

DIFFERENTIAL GEOMETRY AND LIE GROUPS FOR PHYSICISTS

MARIÁN FECKO

Comenius University, Bratislava, Slovakia
and

Slovak Academy of Sciences, Bratislava, Slovakia



CAMBRIDGE
UNIVERSITY PRESS

Contents

| | | |
|---|---|----------------|
| | <i>Preface</i> | <i>page xi</i> |
| | Introduction | 1 |
| 1 | The concept of a manifold | 4 |
| | 1.1 Topology and continuous maps | 4 |
| | 1.2 Classes of smoothness of maps of Cartesian spaces | 6 |
| | 1.3 Smooth structure, smooth manifold | 7 |
| | 1.4 Smooth maps of manifolds | 11 |
| | 1.5 A technical description of smooth surfaces in \mathbb{R}^n | 16 |
| | Summary of Chapter 1 | 20 |
| 2 | Vector and tensor fields | 21 |
| | 2.1 Curves and functions on M | 22 |
| | 2.2 Tangent space, vectors and vector fields | 23 |
| | 2.3 Integral curves of a vector field | 30 |
| | 2.4 Linear algebra of tensors (multilinear algebra) | 34 |
| | 2.5 Tensor fields on M | 45 |
| | 2.6 Metric tensor on a manifold | 48 |
| | Summary of Chapter 2 | 53 |
| 3 | Mappings of tensors induced by mappings of manifolds | 54 |
| | 3.1 Mappings of tensors and tensor fields | 54 |
| | 3.2 Induced metric tensor | 60 |
| | Summary of Chapter 3 | 63 |
| 4 | Lie derivative | 65 |
| | 4.1 Local flow of a vector field | 65 |
| | 4.2 Lie transport and Lie derivative | 70 |
| | 4.3 Properties of the Lie derivative | 72 |
| | 4.4 Exponent of the Lie derivative | 75 |
| | 4.5 Geometrical interpretation of the commutator $[V, W]$, non-holonomic frames | 77 |
| | 4.6 Isometries and conformal transformations, Killing equations | 81 |
| | Summary of Chapter 4 | 91 |

| | | |
|----|--|-----|
| 5 | Exterior algebra | 93 |
| | 5.1 Motivation: volumes of parallelepipeds | 93 |
| | 5.2 p -forms and exterior product | 95 |
| | 5.3 Exterior algebra ΛL^* | 102 |
| | 5.4 Interior product i_v | 105 |
| | 5.5 Orientation in L | 106 |
| | 5.6 Determinant and generalized Kronecker symbols | 107 |
| | 5.7 The metric volume form | 112 |
| | 5.8 Hodge (duality) operator $*$ | 118 |
| | Summary of Chapter 5 | 125 |
| 6 | Differential calculus of forms | 126 |
| | 6.1 Forms on a manifold | 126 |
| | 6.2 Exterior derivative | 128 |
| | 6.3 Orientability, Hodge operator and volume form on M | 133 |
| | 6.4 V -valued forms | 139 |
| | Summary of Chapter 6 | 143 |
| 7 | Integral calculus of forms | 144 |
| | 7.1 Quantities under the integral sign regarded as differential forms | 144 |
| | 7.2 Euclidean simplices and chains | 146 |
| | 7.3 Simplices and chains on a manifold | 149 |
| | 7.4 Integral of a form over a chain on a manifold | 150 |
| | 7.5 Stokes' theorem | 151 |
| | 7.6 Integral over a domain on an orientable manifold | 153 |
| | 7.7 Integral over a domain on an orientable Riemannian manifold | 159 |
| | 7.8 Integral and maps of manifolds | 161 |
| | Summary of Chapter 7 | 163 |
| 8 | Particular cases and applications of Stokes' theorem | 164 |
| | 8.1 Elementary situations | 164 |
| | 8.2 Divergence of a vector field and Gauss' theorem | 166 |
| | 8.3 Codifferential and Laplace–deRham operator | 171 |
| | 8.4 Green identities | 177 |
| | 8.5 Vector analysis in E^3 | 178 |
| | 8.6 Functions of complex variables | 185 |
| | Summary of Chapter 8 | 188 |
| 9 | Poincaré lemma and cohomologies | 190 |
| | 9.1 Simple examples of closed non-exact forms | 191 |
| | 9.2 Construction of a potential on contractible manifolds | 192 |
| | 9.3* Cohomologies and deRham complex | 198 |
| | Summary of Chapter 9 | 203 |
| 10 | Lie groups: basic facts | 204 |
| | 10.1 Automorphisms of various structures and groups | 204 |

| | | |
|-------|--|-----|
| 10.2 | Lie groups: basic concepts | 210 |
| | Summary of Chapter 10 | 213 |
| 11 | Differential geometry on Lie groups | 214 |
| 11.1 | Left-invariant tensor fields on a Lie group | 214 |
| 11.2 | Lie algebra \mathcal{G} of a group G | 222 |
| 11.3 | One-parameter subgroups | 225 |
| 11.4 | Exponential map | 227 |
| 11.5 | Derived homomorphism of Lie algebras | 230 |
| 11.6 | Invariant integral on G | 231 |
| 11.7 | Matrix Lie groups: enjoy simplifications | 232 |
| | Summary of Chapter 11 | 243 |
| 12 | Representations of Lie groups and Lie algebras | 244 |
| 12.1 | Basic concepts | 244 |
| 12.2 | Irreducible and equivalent representations, Schur's lemma | 252 |
| 12.3 | Adjoint representation, Killing–Cartan metric | 259 |
| 12.4 | Basic constructions with groups, Lie algebras and their representations | 269 |
| 12.5 | Invariant tensors and intertwining operators | 278 |
| 12.6* | Lie algebra cohomologies | 282 |
| | Summary of Chapter 12 | 287 |
| 13 | Actions of Lie groups and Lie algebras on manifolds | 289 |
| 13.1 | Action of a group, orbit and stabilizer | 289 |
| 13.2 | The structure of homogeneous spaces, G/H | 294 |
| 13.3 | Covering homomorphism, coverings $SU(2) \rightarrow SO(3)$ and $SL(2, \mathbb{C}) \rightarrow L_+^\uparrow$ | 299 |
| 13.4 | Representations of G and \mathcal{G} in the space of functions on a G -space, fundamental fields | 310 |
| 13.5 | Representations of G and \mathcal{G} in the space of tensor fields of type $\hat{\rho}$ | 319 |
| | Summary of Chapter 13 | 325 |
| 14 | Hamiltonian mechanics and symplectic manifolds | 327 |
| 14.1 | Poisson and symplectic structure on a manifold | 327 |
| 14.2 | Darboux theorem, canonical transformations and symplectomorphisms | 336 |
| 14.3 | Poincaré–Cartan integral invariants and Liouville's theorem | 341 |
| 14.4 | Symmetries and conservation laws | 346 |
| 14.5* | Moment map | 349 |
| 14.6* | Orbits of the coadjoint action | 354 |
| 14.7* | Symplectic reduction | 360 |
| | Summary of Chapter 14 | 368 |
| 15 | Parallel transport and linear connection on M | 369 |
| 15.1 | Acceleration and parallel transport | 369 |
| 15.2 | Parallel transport and covariant derivative | 372 |
| 15.3 | Compatibility with metric, RLC connection | 382 |
| 15.4 | Geodesics | 389 |

| | | |
|-------|--|-----|
| 15.5 | The curvature tensor | 401 |
| 15.6 | Connection forms and Cartan structure equations | 406 |
| 15.7 | Geodesic deviation equation (Jacobi's equation) | 418 |
| 15.8* | Torsion, complete parallelism and flat connection | 422 |
| | Summary of Chapter 15 | 428 |
| 16 | Field theory and the language of forms | 429 |
| 16.1 | Differential forms in the Minkowski space $E^{1,3}$ | 430 |
| 16.2 | Maxwell's equations in terms of differential forms | 436 |
| 16.3 | Gauge transformations, action integral | 441 |
| 16.4 | Energy–momentum tensor, space-time symmetries and conservation laws due to them | 448 |
| 16.5* | Einstein gravitational field equations, Hilbert and Cartan action | 458 |
| 16.6* | Non-linear sigma models and harmonic maps | 467 |
| | Summary of Chapter 16 | 476 |
| 17 | Differential geometry on TM and T^*M | 478 |
| 17.1 | Tangent bundle TM and cotangent bundle T^*M | 478 |
| 17.2 | Concept of a fiber bundle | 482 |
| 17.3 | The maps Tf and T^*f | 485 |
| 17.4 | Vertical subspace, vertical vectors | 487 |
| 17.5 | Lifts on TM and T^*M | 488 |
| 17.6 | Canonical tensor fields on TM and T^*M | 494 |
| 17.7 | Identities between the tensor fields introduced here | 497 |
| | Summary of Chapter 17 | 497 |
| 18 | Hamiltonian and Lagrangian equations | 499 |
| 18.1 | Second-order differential equation fields | 499 |
| 18.2 | Euler–Lagrange field | 500 |
| 18.3 | Connection between Lagrangian and Hamiltonian mechanics, Legendre map | 505 |
| 18.4 | Symmetries lifted from the base manifold (configuration space) | 508 |
| 18.5 | Time-dependent Hamiltonian, action integral | 518 |
| | Summary of Chapter 18 | 522 |
| 19 | Linear connection and the frame bundle | 524 |
| 19.1 | Frame bundle $\pi : LM \rightarrow M$ | 524 |
| 19.2 | Connection form on LM | 527 |
| 19.3 | k -dimensional distribution \mathcal{D} on a manifold \mathcal{M} | 530 |
| 19.4 | Geometrical interpretation of a connection form: horizontal distribution on LM | 538 |
| 19.5 | Horizontal distribution on LM and parallel transport on M | 543 |
| 19.6 | Tensors on M in the language of LM and their parallel transport | 545 |
| | Summary of Chapter 19 | 550 |
| 20 | Connection on a principal G -bundle | 551 |
| 20.1 | Principal G -bundles | 551 |

| | | |
|------------|---|-----|
| 20.2 | Connection form $\omega \in \Omega^1(P, \text{Ad})$ | 559 |
| 20.3 | Parallel transport and the exterior covariant derivative D | 563 |
| 20.4 | Curvature form $\Omega \in \Omega^2(P, \text{Ad})$ and explicit expressions of D | 567 |
| 20.5* | Restriction of the structure group and connection | 576 |
| | Summary of Chapter 20 | 585 |
| 21 | Gauge theories and connections | 587 |
| 21.1 | Local gauge invariance: “conventional” approach | 587 |
| 21.2 | Change of section and a gauge transformation | 594 |
| 21.3 | Parallel transport equations for an object of type ρ in a gauge σ | 600 |
| 21.4 | Bundle $P \times_{\rho} V$ associated to a principal bundle $\pi : P \rightarrow M$ | 606 |
| 21.5 | Gauge invariant action and the equations of motion | 607 |
| 21.6 | Noether currents and Noether’s theorem | 618 |
| 21.7* | Once more (for a while) on LM | 626 |
| | Summary of Chapter 21 | 633 |
| 22* | Spinor fields and the Dirac operator | 635 |
| 22.1 | Clifford algebras $C(p, q)$ | 637 |
| 22.2 | Clifford groups $\text{Pin}(p, q)$ and $\text{Spin}(p, q)$ | 645 |
| 22.3 | Spinors: linear algebra | 650 |
| 22.4 | Spin bundle $\pi : SM \rightarrow M$ and spinor fields on M | 654 |
| 22.5 | Dirac operator | 662 |
| | Summary of Chapter 22 | 670 |
| Appendix A | Some relevant algebraic structures | 673 |
| A.1 | Linear spaces | 673 |
| A.2 | Associative algebras | 676 |
| A.3 | Lie algebras | 676 |
| A.4 | Modules | 679 |
| A.5 | Grading | 680 |
| A.6 | Categories and functors | 681 |
| Appendix B | Starring | 683 |
| | <i>Bibliography</i> | 685 |
| | <i>Index of (frequently used) symbols</i> | 687 |
| | <i>Index</i> | 690 |