out to be geodesics of this connection.

Hint: $\nabla_U e_\vartheta = 0 = \nabla_U e_\varphi$ (for arbitrary U) results in $\nabla_V V = 0$ for $V = k_1 e_\vartheta + k_2 e_\varphi$; in particular, $\nabla_{e_\varphi} e_\varphi = 0$ says that integral curves of the field e_φ (parallel lines) are geodesics; for general k_1, k_2 the integral curves happen to coincide with loxodromes (3.2.8). □

• The fact that in this particular case there holds $T(e_\vartheta, e_\varphi) = -[e_\vartheta, e_\varphi] \neq 0$ means that the vector which encloses a *geodesic* parallelogram coincides with the vector enclosing the parallelogram made from *integral curves*. This is not an accident. Both procedures of construction of the parallelograms eventually lead to the same result: if we take e_ϑ, e_φ as U, V, a motion along integral curves is the same as the motion along geodesics (the first halves of the construction thus coincide), the parallel transport of the second vector along

[328] This technique can be safely used at the scale of a town, say; as a preparation the reader is invited to use it at a copy-book scale.

the geodesic given by the first one results in the value of the second field at a new place so that also the second halves of the construction give the same result. The effect of torsion thus coincides here with the effect of non-commutativity of the fields e_ϑ and e_φ. However, here the latter is *very clear* visually.



15.8.4 Be sure you understand that the effect of non-commutativity of the fields e_φ and e_ϑ (and consequently also of the non-vanishing torsion of the connection under consideration) consists of the elementary fact that if we move a small distance eastwards and then the same distance southwards, we do not reach (exactly) the same point as if we did the same steps in the opposite order. Try to obtain (by an elementary computation) the difference and check that you get the same result as you get by "scientific" consideration, i.e. by the computation of the term $\epsilon^2 T(e_\varphi, e_\vartheta) \equiv -\epsilon^2[e_\varphi, e_\vartheta]$.

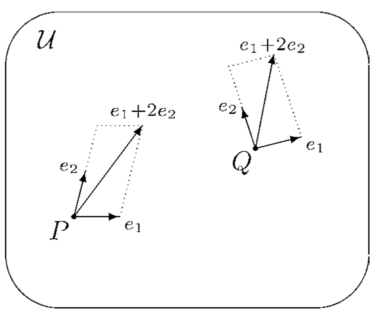
Hint: the distance between meridians gets shorter when we start to move in a direction toward the poles. □

• In the example discussed above an important class of connections has been illustrated, called a *complete parallelism*. This comes into being when in a domain on a manifold there is a *covariantly constant frame field* e_a (alternatively it is known as a *parallel* frame field), i.e. a frame field for which the covariant derivative in an arbitrary direction vanishes, $\nabla_V e_a = 0$ for each V, so that also

$$\nabla e_a = 0 \qquad \text{or} \qquad e_{a;\mu} = 0$$

Such a field may in general be non-holonomic. If it happens to be holonomic (i.e. coordinate), we speak about a *flat connection*.

Yet another name used for a complete parallelism is *teleparallelism*, i.e. parallelism "at a distance." The origin of this terminology will be clear from the next problem.



15.8.5 Let e_a be a covariantly constant frame field in a domain $\mathcal{U}$. Be sure to understand that parallel transport in this domain *does not depend on the path*, so that there exists a natural identification of any two tangent spaces in $\mathcal{U}$. Consequently a comparison of vectors sitting in different (possibly fairly remote) points in $\mathcal{U}$ now *makes* "absolute" *sense* (see the motivation of the concept of a connection at the very beginning of the chapter).

Hint: covariant constancy of the frame field gives the following equation of the parallel transport of (say) a vector field: $0 = \nabla_{\dot\gamma}(A^a e_a) = \dot A^a e_a$, i.e. $A^a(t) = k^a \equiv$ constant, *regardless of the path* along which the vector is transported ⇒

the transport (within the domain where the frame field e_a operates) from a point P to any point Q looks like $k^a e_a(P) \mapsto k^a e_a(Q)$ (the transport thus consists in decomposing the vector at the point P with respect to the basis e_a and thereafter in composing it back *with the same coefficients* at the final point Q; so it works as if we made an "immediate leap" to Q (i.e. transport "at a distance")). Two vectors at different points are regarded as being "equal" if they have equal coefficients with respect to e_a at these two points. □

15.8.6 Check that

(i) for a complete parallelism the connection forms vanish at an appropriate basis, $\omega^a_b = 0$, for a flat connection the Christoffel symbols vanish at appropriate coordinates, $\Gamma^i_{jk} = 0$

(ii) for a complete parallelism the curvature vanishes and for a flat connection both the curvature and torsion vanish

$$\begin{aligned} \nabla_{e_a} e_b &= 0 \ \text{(complete parallelism)} &&\Rightarrow R^a{}_{bcd} = 0 \\ \nabla_{\partial_a} \partial_b &= 0 \ \text{(flat connection)} &&\Rightarrow R^a{}_{bcd} = 0 = T^a_{bc} \end{aligned}$$

Hint: (i) for a covariantly constant frame we have $\nabla_V e_a = \omega^b_a(V) e_b = 0$ or $\nabla_{\partial_a} \partial_b = \Gamma^c_{ba} \partial_c = 0$; (ii) from the Cartan structure equations (15.6.7); or since $\nabla_{e_a} e_b = 0$, we have $R(e_a, e_b) e_c \equiv (\nabla_{e_a} \nabla_{e_b} - \nabla_{e_b} \nabla_{e_a} - \nabla_{[e_a, e_b]}) e_c = 0$; $T(e_a, e_b) \equiv \nabla_{e_a} e_b - \nabla_{e_b} e_a - [e_a, e_b] = -[e_a, e_b]$, which vanishes exactly for $e_a = \partial_a$. □

• Note that both statements in problem (15.8.6) had the form of a one-way implication. The opposite implication

$$\begin{aligned} R^a{}_{bcd} = 0 \quad &\overset{?}{\Rightarrow} \quad \exists e_a : \ \nabla_{e_a} e_b = 0 \ \text{(complete parallelism)} \\ R^a{}_{bcd} = 0 = T^a_{bc} \quad &\overset{?}{\Rightarrow} \quad \exists x^a : \nabla_{\partial_a} \partial_b = 0 \ \text{(flat connection)} \end{aligned}$$

is a priori not clear and the issue needs a special analysis. One line of thought might be based on the way in which the curvature tensor occurred: its vanishing guarantees the triviality of parallel transport around particular infinitesimal loops. This can also be extended to bigger (finite) loops and one may infer from that the possibility of transport of a frame from the point P to its neighborhood *independent* of path, which implies the existence of the covariantly constant frame field being sought.

A different line of thought goes as follows: vanishing curvature means that *for each* frame field we have $0 = \Omega \equiv d\omega + \omega \wedge \omega$. This does not necessarily also mean $\omega = 0$ (covariantly constant e_a). A general change of frame field by a matrix A results in $\omega \mapsto \hat{\omega} = A^{-1}(\omega A + dA)$, so that the question of whether there exists a frame field $\hat{e}_a$ such that $\hat{\omega} = 0$ leads to the formulation of the problem of whether there exists a non-singular matrix field $A(x)$ which obeys a system of (partial differential) equations $\omega A + dA = 0$. The problem may be solved and the answer is *yes*.

The third possibility (which will be adopted here) is to postpone the discussion until Chapter 20, see (20.4.11), to the general context of connection theory. In this theory the notion of a covariantly constant frame field takes an interesting geometrical interpretation in terms of "integrable distributions"; by referring to the "Frobenius integrability condition" we learn that vanishing curvature is indeed also a *sufficient* condition for the existence of a covariantly constant frame field[329] (so the first implication *holds*), from which then already the *validity of the second* implication follows immediately.[330] Vanishing of both the curvature and torsion tensors thus means that the connection is flat.

15.8.7 Check that

(i) the ordinary RLC connection in (pseudo-)Euclidean space $E^{r,s}$ is flat
(ii) the connection on a Lie group G which we mentioned in problem (15.4.15) (parallel transport being given by left translation) is a complete parallelism, but in general it is not flat.

Hint: (i) $e_a = \partial_a =$ *Cartesian* frame field; (ii) $e_a =$ *left-invariant* basis; the latter fails to be holonomic for non-Abelian groups. □

15.8.8 * Define a connection on a Lie group G by the formula

$$\nabla_{L_X} L_Y := \lambda[L_X, L_Y] \equiv \lambda L_{[X,Y]} \qquad L_X, L_Y \text{ left-invariant fields}$$

(λ being a real parameter to be specified later). Check that the explicit expressions for the curvature and torsion of this connection read

$$R(L_X, L_Y)L_Z = \lambda(\lambda - 1)L_{[[X,Y],Z]} \qquad \text{i.e.} \quad R^a{}_{bcd} = \lambda(\lambda - 1)c^a_{fb}c^f_{cd}$$
$$T(L_X, L_Y) = (2\lambda - 1)L_{[X,Y]} \qquad \text{i.e.} \quad T^a_{bc} = (2\lambda - 1)c^a_{bc}$$

(e_a is a left-invariant frame field) and we may identify the following special cases:

$\lambda = \frac{1}{2}$	$R^a{}_{bcd} \neq 0$	$T^a_{bc} = 0$	RLC connection for Killing metric $\mathcal{K}$ on G
$\lambda = 0$	$R^a{}_{bcd} = 0$	$T^a_{bc} \neq 0$	parallel transport is *left* translation (15.4.15)
$\lambda = 1$	$R^a{}_{bcd} = 0$	$T^a_{bc} \neq 0$	parallel transport is *right* translation

This connection is not flat, but for $\lambda = 0$ as well as $\lambda = 1$ we have complete parallelism.

Hint: computation of R and T right from the definitions; for any λ the connection turns out to be metric with respect to the Killing metric:

$$\begin{aligned}
\nabla_{L_Z}\{\mathcal{K}(L_X, L_Y)\} &\overset{1}{=} L_Z K(X, Y) = 0 \\
&\overset{2}{=} (\nabla_{L_Z}\mathcal{K})(L_X, L_Y) + \mathcal{K}(\nabla_{L_Z} L_X, L_Y) + \mathcal{K}(L_X, \nabla_{L_Z} L_Y) \\
&= (\nabla_{L_Z}\mathcal{K})(L_X, L_Y) + \lambda\{K([Z, X], Y) + K(X, [Z, Y])\} \\
&= (\nabla_{L_Z}\mathcal{K})(L_X, L_Y) \quad \text{due to (12.3.9)}
\end{aligned}$$

Then (15.8.7) and (15.4.15). □

[329] The notion of curvature itself leans heavily on the integrability condition mentioned above. Namely the curvature is introduced *so that* integrability would (by definition) mean *vanishing* curvature. It turns out that a covariantly constant frame field corresponds to a "horizontal section" and that the latter exists if and only if a horizontal distribution happens to be integrable.

[330] If $R^a{}_{bcd} = 0$ gives $\nabla_{e_a} e_b = 0$, then *with respect to this* frame field we have $0 = T^a_{bc} = \cdots = -\langle e^a, [e_b, e_c]\rangle \Rightarrow [e_b, e_c] = 0$, i.e. $e_a = \partial_a$.

Summary of Chapter 15

In many applications (e.g. in the computation of acceleration of a point mass in elementary mechanics) one performs linear combinations (in particular, the difference in the case of acceleration) of vectors (or more generally tensors) sitting at different points of a manifold. This is not possible on a "bare" manifold. The structure which makes it legal is a (linear) *connection* ∇ on M. The connection enables one to transport vectors along a given path (the transport being path-dependent in general) and consequently to perform the above-mentioned comparison (vector in x is compared with the one being *transported to* x from y). This transport is *by definition* called *parallel* (in the sense of the connection ∇). A connection is frequently defined by postulating the properties of a derived object, the *covariant derivative*. One can introduce the concept of a straight line (geodesic) on (M, ∇). Two tensor fields are associated with a linear connection, the curvature and torsion tensors. It is shown that the requirements of compatibility of a connection with the metric (conservation of any scalar product upon any parallel transport) together with vanishing of its torsion result in a unique connection, the Riemannian or Levi-Civita (RLC) connection. The curvature tensor encodes the local information of "how much" (if ever) the parallel transport (along infinitesimal paths) is path-dependent; it also displays itself in the behavior of nearby geodesics, causing their deviation (Jacobi's equation). A non-zero torsion implies non-closure of a geodesic parallelogram. An efficient tool for working with a connection is provided by the machinery of differential forms. Basic objects are encoded into forms and relations between them are given by the Cartan structure equations. A connection is called a complete parallelism if there exists a covariantly constant frame field. Then the curvature tensor turns out to vanish and moreover a comparison of vectors (as well as tensors) in different (possibly remote) points makes sense. A connection is said to be flat if the covariantly constant frame field happens to be holonomic (coordinate). Then both the curvature and torsion tensors turn out to vanish.

$\nabla_a e_b =: \Gamma^c_{ba} e_c$	Coefficients of connection with respect to e_a	(15.2.1)
$\nabla_j \partial_i =: \Gamma^k_{ij} \partial_k$	Christoffel symbols of the second kind	(15.2.3)
$\dot{V}^i + \Gamma^i_{jk} \dot{x}^k V^j = 0$	Equations of parallel transport of vector	(15.2.6)
$\nabla g = 0 \quad (g_{ij;k} = 0)$	Connection ∇ is metric	(15.3.1)
$T(U, V) := \nabla_U V - \nabla_V U - [U, V]$	Torsion tensor induced by ∇	(15.3.3)
$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})$	Riemann/Levi-Civita connection (RLC)	(15.3.4)
$\nabla_{\dot\gamma} \dot\gamma = 0 \quad \left(\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0\right)$	Geodesic equation	(15.4.1)
$\exp v := \gamma_v(1), \quad \dot\gamma_v(0) = v \in T_P M$	Exponential map centered at $P \in M$	(15.4.10)
$\langle \alpha, ([\nabla_U, \nabla_V] - \nabla_{[U,V]}) W \rangle$	Riemann curvature tensor	(15.5.5)
$R_{ab} := R^c{}_{acb}, \quad R := R^a{}_a \equiv R^{ab}{}_{ab}$	Ricci tensor and scalar curvature	Sec. 15.5
$\nabla_V e_a = \omega^b_a(V) e_b \quad \left(\omega^a_b = \Gamma^a_{bc} e^c\right)$	Connection forms ω^b_a with respect to e_a	(15.6.1)
$\omega' = A^{-1} \omega A + A^{-1} dA$	Transformation law for ω under $e' = eA$	(15.6.2)
$de + \omega \wedge e = T, \quad d\omega + \omega \wedge \omega = \Omega$	Cartan structure equations	(15.6.7)
$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0, \ \Omega \wedge e = 0$	Bianchi and Ricci identities (for RLC)	(15.6.16)
$\nabla^2_{\dot\gamma} \xi = R(\dot\gamma, \xi) \dot\gamma$	Jacobi's equation for geodesic deviation	(15.7.2)
$R^a{}_{bcd} = 0 = T^a_{bc}$	Flat connection	(15.8.6)