PREFACE

This is an **introductory** text dealing with a part of **mathematics**, modern differential geometry and the theory of Lie groups. It is written **from the perspective** of and **mainly for the needs** of **physicists**. The orientation on physics makes itself felt in the choice of material, in the way it is presented (e.g. with no use of definition-theorem-proof scheme) as well as in the content of exercises (often they are closely related to physics).

Its potential **readership** does **not**, however, consist of **physicists alone**. Since the book is about mathematics and since physics has served for a fairly long time as a rich source of inspiration for mathematics, it might be useful for the mathematical community as well. More generally, it is suitable for anybody who has some (rather modest) preliminary background knowledge (to be specified in a while) and who desires to become familiar **in a comprehensible way** with this interesting, important and living subject, which penetrates increasingly into various branches of modern theoretical physics, "pure" mathematics itself as well as into its numerous applications.

So, what is the **minimal background knowledge** necessary for a meaningful study of this book? As mentioned above, the demands are fairly modest, indeed. The required mathematical background knowledge does not go beyond what should be familiar from standard introductory **undergraduate mathematics courses** taken by physics or **even engineering** majors. This in particular includes some calculus as well as linear algebra (the reader should be familiar with stuff like partial derivatives, several variables Taylor expansion, multiple Riemann integral, linear maps versus matrices, bases and subspaces of a linear space and so on). Some experience in writing and solving simple systems of ordinary differential equations, as well as a clear understanding of what is actually behind this activity, is highly desirable. Necessary basics in algebra in the form used in the main text are concisely summarized in Appendix A at the end of the book, enabling the reader to fill particular gaps "on the run", too.

The book is intentionally written in a form which makes it possible to be fully grasped also by a **self-taught** person - anybody who is attracted by **tensor** and **spinor fields** or by **fiber bundles**, who would like to learn how **differential forms** are **differentiated** and **integrated**, who wants to see how symmetries are related to **Lie groups and algebras** as well as to their **representations**, what is **curvature** and **torsion**, why **symplectic geometry** is useful in **Lagrangian** and **Hamiltonian mechanics**, in what sense **connections** and **gauge fields** realize the same idea, how **Noetherian currents** emerge and how they are related to **conservation laws** etc.

Clearly, it is highly advantageous, as the scope of the book indicates, to be familiar (at least superficially) with the relevant parts of physics, on which the applications of various techniques are illustrated. However, one may derive profit from the book (in term of geometry alone) even with no background from physics. If we have never seen, say, **Maxwell's equations** and we are not aware at all of their role in physics, then although we will not be able to understand *why* such attention is paid to them, nevertheless we will understand perfectly *what* we do with these equations here from the technical point of view. We will see how these **partial differential equations** may be reformulated in terms of differential forms, what the action integral looks like in this particular case, how **conservation laws** may be derived from it by means of the **energy-momentum tensor** and so on. And if we find it interesting, we may hopefully learn some "traditional" material on electrodynamics behindhand.

If we, in like manner, know nothing about general relativity, then although we will not understand from where the concept of a **"curved" space-time** endowed with a **metric tensor** emerged, still we will learn the basics of what space-time is from a geometrical point of view and what is standardly done there. We will not penetrate into the physical heart of the **Einstein equations** for **gravitational field**, we will see, however, their formal structure and we will learn some simple, though at the same time powerful techniques for routine manipulations with these equations. Mastering this machinery then greatly **facilitates** to grasp the **physical** side of the theory, if we will later read about general relativity something written from the physical perspective.

The key qualification asked of the future reader is a real **interest in learning** the subject treated in the book not only in a Platonic way (say, for the sake of an intellectual conversation at a party) but rather at a **working level**. Needless to say, one then has to accept a natural consequence: it is not possible to achieve this objective by a passive reading of a "noble science" alone. On the contrary, a fairly large amount of "dirty" self-activity is needed (an ideal potential reader should be *pleased* by reading this fact), inevitably combined with due investment of time. Formal organization of the book strongly promotes this way of study.

Namely, a specific feature of the book is its strong emphasis on developing the general theory through a **large number of simple exercises** (more than a thousand of them), in which the reader analyzes **"in a hands-on**

fashion" various details of a "theory" as well as plenty of **concrete examples** (the proof of the pudding is in the eating). This style is highly appreciated, according to my teaching experience, by many students.

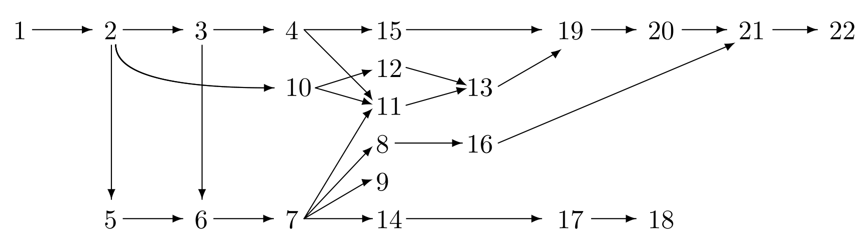
The beginning of an exercise is indicated by a box containing its number (as an example, 14.4.3 denotes the third exercise in Section 4, Chapter 14), the end of the exercise is marked by a square □. The majority of exercises (around nine hundred) are endowed with a **hint** (often quite detailed) and some of them, around fifty, with a **worked solution**. The symbol • marks the beginning of "text", which is not an exercise (a "theory" or a comment to exercises). Starred sections (like 12.6.*) as well as starred exercises may be omitted at the first reading (they may be regarded as a complement to the "hard core" of the book; actually they need not be harder but more specific material is often treated there).

This book contains a fairly large amount of material, so that a few words might be useful on how to read it efficiently. There are several ways in how to proceed, depending on what we actually need and how much time and effort we are willing to devote to the study.

The basic way, which we recommend the most, consists in systematic reading from cover to cover, solving step by step (nearly) all problems. This is the way in which we may make full use of the text. The subject may be understood in sufficient broadness, with a lot of interrelations and applications. This needs, however, enough motivation and patience.

If we lack either, we may proceed differently. Namely, we will solve in detail only those problems which we, for some reason, regard as particularly interesting or from which we crucially need the result. Proceeding this way, it may happen here and there that we will not be able to solve some problem; we are lacking some vital link (knowledge or possibly a skill) treated in the material being omitted. If we are able to locate the missing link (the numbers of useful previous exercises, mentioned in hints, might help in doing so), we simply fill this gap behindhand.

Yet more quickly will proceed a reader, who decides to restrict the study to a particular direction of interest and who is interested in the rest part of the book only to the extent that it is important for his or her preferred direction. As an aid to a reader of this category we present here the scheme of **logical dependence of the chapters**:



(The scheme does not represent the dependence completely; several sections, short parts or even individual exercises would require drawing additional arrows into it, making the scheme then, however, virtually worthless.)

To be more explicit, one could mention the following possible particular directions of interest.

1. The geometry needed for the fundamentals of **general relativity** (**covariant derivatives**, **curvature tensor**, **geodesics**, etc.).

One should follow the line 1 - 2 - 3 - 4 - 15 (similar stuff goes well with advanced **continuum mechanics**). If we want to master working with forms, too (to grasp, as an example, section 15.6., dealing with the computation of the Riemann tensor in terms of Cartan's structure equations, or section 16.5. on Einstein's equations and their derivation from an action integral), we have to add chapters 5 - 6 - 7.

2. **Elementary** theory of **Lie groups** and their **representations** ("(differential) geometry-free mini-course").

The route might contain the chapters (or only the explicitly mentioned sections of some of them) 1 - 2.4 - 10 - 11.7 - 12 - 13.1,2,3

3. **Hamiltonian** mechanics and **symplectic manifolds**.

The minimal itinerary contains chapters 1 - 2 - 3 - beginning of 4 - 5 - 6 - 7 - 14. Its extension (the formulation of Lagrangian and Hamiltonian mechanics on the fiber bundles TM and T^*M respectively) takes place in chapters 17 - 18. If we have the ambition to follow the more advanced sections on symmetries (14.5.- 14.7. and 18.4.), we need to understand the geometry on Lie groups and the actions of Lie groups on manifolds (chapters 11 - 13).

4. Basics of working with **differential forms**.

The route could be 1 - 2 - 3 - beginning of 4 - 5 - 6 - 7 - 8 - 9, or perhaps adding the beginning of chapter 16.

This book stems from (and in turn covers) several courses I started to give roughly fifteen years ago for theoretical physics students at the Faculty of Mathematics and Physics in Bratislava. It has been, however, extended (for the convenience of those smart students who are interested in a broader scope on the subject) as well as polished a bit (although its presentation often still resembles more the style of informal lectures than that of a dry "noble-science monograph"). In order to mention an example of **how the book may be used by a teacher**, let me briefly note, what **four** particular formal **courses** are covered by the book. The first, fairly broad one, is compulsory and it corresponds roughly to (parts of) Chapters 1-9 and 14-16. Thus it is devoted to the essentials of general differential geometry and an outline of its principal applications. The other three courses are optional and they treat more specific parts of the subject. Namely, (elementary) Lie groups and algebras and their representations (it reproduces more or less the "particular direction of interest" number 2, mentioned above), geometrical methods in classical mechanics (the rest of Chapter 14 and Chapters 17-18) and connections and gauge fields (Chapters 19-21).

I have benefited from numerous discussions about geometry in physics with colleagues from the Department of Theoretical Physics, in particular with Paľo Ševera and Vlado Balek.

I thank Pavel Bóna for his critical comments on the Slovak edition of the book, Vlado Bužek and Vlado Černý for constant encouragement during the course of the work and the former also for the idea to publish it abroad.

Thanks are due to E.Bartoš, J.Buša, V.Černý, J.Hitzinger, J.Chlebíková, E.Masár, E.Saller, S.Slisz and A.Šurda for helping me navigate the troubled waters of computer typesetting (in particular through the subtleties of TEX) and to my sons, Stanko and Mirko, for drawing the figures (in TEX).

I would like to thank the helpful and patient people of Cambridge University Press, particularly Tamsin van Essen, Vincent Higgs, Emma Pearce and Simon Capelin. I would also like to thank all the (anonymous) referees of Cambridge University Press for valuable comments and suggestions (e.g. for the idea to complement the summaries of the individual chapters by a list of the most relevant formulas).

I am indebted to Arthur Greenspoon for careful reading of the manuscript. He helped to smooth out various pieces of the text which had hardly been continuous before.

Finally, I wish to thank my wife, Ľubka, and my children, Stanko, Mirko and Danka, for the considerable amount of patience displayed during the years it took me to write this book.

I tried hard to make Differential geometry and Lie groups for physicists error-free, but spotting mistakes in one's own writing can be difficult in a book-length work. If you notice any errors in the book or have suggestions for improvements, please let me know (fecko@fmph.uniba.sk). Errors reported to me (or found by myself) will be listed at my web page

http://sophia.dtp.fmph.uniba.sk/~fecko

Bratislava

Marián Fecko

0. INTRODUCTION

In physics every now and then one needs something to differentiate or integrate. This is the reason why a novice in the field is simultaneously initiated into the secrets of differential and integral calculus.

One starts with functions of a single variable, then several variables occur. Multiple integrals and partial derivatives arrive on the scene, and one calculates plenty of them on the drilling ground in order to survive in the battlefield.

However, if we scan carefully the structure of expressions containing partial derivatives in real physics formulas, we observe that some combinations are found fairly often, but other ones practically never occur. If, for example, the frequency of the expressions

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \qquad \text{and} \qquad \frac{\partial^3 f}{\partial x^3}+\frac{\partial^2 f}{\partial y \partial z}+4\frac{\partial f}{\partial z}$$

is compared, we come to the result that the first one (Laplace operator applied to a function f) is met very often, the second one may be found only in problem books on calculus (where it occurs because of didactic reasons alone). Combinations which do enter real physics books, result, as a rule, from a computation which realizes some *visual local geometrical* conception corresponding to the problem under consideration (like a phenomenological description of diffusion of matter in a homogeneous medium). These very conceptions constitute the subject of a systematic study of *local differential geometry.* In accordance with physical experience it is observed there that there is a fairly small number of truly interesting (and, consequently, frequently met) operations to be studied in detail (good news - they can be mastered in a reasonably short time).

We know from our experience in general physics that the same situation may be treated using *various kinds of coordinates* (Cartesian, spherical polar, cylindrical,...) and it is clear from the context that the *result* certainly *does not depend* on the choice of coordinates (which is, however, far from being true concerning the *sweat involved* in the computation; *just that* is the reason a careful choice of coordinates is a part of wise strategy in solving problems). Thus both objects and operations on them are independent of the choice of coordinates used to describe them. It should be not surprising, then, that in a properly built formalism a great deal of the work may be performed using *no coordinates* whatsoever (just what part of computation it is depends both on problem and the mastership of particular user). There are several advantages which should be mentioned in favor of these "abstract" (coordinate-free) computations. They tend to be considerably shorter and more transparent, making repeated check, as an example, much easier, individual steps may be better understood visually and so on. Consider, in order to illustrate this fact, the following equations

$$\begin{aligned}
\mathcal{L}_\xi g = 0 \qquad &\leftrightarrow \qquad \xi^k g_{ij,k} + \xi^k{}_{,i} g_{kj} + \xi^k{}_{,j} g_{ik} = 0 \\
\nabla_{\dot\gamma}\dot\gamma = 0 \qquad &\leftrightarrow \qquad \ddot x^i + \Gamma^i_{jk}\dot x^j \dot x^k = 0 \\
\nabla g = 0 \qquad &\leftrightarrow \qquad g_{ij,k} - \Gamma_{ijk} - \Gamma_{jik} = 0
\end{aligned}$$

We will learn step by step in this book that the pairs of equations standing on the left and on the right side of the same line always tell us *just the same*: the expression on the right may be regarded as being obtained from that on the left by expressing it in (arbitrary) coordinates.

(The first line represents *Killing equations*; they tell us that the Lie derivative of g along ξ vanishes, i.e. that the metric tensor g has a *symmetry* given by a vector field ξ. The second one defines particular curves called geodesics, representing uniform motion in a straight line (= its acceleration vanishes). The third one encodes the fact that a linear connection is metric; it says that a scalar product of vectors remains unchanged under parallel translation.)

In spite of the highly efficient way of writing of the coordinate versions of the equations (summation convention, partial derivatives via commas), it is clear that they can hardly compete with the left side's brevity. Thus if we will be able to *reliably manipulate* the objects occurring on the left, we gain an ability of manipulating (indirectly) fairly complicated expressions containing partial derivatives, always keeping under control what we *actually* do.

At the introductory level calculus used to be developed in Cartesian space $\mathbb{R}^n$ or in open domains in $\mathbb{R}^n$. In numerous cases, however, we apply the calculus in spaces which *are not* open domains in $\mathbb{R}^n$, although they are "very close" to them.

In analytical mechanics, as an example, we study the motion of pendulums by solving (differential) Lagrange equations for coordinates introduced in the pendulum's configuration spaces, regarded as functions of time. These configuration spaces are not, however, open domains in $\mathbb{R}^n$. Take a simple pendulum swinging in a plane. Its configuration space is clearly a *circle* S^1. Although this is a one-dimensional space, it is intuitively clear (and one may prove) that it is essentially *different* from (an open set in) $\mathbb{R}^1$. Similarly the configuration space of a spherical pendulum happens to be the two-dimensional sphere S^2, which differs from (an open set in) $\mathbb{R}^2$.

Notice, however, that a sufficiently *small neighborhood* of an arbitrary point on S^1 or S^2 is practically indistinguishable from a sufficiently small neighborhood of an arbitrary point in $\mathbb{R}^1$ or $\mathbb{R}^2$ respectively; they are in a sense "locally equal", the difference being "only global". Various applications of mathematical analysis (including those in physics) thus strongly motivate its extension to more general spaces than those which are simple open domains in $\mathbb{R}^n$.

Such more general spaces are provided by *smooth manifolds*. Loosely speaking they are spaces which a *short-sighted observer* regards as $\mathbb{R}^n$ (for suitable n), but globally ("topologically", when a pair of spectacles are found at last) their structure may differ profoundly from $\mathbb{R}^n$.

We can regard as an enjoyable bonus that the formalism, which will be developed in order to perform coordinate-free computations, happens to be at the same time (free of charge) well suited to treat *global* geometrical problems, too, i.e. we may study the objects and operations on them, being well defined *on the manifold* as a whole. Therefore, we speak sometimes about *global analysis*, or the analysis on manifolds. All the above-mentioned equations $\mathcal{L}_\xi g = 0$, $\nabla_{\dot\gamma}\dot\gamma = 0$ and $\nabla g = 0$ represent, to give an example, equations on manifolds and their solutions may be defined as objects living on manifolds, too.

The key concept of a manifold itself will be introduced in Chapter 1. The exposition is mainly at the intuitive level. A good deal of material treated in detail in mathematical texts on differential *topology* will only be mentioned in a fairly informative way or will be even omitted completely. The aim of this introductory chapter is to provide the reader a minimal amount of material which is necessary to grasp (fully, already at the working level) the main topic of the book, which is differential *geometry* on manifolds.

Summary of Chapter 1

The *smooth manifold* is *the* basic playing field in differential geometry. It is a generalization of the Cartesian space $\mathbb{R}^n$ (or an open domain in the latter) to a more elaborate object, which (only) *locally* looks like $\mathbb{R}^n$, but its global structure can be much more complicated. It is, however, always possible to contemplate it as a whole in which *several* pieces homeomorphic to $\mathbb{R}^n$ are glued together; the number n, which is the same for all pieces, is called the dimension of the manifold. The technical realization of these ideas is achieved by the concepts of a *chart* (local coordinates) and an *atlas* (consisting of several charts). The Cartesian product $M \times N$ of two manifolds is a new manifold, constructed from two given ones M and N. Any manifold admits a realization as a surface, which is nicely embedded in a *Cartesian* space of sufficiently large dimension.

$(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$	Euclidean distance between two points $x, y \in \mathbb{R}^n$	1.1.5
$\varphi : \mathcal{O} \to \mathbb{R}^n[x^1, \dots, x^n]$	chart (local coordinates) in a patch $\mathcal{O} \subset (X, \{\tau\})$	1.3
$\varphi_\beta \circ \varphi_\alpha^{-1}$	change of coordinates in a patch $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$	1.3
$(x, y) \mapsto (\varphi_\alpha(x), \psi_a(y)) \in \mathbb{R}^{n+m}$	atlas for the Cartesian product $X \times Y$	1.3.3
$\hat{f} \equiv \psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$	coordinate presentation of $f : M \to N$	1.4
$y^{m+1} = \ldots = y^n = 0$	immersion (some coordinates on N vanish)	1.4
$f(M) \subset N$	$f(M)$ is a submanifold of N (f = embedding)	1.4
$\phi^1(x) = \cdots = \phi^m(x) = 0$	smooth constraints (manifold as a surface in $\mathbb{R}^n$)	1.5
$x^i(u^1, \dots, u^m),\ i = 1, \dots n \geq m$	parametric expression of a manifold	1.5

Summary of Chapter 2

For each point x of an n-dimensional manifold M there is the canonically defined n-dimensional linear space T_xM, the tangent space at the point x. Its elements are called vectors at x. There are several mutually equivalent definitions of this concept, useful in different contexts. A vector *field* on a manifold M is a smooth assignment of a vector to each point $x \in M$. The integral curve of a vector field is the curve whose motion at each point is just that dictated by the vector of the field in this point. Standard constructions of multilinear algebra (construction of tensors of type $\binom{p}{q}$ for a given vector space L) lead to the notion of a *tensor field* of type $\binom{p}{q}$ on a manifold. In particular, one has functions (type $\binom{0}{0}$), vector and covector fields (type $\binom{1}{0}$ and $\binom{0}{1}$), fields of bilinear form (type $\binom{0}{2}$, in the symmetric non-degenerate case the metric tensor) and linear operators (type $\binom{1}{1}$).

$\gamma : \mathbb{R} \to M$	a curve γ on a manifold M	2.1
$f : M \to \mathbb{R}$	a function f on a manifold M	2.1
$e_i := \partial_i\|_P$	coordinate basis of T_PM	2.2.6
$a^i \mapsto a'^i = J^i_j(P)a^j$	transformation of components of a vector in P	2.2.6
$V(fg) = (Vf)g + f(Vg)$	Leibniz rule for action of vector fields	2.2.8
$\dot{x}^i = V^i(x) \quad (\dot{\gamma} = V)$	equations for finding integral curves of V	2.3.1
$v = \sum_{b=1}^{n} v^b e_b \equiv v^b e_b$	summation convention	2.4.2
$\langle e^a, e_b\rangle = \delta^a_b$	the base e^a is dual with respect to e_a	2.4.2
$t^{c\dots d}_{a\dots b} := t(e_a, \dots, e_b; e^c, \dots, e^d)$	components of tensor $t \in T^p_q(L)$	2.4.6
$v_a := g_{ab}v^b\ ,\ \alpha^a := g^{ab}\alpha_b$	lowering and raising of indices by means of g	2.4.13
$\langle df, V\rangle := Vf$	gradient of a function f as a covector field	2.5.3
$T = (1/2)h(\dot{\Gamma}, \dot{\Gamma})$	kinetic energy of a system of N point masses	2.6.7
$l[\gamma] := \int_{t_1}^{t_2} dt\sqrt{g(\dot{\gamma}, \dot{\gamma})}$	functional of the length of a curve γ	2.6.9
$(\nabla f)^i := g^{ij} f_{,j} \quad (\nabla f := \sharp_g df)$	gradient of a function f as a vector field	2.6

Summary of Chapter 3

Each (smooth) mapping of the points of manifolds $f : M \to N$ induces a mapping of tensors living on them. It is denoted by f_* if it pushes tensors forward (in the same direction as f, from M to N) and f^* if it pulls tensors back (in the opposite direction, from N to M). For diffeomorphisms it is possible to define both f_* and f^* for tensor fields of arbitrary type; if f is not the diffeomorphism, several kinds of problems may occur. There always exists a pull-back map f^* for tensor fields of type $\binom{0}{p}$. In particular, one can induce (via pull-back) a metric tensor on M from a Riemannian manifold (N, h), giving rise to a Riemannian manifold (M, g), $g = f^*h$. The most common instance of this procedure is that one induces a metric tensor onto a submanifold M of the Euclidean space $N = E^n$ (or more generally $E^{r,s}$), starting from the canonical metric tensor $h = \eta$ on N.

$f^*\psi := \psi \circ f$	pull-back of a function ψ	3.1.1
$f_*[\gamma] := [f \circ \gamma]$	push-forward of a vector $[\gamma]$	3.1.2
$(f_*V)\psi := V(f^*\psi)$	push-forward of a vector V	3.1.2
$(f^*t)(U, \alpha) := t(f_*U, (f^{-1})^*\alpha)$	pull-back of a tensor field	3.1.6
$(g \circ f)^* = f^* \circ g^*$	pull-back for the composition of maps	3.1.6
$(g \circ f)_* = g_* \circ f_*$	push-forward for the composition of maps	3.1.6
$f^* \circ C = C \circ f^*$	pull-back commutes with contractions	3.1.7
$df^* = f^*d$	pull-back commutes with gradient	3.1.9
$g := f^*h$	induced metric tensor ($f\colon M \to (N, h)$)	3.2.1
$g_{ij} = J^a_i h_{ab} J^b_j \equiv y^a{}_{,i} h_{ab} y^b{}_{,j}$	induced metric tensor (components)	3.2.1
$T = (1/2)g(\dot\gamma, \dot\gamma)$	kinetic energy on a configuration space	3.2.9

Summary of Chapter 4

Each vector field V on M naturally induces a map $\Phi_t : M \to M$, which translates a point x along the integral curve starting in x by the parametric distance t. It is called the flow generated by V, or taking into account its composition property $\Phi_{t+s} = \Phi_t \circ \Phi_s$ a one-parameter group of transformations. According to the results of Chapter 3 the map Φ_t of a manifold M onto itself induces a mapping of tensor fields Φ_t^*, which is called the *Lie transport* of tensors (along the integral curves of the field V). The natural measure of sensitivity of a tensor field A to Lie transport is the *Lie derivative*. One can assign to any two vector fields V, W a third one, their commutator $[V, W]$ (which happens to coincide with $\mathcal{L}_V W$). Two fields commute if and only if their flows do; non-commuting of vector fields thus results in anholonomy phenomena (dependence on the path). A Killing vector is a vector field with respect to which the metric tensor is Lie constant. The flow of a Killing vector is the isometry of a Riemannian manifold (M, g), i.e. a map of M onto itself which preserves all lengths and angles. If the angles alone are preserved, we speak of conformal transformations and the corresponding generators are called conformal Killing vectors.

$\Phi_{t+s} = \Phi_t \circ \Phi_s$	"composition" property of a flow	4.1.2
$\Phi_t^* A = A$	A is Lie invariant (dragged)	4.2
$\mathcal{L}_V A := (d/dt)_0 \Phi_t^* A$	Lie derivative of A along $V \leftrightarrow \Phi_t$	4.2
$\mathcal{L}_V(A + \lambda B) = \mathcal{L}_V A + \lambda \mathcal{L}_V B$	Lie derivative of a linear combination	4.3.1
$\mathcal{L}_V(A \otimes B) = \mathcal{L}_V A \otimes B + A \otimes \mathcal{L}_V B$	Lie derivative of a tensor product	4.3.1
$\mathcal{L}_V \circ C = C \circ \mathcal{L}_V$	Lie derivative commutes with contractions	4.3.1
$\mathcal{L}_V W = [V, W]$	Lie derivative of W along V	4.3.6
$\mathcal{L}_{V+\lambda W} = \mathcal{L}_V + \lambda \mathcal{L}_W$	Lie derivative along a linear combination	4.3.8
$\mathcal{L}_{[V,W]} = [\mathcal{L}_V, \mathcal{L}_W]$	Lie derivative along a commutator	4.3.8
$\Phi_t^* = e^{t\mathcal{L}_V} \equiv 1 + t\mathcal{L}_V + \dots$	exponent of the Lie derivative	4.4.2
$\Phi_{-\epsilon}^W \circ \Phi_{-\epsilon}^V \circ \Phi_{-\epsilon^2}^{[V,W]} \circ \Phi_\epsilon^W \circ \Phi_\epsilon^V = \hat{1} + \dots$	interpretation of the commutator $[V, W]$	4.5.2
$l[f \circ \gamma, g] = l[\gamma, f^* g]$	behavior of the length functional	4.6.1
$f^* g = g$	f is an isometry of (M, g)	4.6.2
$f^* g = \sigma g$	f is a conformal transformation of (M, g)	4.6.3
$\mathcal{L}_\xi g = 0$	Killing equations (ξ generates isometries)	4.6.5
$f^* \eta = \eta$	f is the Poincaré transformation	4.6.10
$\mathcal{L}_\xi g = \chi g$	conformal Killing equations	4.6.16
$\varepsilon = (1/2)\mathcal{L}_{\mathbf{u}} g$	strain tensor (elastic continuum)	4.6.24
$(1/2)\mathcal{L}_{\mathbf{v}} g$	strain-rate tensor (viscous fluids)	4.6.25

Summary of Chapter 5

The computation of volumes of parallelepipeds (and consequently the integration procedure, where the values of functions are multiplied by the volumes of *infinitesimal* parallelepipeds) singles out completely antisymmetric fully covariant tensors, usually called forms. This chapter makes the reader acquainted with forms at the level of linear algebra. Forms enjoy several important unique properties (not shared with general tensors). They are naturally $\mathbb{Z}$-graded, one can multiply them one with another via the (graded commutative) exterior (wedge) product $\wedge$ (giving rise to a graded *exterior* = Grassmann algebra) and with vectors via the interior product i_v (which turns out to be a derivation of degree -1 of the exterior algebra). If a vector space is endowed with a metric tensor and orientation, there are also the canonical volume form and Hodge star operator $*$ on forms available. The determinant is naturally related to these concepts.

$\wedge := \dfrac{(p+q)!}{p!q!}\,\pi^A \circ \otimes$	exterior (wedge) product of forms	5.2.4
$(\beta + \lambda\tau)\wedge\alpha = \beta\wedge\alpha + \lambda\tau\wedge\alpha$		
$\alpha\wedge(\beta+\lambda\tau) = \alpha\wedge\beta + \lambda\alpha\wedge\tau$	bilinearity of $\wedge$	5.2.4
$(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$	associativity of $\wedge$	5.2.4
$\alpha\wedge\beta = (-1)^{pq}\beta\wedge\alpha$	$\mathbb{Z}$-graded commutativity of $\wedge$	5.2.4
$\alpha = (1/p!)\ \alpha_{a\dots b}\ e^a\wedge\dots\wedge e^b$	expression of a p-form in terms of e^a	5.2.9
$\hat{\eta}\alpha := (-1)^{\deg\,\alpha}\alpha$	main automorphism of ΛL^*	5.3.3
$(i_v\alpha)(u,\dots,w) := \alpha(v,u,\dots,w)$	interior product (of v and α)	5.4.1
$(i_v\alpha)_{a\dots b} = v^c\alpha_{ca\dots b}$	component expression of i_v	5.4.1
$i_v(\alpha\wedge\beta) = (i_v\alpha)\wedge\beta + (\hat{\eta}\alpha)\wedge(i_v\beta)$	graded Leibniz rule for i_v	5.4.2
$\delta^{a\dots b}_{c\dots d} = \delta^a_{[c}\dots\delta^b_{d]} \equiv \delta^{[a}_c\dots\delta^{b]}_d \equiv \delta^{[a}_{[c}\dots\delta^{b]}_{d]}$	p-delta (generalized Kronecker) symbol	5.6.2
$n!\ \det A = \varepsilon_{a\dots b}\varepsilon^{c\dots d}\ A^a_c\dots A^b_d$	determinant and Levi-Civita symbol	5.6.2
$\omega_g = o(f)\sqrt{\lvert g\rvert}\ f^1\wedge\dots\wedge f^n$	metric volume form	5.7.3
$\mathrm{vol}(Au,\dots,Av) =: (\det A)\ \mathrm{vol}(u,\dots,v)$	determinant of a linear map A	5.7.6.
$p!(*\alpha)_{a\dots b} := \alpha^{c\dots d}\ \omega_{c\dots da\dots b}$	Hodge star (duality) operator	5.8.1
$*_g *_g = \operatorname{sgn} g\ (-1)^{p(n+1)}$	star squared is $\pm$ the unity	5.8.2
$\alpha\wedge *_g\beta =: (\alpha,\beta)_g\omega_g$	scalar product $(\alpha,\beta)_g$ of forms	5.8.4
$p!(\alpha,\beta)_g = \alpha_{a\dots b}\ \beta^{a\dots b}$	component expression of $(\alpha,\beta)_g$	5.8.4

Summary of Chapter 6

Forms are treated as fields on a manifold (*differential* forms). All the algebraic constructions known from Chapter 5 still work, but a new differential operation of crucial importance enters the scene, the *exterior derivative*. It turns to be a nilpotent ($dd = 0$) derivation of degree $+1$ of the Cartan algebra $\Omega(M)$ of forms on a manifold. A simple (but useful) generalization of ordinary forms is provided by arbitrary vector space valued forms (the ordinary ones being $\mathbb{R}$-valued).

$\alpha = (1/p!)\ \alpha_{i\dots j}(x) dx^i \wedge \dots \wedge dx^j$	coordinate expression of a form	6.1.1
$D_k(a_i b) = (D_k a_i) b\ +\ (-1)^{ik} a_i (D_k b)$	D_k is a derivation of degree k	6.1.7
$(d\alpha)_{i\dots jk} := (-1)^p\ (p+1)\ \alpha_{[i\dots j,k]}$	exterior derivative in coordinates	6.2.5
$dd = 0$	exterior derivative is nilpotent	6.2.5
$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (\hat{\eta}\alpha) \wedge d\beta$	graded Leibniz's rule for d	6.2.5
$\mathcal{L}_V = i_V\ d + d\ i_V$	Cartan's identity	6.2.8
$[d, \mathcal{L}_V] \equiv d\ \mathcal{L}_V - \mathcal{L}_V\ d = 0$	exterior and Lie derivatives commute	6.2.10
$[d, f^*] \equiv d\ f^* - f^*\ d = 0$	exterior derivative commutes with pull-back	6.2.11
$d\alpha(U, V) = \dots$	Cartan formula (for $p = 1$)	6.2.13
$d\beta(U, V, W) = \dots$	Cartan formula (for $p = 2$)	6.2.13
$\alpha = \alpha^A E_A$	V-valued form on M	6.4.1

Summary of Chapter 7

Inspection of several simple examples and facts from elementary integral calculus leads to the conclusion that the objects under the integral sign should be treated as differential forms from Chapter 6. The crucial concept of the integral of a form over a chain is introduced, assuming the standard background knowledge on basics of the Riemann multiple integral. Stokes' theorem for differential forms is presented. It relates the integral of a form over the *boundary* of a chain to the integral of the *exterior derivative* of the form over the chain itself. Reinterpretation of the integral over a domain on an oriented manifold in terms of the integral over the chain (including Stokes' theorem) is given and particular features of integration over a *Riemannian* manifold are mentioned. The remarkably simple behavior of the integral with respect to maps between manifolds is revealed.

$c = c_i s_p^i$	Euclidean p-chain	7.2
$\partial(P_0, \dots, P_p) = \dots$	action of the boundary operator on a simplex	7.2.2
$\partial\partial = 0$	boundary has no boundary	7.2.2
$\int_c d\alpha = \int_{\partial c} \alpha$	Stokes' theorem	7.5
$\text{vol } (D) := \int_D \omega$	volume of a domain D on (M, ω)	7.6
$\epsilon \int_D i_V \alpha = \int_{D_{\epsilon V}} \alpha$	a "coin interpretation" of the form $i_V \alpha$	7.6.11
$\int_D f := \int_D f \omega_g$	integral of the first kind on (M, g, o)	7.7
$\int \sqrt{\det(g_{\mu\nu} x^\mu{}_{,a} x^\nu{}_{,b})}\ du^1 \wedge du^2$	area of a two-dimensional surface	7.7.5
$\langle \rho \rangle_D := \frac{\int_D \rho \omega_g}{\int_D \omega_g}$	mean value of the (scalar) quantity ρ over D	7.7
$\int_{f(c)} \alpha = \int_c f^* \alpha$	integral and maps of manifolds	7.8.1

Summary of Chapter 8

One often encounters the general Stokes' theorem for differential forms from Chapter 7 as hidden behind one of its numerous classical versions. Here we demonstrate this, in particular, for the divergence (Gauss') theorem, Green's identities, the "common" Stokes' theorem known from vector analysis or some well-known facts from elementary complex analysis. The codifferential δ is introduced (as the operator adjoint to the differential d = exterior derivative) and the self-adjoint combination $\Delta = -(d\delta + \delta d)$, the Laplace-deRham operator (a generalization of the Laplace operator on functions to forms of arbitrary degree). In the section devoted to standard vector analysis we learn that the essence of the well-known operations of gradient, curl and divergence is simply the exterior derivative acting on forms of all non-trivial degrees in three-dimensional space.

$\mathcal{L}_V \omega_g =: \ (\operatorname{div} V)\ \omega_g$	definition of the divergence of V	8.2.1
$\operatorname{div} V = \frac{1}{\sqrt{\mid g \mid}}(\sqrt{\mid g \mid}\ V^k)_{,k}$	coordinate expression of $\operatorname{div} V$	8.2.1
$\langle \operatorname{div} V \rangle_D = \frac{d}{dt}\Big\vert_{t=0} \frac{\text{vol } D(t)}{\text{vol } D}$	interpretation of $\operatorname{div} V$	8.2.2
$\langle \operatorname{div} V \rangle_D = \frac{\text{the flux of } V \text{ for } \partial D}{\text{the volume of } D}$	another interpretation of $\operatorname{div} V$	8.2.9
$\int_D (\operatorname{div} V)\ \omega_g = \int_{\partial D} V^i d\Sigma_i \vert_{\partial D}$	Gauss' theorem	8.2.7
$\langle \alpha, \beta \rangle := \int_D \alpha \wedge * \beta$	scalar product of forms on (M, g)	8.3.1
$\delta := *^{-1} d * \hat{\eta}$	definition of the codifferential δ	8.3.2
$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_{\partial D} \alpha \wedge * \beta$	basic property of the codifferential δ	8.3.2
$\triangle := -(\delta d + d\delta) \equiv -(d^+ d + d d^+)$	Laplace-deRham operator	8.3.3
$\Delta f = -\delta df \equiv \frac{1}{\sqrt{\mid g \mid}}(\sqrt{\mid g \mid} g^{kj} f_{,j})_{,k}$	Laplace-Beltrami operator	8.3.5
$\langle du, dv \rangle + \langle u, \Delta v \rangle = \int_{\partial D} u * dv$	"ordinary" Green identity	8.4.1
$\langle u, \Delta v \rangle - \langle v, \Delta u \rangle = \int_{\partial D} (u * dv - v * du)$	"symmetric" Green identity	8.4.1
$f,\ \mathbf{A}.d\mathbf{r},\ \mathbf{B}.d\mathbf{S},\ hdV$	differential forms on E^3	8.5.2
$d(\mathbf{A}.d\mathbf{r}) = (\operatorname{curl} \mathbf{A}).d\mathbf{S}$	a definition of $\operatorname{curl} \mathbf{A}$	8.5.4
$(\mathbf{A}.d\mathbf{r}) \wedge (\mathbf{B}.d\mathbf{r}) = (\mathbf{A} \times \mathbf{B}).d\mathbf{S}$	how the vector (cross) product appears	8.5.8
$g = h_1^2 dx^1 \otimes dx^1 + \dots$	Lamé coefficients	8.5.9
$d(f(z)dz) = 0$	why the Cauchy theorem holds	8.6.5

Summary of Chapter 9

A form is closed if its exterior derivative vanishes, and exact if it is itself the exterior derivative of some other form (its *potential*). Since the operator d is nilpotent (i.e. $dd = 0$), each exact form is necessarily closed. Simple counterexamples show that the converse of this statement, freely used in elementary physics, does not hold *in general*. It does hold, however, on *contractible* manifolds. In particular it holds *locally*, i.e. within a sufficiently small neighborhood of any point on any manifold; this statement is known as the Poincaré lemma. An explicit formula for the potential is then given. A more subtle treatment of the issue is provided by cohomology theory, namely by cohomologies of the *deRham* complex.

$\hat{h} = -\int_0^\infty dt \Phi_t^* i_\xi$	homotopy operator	9.2.3
$d \circ \hat{h} + \hat{h} \circ d = \hat{1}$	essential property of $\hat{h}$	9.2.3
$\alpha = d(\hat{h}\alpha) \equiv d\beta$	$\beta \equiv \hat{h}\alpha$ is a potential of α	9.2.4
$x^k \int_0^1 d\lambda \lambda^{p-1} \alpha_{ki\dots j}(\lambda x)$	coordinate expression of $(\hat{h}\alpha)_{i\dots j}(x)$	9.2.7
$[e_a, e_b] = c^c_{ab}(x) e_c$	coefficients of anholonomy of e_a	9.2.10
$e_a = \partial_a \quad \Leftrightarrow \quad [e_a, e_b] = 0$	when a frame field is holonomic (coordinate)	9.2.11
$e^a = dx^a \quad \Leftrightarrow \quad de^a = 0$	when a coframe field is holonomic	9.2.11
$Z^p := \operatorname{Ker} d_p$	p-cocycles	9.3.1
$B^p := \operatorname{Im} d_{p-1}$	p-coboundaries	9.3.1
$H^p := Z^p / B^p$	p-th cohomology group	9.3
$b^p := \dim H^p$	p-th Betti number	9.3
$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M)$	deRham complex of a manifold M	9.3.2

Summary of Chapter 10

Groups enter into play both in physics and mathematics as *symmetry* groups, i.e. (in mathematical terms) as groups of *automorphisms* of various structures. Several structures leading to the common "classical groups" (general linear, orthogonal, symplectic, unitary, ...) are discussed from this point of view. The *Lie group* combines in a single object the algebraic concept of a group with the differential-topological notion of a manifold. All of the above-mentioned groups (as well as some others) are examples of Lie groups.

$G = \mathrm{Aut}\,(X, \mathrm{s})$	group of automorphisms of a structured set	10.1
$h(Av, Aw) = h(v, w)$	A preserves the bilinear form h	10.1.4
$A^a_c h_{ab} A^b_d = h_{cd}$	component expression of the same fact	10.1.4
$A^T hA = h$	matrix expression of the same fact	10.1.4
$\omega(Av, \dots, Aw) = \omega(v, \dots, w)$	A preserves the volume form ω	10.1.7
$A^a_c \dots A^b_d \varepsilon_{a\dots b} = \varepsilon_{c\dots d}$	component expression of the same fact	10.1.7
$\det\ A = 1$	matrix expression of the same fact	10.1.7
$m(g, h) := gh$	composition law in a group	10.2.5
"Classical" matrix groups introduced	They are summarized in problem	11.7.6

Summary of Chapter 11

Differential geometry turns out to provide an effective tool for studying such sophisticated objects as Lie groups represent. Their rich geometry stems from the compatibility of the two structures involved. A much simpler object, the Lie *algebra* (it is a finite-dimensional vector space) may be associated canonically with each Lie group with the help of the *left-invariant* vector fields. In spite of its simplicity the Lie algebra of a group encodes a great deal (the essential part) of information concerning the group itself. The exponential map from the Lie algebra to the group is introduced.

$L_g h := gh,\ R_g h := hg$	left translation, right translation	11.1.1
$L_g^* T = T$	T is left-invariant tensor field on G	11.1.4
$e_a(g) = L_{g*}E_a,\ E_a = e_a(e)$	left-invariant frame field generated by E_a	11.1.6
$(x^{-1})^i_k dx^k_j \equiv (x^{-1}dx)^i_j$	left-invariant 1-forms on $GL(n,\mathbb{R})$	11.1.9
$x^i_k \partial^k_j \equiv (x\partial)^i_j$	left-invariant vector fields on $GL(n,\mathbb{R})$	11.1.10
$[E_a, E_b] = c^c_{ab}E_c$	structure constants with respect to E_a	11.2.2
$de^a + (1/2)c^a_{bc}e^b \wedge e^c = 0$	Maurer-Cartan formula in terms of e^a	11.2.3
$\langle \theta, L_X \rangle := X,\quad \theta = e^a E_a$	canonical (Maurer-Cartan) 1-form θ on G	11.2.6
$d\theta + (1/2)[\theta \wedge \theta] = 0$	Maurer-Cartan formula in terms of θ	11.2.6
$\gamma(t+s) = \gamma(t)\gamma(s),\quad \gamma(0) = e$	one-parameter subgroup on G	11.3
$\gamma^X(t) = e^{tX}$	one-parameter subgroup in terms of exp	11.4.1
$f(e^X) = e^{f'(X)}$	derived homomorphism f'	11.5.3
$x^{-1}dx$	canonical 1-form on $GL(n,\mathbb{R})$	11.7.19
$j^*(x^{-1}dx) = x^{-1}(z)dx(z)$	canonical 1-form on matrix groups	11.7.21

Summary of Chapter 12

A Lie group often shows its presence via its *representations*, i.e. there exists a homomorphism of the former to the group of invertible linear operators in a vector space and in a given situation we encounter only the image of the group with respect to this homomorphism. Each representation of a Lie group induces automatically a representation of its Lie *algebra* (called the derived representation), the latter meaning in general a homomorphism of a Lie algebra into the Lie algebra of (all) linear operators (in a fixed vector space). If a representation admits a non-trivial invariant subspace it is called *reducible*, since it may be reduced (by restriction) to a (smaller) representation in this subspace. *Irreducible* representations cannot be reduced in this way. Schur's lemma provides a useful criterion of irreducibility. If the invariant subspace admits an invariant *complement* as well, the representation is equivalent to a direct sum of two simpler ones. Such a complement sometimes happens to be *orthogonal* with respect to an *invariant* scalar product (if it does exist; on compact groups its existence is guaranteed and the procedure for its construction is given here). One can perform some standard constructions with representations, such as the dual (contragredient) one and the direct sum and the direct product; combining these two with the restriction to invariant subspaces in the resulting spaces, a lot of new representations may be obtained from a small number of them at the beginning (like all "tensor" representations ρ^p_q from a single "vector" one ρ^1_0; in some cases even all irreducible representations from just a single one, see section 13.3). Invariant tensors and related intertwining operators enable one to "transmute the type" of quantities, i.e. associate with vectors acted by a representation ρ_1 vectors acted by a representation ρ_2 (of the same group). A representation of a Lie algebra induces a complex; we study its cohomologies for a while.

$\rho(1+\epsilon X) = 1 + \epsilon\rho'(X)$	computation of the derived representation ρ'	12.1.6
$\rho'(E_i)E_a =: \rho^b_{ai}E_b$	matrix elements of generators	12.1.6
$\langle\check{\rho}(g)\alpha, v\rangle := \langle\alpha, \rho(g^{-1})v\rangle$	contragredient (dual) representation $\check{\rho}$	12.1.8
$h(\rho(g)v, \rho(g)w) = h(v, w)$	scalar product h is ρ-invariant	12.1.10
$h_{bc}\rho^c_{ai} + h_{ac}\rho^c_{bi} = 0$	component expression of the same fact	12.1.10
$\rho_2(g)A = A\rho_1(g)$	A is intertwining operator for ρ_1 and ρ_2	12.2
$ge^X g^{-1} = e^{\mathrm{Ad}_g X}$	adjoint representation Ad of G	12.3.1, 2
$\mathrm{Ad}_A X = AXA^{-1}$	explicit expression of Ad for matrix groups	12.3.1
$\mathrm{ad}_X Y = [X, Y] \quad (\mathrm{ad} \equiv \mathrm{Ad}')$	adjoint representation ad of $\mathcal{G}$	12.3.5
$\mathrm{ad}_{E_i} E_j = c^k_{ij} E_k$	component expression of ad	12.3.5
$K(X, Y) := \mathrm{Tr}\,(\mathrm{ad}_X \mathrm{ad}_Y)$	Killing-Cartan form on $\mathcal{G}$	12.3.8
$\hat{C}_2 := k^{ij}\rho'(E_i)\rho'(E_j)$	quadratic Casimir operator	12.3.13
$(g_1, h_1) \circ (g_2, h_2) := (g_1 g_2, h_1 h_2)$	direct product of groups	12.4.7
$(\rho_1 \otimes \rho_2)(g) := \rho_1(g) \otimes \rho_2(g)$	direct product of representations of G	12.4.11
$(\rho_1 \otimes \rho_2)' = \rho'_1 \otimes \hat{1} + \hat{1} \otimes \rho'_2$	derived representation for $\rho_1 \otimes \rho_2$	12.4.11

Summary of Chapter 13

From the point of view of differential geometry, the most interesting actions of groups are their smooth actions on manifolds. As a rule, there is some additional structure on the manifold and the action preserves this structure (e.g. actions via isometries on Riemannian manifolds or symplectic actions on symplectic manifolds). An action of a Lie group induces at the infinitesimal level an action of its Lie algebra, generated by the *fundamental* (vector) *fields* (the generators). An action on points of a manifold results (using the tools of (3.1)) in an action on functions on the manifold (and more generally on tensor fields on the latter). By this simple method we obtain a construction of (∞-dimensional) representations of groups and their algebras (tensor fields, in particular functions, are naturally endowed with a linear space structure). Upon restriction to invariant subspaces finite-dimensional representations are also obtained by this method. Restriction to a G-invariant subspace of functions (tensor fields) is a standard useful way to solve complicated differential equations (an ansatz with some symmetry properties).

$L_{gh} = L_g \circ L_h,\ R_{gh} = R_h \circ R_g$	left action, right action of G on M	13.1
$L_{\hat{g}}[g] := [\hat{g}g]$	left action of G on the homogeneous space G/H	13.2.5
$[g][\tilde{g}] := [g\tilde{g}]$	multiplication in the factor group G/H	13.2.10
$gHg^{-1} = H$	H is a normal (invariant) subgroup of G	13.2.10
$G/\mathrm{Ker}\, f = \mathrm{Im}\, f$	homomorphism theorem	13.2.12
$e^{-\frac{i}{2}\alpha\mathbf{n}.\boldsymbol{\sigma}} \mapsto e^{\alpha\mathbf{n}.\mathbf{l}}$	universal two-sheet covering of $SO(3)$ by $SU(2)$	13.3.6
$\rho(g)\psi := \psi \circ R_g \equiv R_g^*\psi$	representation of G in $\mathcal{F}(M)$	13.4.1
$\xi_X(m) := (d/dt)\vert_0\, R_{\exp tX} m$	fundamental field (generator) of the action R_g	13.4.3
$\rho'(X) = \xi_X$	derived representation in $\mathcal{F}(M)$	13.4.3
$\xi_j = -\epsilon_{jkm} x_k \partial_m \equiv (-\mathbf{r} \times \boldsymbol{\nabla})_j$	generators of the rotations in $\mathbb{R}^3$	13.4.6
$\rho(g)\psi := \hat{\rho}(g) \circ \psi \circ R_g$	representation of G in $\mathcal{F}(M, V)$	13.4.11
$\rho'(X) = \xi_X + \hat{\rho}'(X)$	derived representation in $\mathcal{F}(M, V)$	13.4.12
$R_g^* A = \hat{\rho}(g^{-1}) A$	A is a tensor field of type $\hat{\rho}$	13.5.2
$\rho(g) := \hat{\rho}(g) \circ R_g^*$	representation of G in $\mathcal{T}_s^r(M, V)$	13.5.3

Summary of Chapter 14

An appropriate relabelling of coordinates reveals that there is an elegant geometrical structure hidden behind the Hamilton canonical equations. Its essential part is a closed non-degenerate 2-form ω on the phase space of the system, the *symplectic* form. It enables us to raise and lower the indices in a similar manner as we did before with the metric tensor. The vector field which is the counterpart of the gradient field in the Riemannian case (i.e. which is obtained by raising an index on the gradient of a function f understood as a *co*vector field) is called the *Hamiltonian* field generated by the function f. The Hamilton equations turn out to be simply the equations for the integral curves of the Hamiltonian field generated by a distinguished function H, the *Hamiltonian* of the system. Thus we come to the notion of the Hamiltonian *system* (M, ω, H). The vector fields which generate automorphisms of a Hamiltonian system (they preserve the symplectic form as well as the Hamiltonian) are called Cartan symmetries and, those obeying a specific additional property, *exact* Cartan symmetries. There is a one-to-one correspondence between the exact Cartan symmetries and the conserved quantities of the system. More details can be found in sections devoted to the moment map and symplectic reduction. The orbits of the coadjoint action (which is an action of the group G on the dual space $\mathcal{G}^*$ of its own Lie algebra $\mathcal{G}$) provide a rich source of interesting (G-invariant) symplectic manifolds (there is a canonical symplectic structure on them).

$\zeta_f = \mathcal{P}(df, \ .\ )$	Hamiltonian field in terms of $\mathcal{P}$	14.1.1
$\{f, g\} = \mathcal{P}(df, dg)$	Poisson bracket in terms of the Poisson tensor $\mathcal{P}$	14.1.1
$\dot{\gamma} = \zeta_H$	Hamilton equations - coordinate-free version	14.1.1
$i_{\zeta_f}\omega = -df$	Hamiltonian field in terms of symplectic form ω	14.1.6
$\omega = dp_a \wedge dq^a$	symplectic form in canonical (Darboux) coordinates	14.2.2
$\Omega_\omega := \text{const.}\ \omega \wedge \cdots \wedge \omega$	Liouville volume form on (M, ω)	14.3.6
$\int_{\Phi_t(\mathcal{D})} \Omega_\omega = \int_{\mathcal{D}} \Omega_\omega$	Liouville's theorem	14.3.6
$i_V\omega = -dF, \quad VH = 0$	V is exact Cartan symmetry of (M, ω, H)	14.4.2
$\gamma_s(t) := \Phi_s^V(\gamma(t))$	a new solution generated by a symmetry flow	14.4.6
$\langle P(x), X\rangle := P_X(x)$	moment map corresponding to the Poisson action	14.5.3
$\omega_{Z^*}(\xi_X, \xi_Y) := \langle Z^*, [X, Y]\rangle$	canonical symplectic form on coadjoint orbits	14.6.3

Summary of Chapter 15

In many applications (e.g. in the computation of acceleration of a point mass in elementary mechanics) one performs linear combinations (in particular the difference in the case of acceleration) of vectors (or more generally tensors) sitting at different points of a manifold. This is not possible on a "bare" manifold. The structure which makes it legal is a (linear) *connection* ∇ on M. The connection enables one to transport vectors along a given path (the transport being path-dependent in general) and consequently to perform the above-mentioned comparison (vector in x is compared with the one being *transported to* x from y). This transport is *by definition* called *parallel* (in the sense of the connection ∇). A connection is frequently defined by postulating the properties of a derived object, the *covariant derivative*. One can introduce the concept of a straight line (geodesic) on (M, ∇). Two tensor fields are associated with a linear connection, the curvature and torsion tensors. It is shown that the requirements of compatibility of a connection with the metric (conservation of any scalar product upon any parallel transport) together with vanishing of its torsion result in a unique connection, the Riemannian or Levi-Civita (RLC) connection. The curvature tensor encodes the local information of "how much" (if ever) the parallel transport (along infinitesimal paths) is path-dependent; it also displays itself in the behavior of nearby geodesics, causing their deviation (Jacobi's equation). A non-zero torsion implies non-closure of a geodesic parallelogram. An efficient tool for working with a connection is provided by the machinery of differential forms. Basic objects are encoded into forms and relations between them are given by the Cartan structure equations. A connection is called a complete parallelism if there exists a covariantly constant frame field. Then the curvature tensor turns out to vanish and moreover a comparison of vectors (as well as tensors) at different (possibly remote) points makes sense. A connection is said to be flat if the covariantly constant frame field happens to be holonomic (coordinate). Then both the curvature and torsion tensors turn out to vanish.

$\nabla_a e_b =: \Gamma^c_{ba} e_c$	coefficients of connection with respect to e_a	15.2.1
$\nabla_j \partial_i =: \Gamma^k_{ij} \partial_k$	Christoffel symbols of the second kind	15.2.3
$\dot{V}^i + \Gamma^i_{jk} \dot{x}^k V^j = 0$	equations of parallel transport of vector	15.2.6
$\nabla g = 0 \quad (g_{ij;k} = 0)$	connection ∇ is metric	15.3.1
$T(U,V) := \nabla_U V - \nabla_V U - [U,V]$	torsion tensor induced by ∇	15.3.3
$\Gamma^i_{jk} = (1/2) g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})$	Riemann/Levi-Civita connection (RLC)	15.3.4
$\nabla_{\dot\gamma}\dot\gamma = 0 \quad (\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j \dot{x}^k = 0)$	geodesic equation	15.4.1
$\exp v := \gamma_v(1), \quad \dot\gamma_v(0) = v \in T_P M$	exponential map centered at $P \in M$	15.4.10
$\langle \alpha, ([\nabla_U, \nabla_V] - \nabla_{[U,V]}) W \rangle$	Riemann curvature tensor	15.5.5
$R_{ab} := R^c{}_{acb}, \quad R := R^a{}_a \equiv R^{ab}{}_{ab}$	Ricci tensor and scalar curvature	15.5
$\nabla_V e_a = \omega^b_a(V) e_b \ \ (\omega^a_b = \Gamma^a_{bc} e^c)$	connection forms ω^b_a with respect to e_a	15.6.1
$\omega' = A^{-1}\omega A + A^{-1} dA$	transformation law for ω under $e' = eA$	15.6.2
$de + \omega \wedge e = T, \quad d\omega + \omega \wedge \omega = \Omega$	Cartan structure equations	15.6.7
$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0, \ \Omega \wedge e = 0$	Bianchi and Ricci identities (for RLC)	15.6.16
$\nabla^2_{\dot\gamma} \xi = R(\dot\gamma, \xi)\dot\gamma$	Jacobi's equation for geodesic deviation	15.7.2
$R^a{}_{bcd} = 0 = T^a{}_{bc}$	flat connection	15.8.6

Summary of Chapter 16

The (four)tensor version of the Maxwell equations in Minkowski space(-time) reveals that the tensors involved are rather special - they may actually be regarded as *differential forms*. That is why the most natural way of formulating four-dimensional electrodynamics is provided by the language of differential forms. Forms in Minkowski space exhibit additional particular structure (as a consequence of the splitting of the space-time into "time" and "space") : one can express any form (in an observer-dependent way) in terms of a pair of *spatial* forms. Such an expression of forms (as well as of operations on them) offers a convenient bridge between a four-dimensional and the (original) three-dimensional formulation of electrodynamics. Forms are not only useful in electrodynamics, but rather in field theory in general. The *action integrals* are simply expressed (since the objects under the integral sign are always forms) and their extrema, providing the equations of motion, are simply computed, too (the codifferential appears naturally). There is a deep link between the space-time symmetries and the *energy-momentum tensor* of the field, which may be defined via variation of the action functional with respect to the metric tensor. The energy-momentum tensor of matter occurs (as a source) in the Einstein equations of the gravitational field, too. Both the Hilbert and Cartan approaches to the derivation of the Einstein equations from a variational principle are discussed. In the former approach, the metric tensor is the key independent field variable (with respect to which small variations are to be performed); the latter approach makes use of (co)frame (tetrad) fields and connection forms. In non-linear sigma models mappings of two Riemannian manifolds are regarded as field variables. There is a natural action integral for such mappings. Harmonic maps are extremals of this action. They correspond to "minimal surfaces", representing e.g. soap bubbles, but also the world-sheets in string theory. There is a technical trick enabling one to get rid of a "square root" action by means of a variation of the "quadratic" one with respect to one of two metric tensors (then called "auxiliary").

$\alpha = dt \wedge \hat{s} + \hat{r}$	decomposition of forms in Minkowski space	16.1.1
$d\alpha = dt \wedge (\partial_t \hat{r} - \hat{d}\hat{s}) + \hat{d}\hat{r}$	action of d on a decomposed form	16.1.4
$* \alpha = dt \wedge (\hat{*}\hat{r}) + \hat{*}\hat{\eta}\hat{s}$	action of the Hodge star $*$ on a decomposed form	16.1.5
$\delta\alpha = dt \wedge (\hat{\delta}\hat{s}) + (-\partial_t \hat{s} - \hat{\delta}\hat{r})$	action of the codifferential δ on a decomposed form	16.1.6
$F := dt \wedge \mathbf{E}.d\mathbf{r} - \mathbf{B}.d\mathbf{S}$	2-form of the electromagnetic field	16.2.1
$j = \rho dt - \mathbf{j}.d\mathbf{r} \equiv j_\mu dx^\mu$	1-form of current	16.2.2
$\delta F = -j, \quad dF = 0$	Maxwell equations	16.2.1, 2
$F = dA$	A is a potential for F	16.3.1
$-(1/2)\langle dA, dA\rangle - \langle A, j\rangle$	action integral $S[A]$ for an electromagnetic field	16.3.2
$(1/2)\langle d\phi, d\phi\rangle - (m^2/2)\langle\phi, \phi\rangle$	action integral $S[\phi]$ for a free scalar field	16.3.7
$T^{\mu\nu}{}_{;\nu} = 0$	energy-momentum tensor is divergence-free	16.4.1
$R_{ab} - (1/2)Rg_{ab} = 8\pi T_{ab}$	Einstein equations	16.5

Summary of Chapter 17

In this chapter the concept of a fiber bundle is introduced. Rather than develop a general theory at the very beginning, instead we begin with a fairly detailed treatment of two paradigmatic examples of fiber bundles, in order to motivate the definition. Namely, we show that with each manifold M two other manifolds of double the dimension, TM and T^*M, may be canonically associated. Both of them are endowed with a remarkable geometrical structure even if M happens to be just a "bare" smooth manifold. For example, they turn out to represent the total spaces of vector bundles over M and carry various canonical tensor fields (in particular, T^*M always carries a symplectic structure), several objects may be lifted from M to the total spaces, etc. In analytical mechanics they serve as the playing fields for Lagrangian and Hamiltonian formulation of the dynamics respectively; this will be discussed in more detail in the following chapter, this one provides the necessary preliminaries.

$\pi : (x^a, v^a) \mapsto x^a, \quad \tau : (x^a, p_a) \mapsto x^a$	canonical projections on TM and T^*M	17.1.7
$T(f \circ g) = Tf \circ Tg$	a property of the tangent map Tf	17.3.2
$\gamma(t) \mapsto \dot{\gamma}(t)$	natural lift of a curve from M to TM	17.5.1
$\Phi_t \mapsto T\Phi_t$	lift of a flow from M to TM	17.5.5
$\triangle = v^a\partial/\partial v^a \quad (\triangle = p_a\partial/\partial p_a)$	Liouville dilation field on TM (T^*M)	17.6.1
$S := 1^{\uparrow} = dx^a \otimes \partial/\partial v^a$	vertical endomorphism on TM	17.6.4
$\langle \theta_p, W \rangle := \langle p, \tau_* W \rangle$	canonical 1-form $\theta = p_a dx^a$ on T^*M	17.6.5
$\omega = d\theta = dp_a \wedge dx^a$	canonical symplectic form on T^*M	17.6.7

Summary of Chapter 18

It is shown how classical mechanics may be formulated on TM and T^*M. In the non-degenerate case, both dynamics turn out to be completely equivalent geometrically: they realize a standard "symplectic" dynamics we studied in Chapter 14, i.e. a motion along integral curves of the Hamiltonian field. On T^*M the canonical symplectic structure is available from the outset so that the choice of a function H is the only step to be made. On TM the situation is a bit more complicated; rather than a symplectic form there is a canonical $\binom{1}{1}$-type tensor field available and the required symplectic structure is given only after the latter is combined with the (non-degenerate) Lagrangian, regarded as a function on TM. The standard Lagrange equations result by the projection of the symplectic dynamics onto the base M. The projection adds one order, so that the final equations are second-order differential equations on M. Hamilton's equations live directly on the total space T^*M and that is why they are (as is always the case for equations for integral curves) only first-order differential equations. Making use of the Lagrangian L one may construct the Legendre map $TM \to T^*M$, which serves as a bridge between the two dynamics. If the Hamiltonian (or Lagrangian) depends explicitly on time, a modification of the formalism is needed since the carrier manifold is now odd-dimensional. The distinguished 1-form $pdq - Hdt$ enters the equations and it turns out that this form also plays a decisive role in a construction of the action functional.

$\theta_L := S(dL), \quad \omega_L := d\theta_L$	Cartan 1-form, Cartan 2-form	18.2.3
$E_L := \triangle L - L$	energy corresponding to the Lagrangian L	18.2
$\dot\gamma = \zeta_{E_L}, \quad i_{\zeta_{E_L}}\omega_L = -dE_L$	Lagrange's equations (on TM yet)	18.2.6
$\langle \hat L(v), w\rangle := (d/dt)\vert_0\, L(v+tw)$	Legendre map $\hat L : TM \to T^*M$	18.3.1
$\hat L \circ \Phi^L_t = \Phi^H_t$	Lagrangian and Hamiltonian flows related	18.3.4
$TR_g, \quad T^*R_{g^{-1}}$	lifts of action R_g on M to TM and T^*M	18.4.1
$\tilde\xi_X$	generators of the lifted actions	18.4.1
$P_X = \langle \theta_L, \tilde\xi_X\rangle, \quad P_X = \langle \theta, \tilde\xi_X\rangle$	"Hamiltonians" of the lifted actions	18.4.1
$L = (1/2)\overset{\circ}{g} - \overset{\circ}{\phi}$	natural Lagrangian on TM	18.4.6
$\int_\gamma (\hat\theta - Hdt) \equiv \int_\gamma (pdq - Hdt)$	action integral for the Hamiltonian dynamics	18.5.6
$\int_{\hat\gamma} (\hat\theta_L - \hat E_L dt) \equiv \int_{t_1}^{t_2} L(\hat\gamma(t))dt$	action integral for the Lagrangian dynamics	18.5.6

Summary of Chapter 19

In order to pave the way for a possible generalization of the theory of linear connection well known from Chapter 15 (to be done in the next chapter) we reformulate it in a new language. The new description takes place on a new playing field, a manifold LM which may be canonically assigned to any manifold M. The points of LM are all frames at all points of M. There is a fairly rich structure on LM even prior to introducing the connection on M: the manifold LM namely turns out to be a total space of a principal $GL(n,\mathbb{R})$-bundle over M. A connection on M adds more structure on LM, a $GL(n,\mathbb{R})$-invariant horizontal distribution. We may reformulate the procedure of parallel transport of a frame along a curve γ on M in terms of the horizontal lift γ^h of the curve γ. There is also an interesting possibility of treating a wide class of geometrical objects on M (in particular tensor fields and more generally fields of type ρ) in terms of equivariant functions Φ on LM. Their parallel transport is discussed and it is shown that an appropriate directional derivative of Φ corresponds to the covariant derivative on M of the geometrical object described by Φ.

$\omega \equiv \omega^a_b E^b_a$	connection form on the frame bundle LM	19.2.1
$R^*_A\omega = A^{-1}\omega A, \quad \langle \omega, \xi_C \rangle = C$	crucial properties of the connection form	19.2.4
$U, V \in \mathcal{D} \Rightarrow [U, V] \in \mathcal{D}$	$\mathcal{D}$ is integrable (Frobenius' criterion)	19.3
$\theta^i\big\|_{\mathcal{D}} = 0 \;\Rightarrow\; d\theta^i\big\|_{\mathcal{D}} = 0$	alternative formulation of the criterion	19.3
$V \in \mathcal{D}^h \;\Leftrightarrow\; \langle \omega, V \rangle = 0$	horizontal distribution on LM	19.4.3
$T_e LM = \mathrm{Ver}_{\,e} LM \oplus \mathrm{Hor}_{\,e} LM$	decomposition induced by a connection	19.4.5
$\langle \omega, \dot{\hat{\gamma}} \rangle = 0$	$\hat{\gamma}$ corresponds to autoparallel frame field	19.5.1
$\Phi \circ R_A = \rho(A^{-1}) \circ \Phi$	Φ is a quantity of type ρ	19.6
$\Phi(\gamma^h(t)) = \text{ const.}$	autoparallel field of quantities of type ρ	19.6.5

Summary of Chapter 20

The translation of the concepts related to the *linear* connection into the language of the frame bundle, which was performed in Chapter 19, clearly indicates a possible generalization. One should simply replace $\pi : LM \to M$ by $\pi : P \to M$, a general principal G-bundle. A connection in this bundle is then (by definition) any G-invariant horizontal distribution on the total space P. The distribution may be conveniently encoded in a connection form ω, a $\mathcal{G}$-valued 1-form on P ($\mathcal{G}$ being the Lie algebra of G). The points of P are now the natural counterparts of frames, and their parallel transport along a curve on M is defined as the horizontal lift of the curve to P. The (local) dependence of parallel transport on path may be rephrased in terms of integrability of the horizontal distribution and the $\mathcal{G}$-valued *curvature* 2-form Ω enters the scene (via the Frobenius integrability condition) just as a measure of this non-integrability. As a convenient formal tool for the explicit computation of the curvature form one introduces the exterior *covariant* derivative D; we then prove the formula $\Omega = D\omega = d\omega + (1/2)[\omega \wedge \omega]$. Similarly we compute the action of D on another important class of objects, horizontal forms of type ρ, where we get $D\alpha = d\alpha + \rho'(\omega)\dot{\wedge}\alpha$. Applying D twice, Bianchi and Ricci identities arise. The last, starred, section is devoted to an interesting relation between subbundles, structures on M and connections compatible with the structures. It explains, for example, how special connections in $\pi : P \to M$ (say, metric connection in $\pi : LM \to M$) may be regarded as being extended from appropriate subbundles $\pi : Q \to M$, $Q \subset P$ (from $\pi : OM \to M$ in the case of the metric connection).

$R_{g*}\mathrm{Hor}_{\,p}P = \mathrm{Hor}_{\,pg}P$	horizontal distribution is G-invariant	20.2.1
$\omega_p := \Psi_p^{-1} \circ \mathrm{ver}\ :\ T_pP \to \mathcal{G}$	connection 1-form in $p \in P$	20.2.4
$R_g^*\omega = \mathrm{Ad}_{\,g^{-1}}\omega, \quad \langle \omega, \xi_X \rangle = X$	crucial properties of the connection form	20.2.5
$\pi \circ \gamma^h = \gamma, \gamma^h(0) = p, \langle \omega, (\dot{\gamma^h}) \rangle = 0$	horizontal lift of γ starting in $p \in P$	20.3.2
$(\mathrm{hor}\,\alpha)(U, \dots) := \alpha(\mathrm{hor}\,U, \dots)$	horizontal part of a form	20.3.4
$D\alpha := \mathrm{hor}\,d\alpha$	exterior covariant derivative of a form	20.3.5
$\Omega := D\omega = \Omega^i E_i$	curvature 2-form on P	20.4.1, 3
$\Omega = d\omega + (1/2)[\omega \wedge \omega]$	Cartan structure equation	20.4.3
$D\alpha = d\alpha + \rho'(\omega)\dot{\wedge}\alpha$	action of D on horizontal forms of type ρ	20.4.6
$DD\omega \equiv D\Omega = d\Omega + [\omega \wedge \Omega] = 0$	Bianchi identity	20.4.4, 7
$DD\alpha = \rho'(\Omega)\dot{\wedge}\alpha$	Ricci identity	20.4.8
$\Omega = 0 \ \Rightarrow\ \exists \sigma\ :\ \sigma^*\omega = 0$	zero curvature $\Rightarrow$ complete parallelism	20.4.11

Summary of Chapter 21

A link between connections on a principal G-bundle and gauge field theory (known from physics) is systematically built here. First a standard "physical" approach is briefly introduced for the convenience of the reader who is not familiar with this stuff from physics. Namely, a "global" G-symmetry of an action is made "local". This is achieved by adding new fields with quite definite transformation properties and interaction with the initial fields. It is shown that all the building blocks of the gauge scheme possess a natural interpretation in terms of connection theory. In particular, fixing of the gauge is given by the choice of a local section σ of the principal bundle, gauge potentials (in this gauge) are obtained by pull-back (with respect to the section) of a connection form to the base, gauge transformations correspond to a change of a section, field strength is obtained as the pull-back of the curvature form and a matter field as the pull-back of an equivariant function on P. Parallel transport equations of an arbitrary quantity of type ρ in a gauge σ are derived. The concept of an associated vector bundle is introduced (it arises from a principal bundle as a result of the replacement of its fiber by a representation space of the group G). The structure of the (locally) gauge invariant action is given and the equations of motion are derived (they generalize the Maxwell equations of electrodynamics, which turns out to be a gauge theory with group $U(1)$). Noether's theorem is introduced, providing a link between the symmetries of the action integral and the conserved quantities. The theorem sheds new light on some older results in this direction, the relation between conservation laws and the energy-momentum tensor in the field theory as well as exact Cartan symmetries in Hamiltonian mechanics. In the last section we return to the frame bundle LM and introduce the canonical 1-form θ with values in $\mathbb{R}^n$ which is related to the torsion on M and learn how to use the exterior covariant derivative $\mathcal{D}$ on the base M.

$\phi \mapsto e^{-i\alpha(x)}\phi, \quad A \mapsto A + d\alpha(x)$	$U(1)$-local gauge transformation	21.1.2
$\hat{\sigma}(x) = \sigma(x)S(x) \equiv R_{S(x)}\sigma(x)$	two sections related via $S \in G^{\mathcal{U}}$	21.2.1
$\hat{\phi} = B^{-1}\phi$	local gauge transformation of a matter field	21.2.5
$\hat{\mathcal{A}} = B^{-1}\mathcal{A}B + B^{-1}dB$	the same for the gauge potential	21.2.5
$\hat{\mathcal{F}} = B^{-1}\mathcal{F}B$	the same for the field strength	21.2.5
$\dot{v} + \langle \mathcal{A}, \dot{\gamma}\rangle v = 0$	equation of parallel transport	21.3.2
$S[\phi, \mathcal{A}] = -(1/2)\langle \mathcal{D}\mathcal{A}, \mathcal{D}\mathcal{A}\rangle_k +$ $+ (1/2)\langle \mathcal{D}\phi, \mathcal{D}\phi\rangle_h - (m^2/2)\langle \phi, \phi\rangle_h$	action of coupled system $(\phi, \mathcal{A})$	21.5.6
$\mathcal{D}^+\mathcal{F} = -\mathcal{J}, \quad \mathcal{D}\mathcal{F} = 0,$ $(\mathcal{D}^+\mathcal{D} - m^2)\phi = 0$	corresponding field equations	21.5.6
$S[\rho(e^{\epsilon s(x)})\psi] = S[\psi] + \epsilon\langle ds, j\rangle_k$	computation of Noether currents j	21.6.1
$j_i = T(\xi_{E_i}, \ . \ )$	Noether currents due to Killing vectors	21.6.6
$\Theta := D\theta$	where torsion sits in LM formalism	21.7.2

Summary of chapter 22

The special orthogonal group $SO(p,q)$ admits the two-sheeted universal covering group, which is called the spin group and is denoted by Spin (p,q). An elementary theory of spin groups is systematically developed with the help of Clifford algebras. An isomorphism of these algebras with appropriate matrix algebras (a faithful representation) is constructed. This leads naturally to the concept of a spinor as an element (vector) of the representation space of the Clifford algebra. Since the spin groups are subsets of the Clifford algebras, restriction of the representation of the algebra is automatically a representation of the spin group. Consequently, spinors also carry a representation of spin groups (and also the two-valued representation of the orthogonal groups). This is called the spinor representation. For some particular values of (p,q) special kinds of spinors may exist (Weyl, Majorana, ...). The term spin structure on M is sometimes used as a synonym for a principal bundle over M (the spin bundle), whose total space is a two-sheeted covering of the total space of the bundle of right-handed orthonormal frames and in the fibers of which the spin group acts. There are also manifolds which do not admit the spin structure. Equivariant functions of type ρ on the total space of the spin bundle (as well as their pull-backs to the base with the help of a section), where ρ is the spinor representation, are called spinor fields on M. The Rarita-Schwinger field then corresponds to a 1-form of type ρ. There is a specific first-order operator which acts on spinor fields, the Dirac operator. Its historical origin is in physics, in the quantum theory of the relativistic electron and it enters the Dirac equation.

$e^a e^b + e^b e^a = 2g^{ab}$	fundamental relations in Clifford product	22.1.1
$u = \alpha_1 \dots \alpha_k, \quad g(\alpha_j, \alpha_j) = \pm 1$	elements of the group Pin (p,q)	22.2.1
$u e^a u^{-1} =: (A^{-1})^a_b e^b$	two-sheeted covering Spin $(p,q) \to SO(p,q)$	22.2.3
$(1/2) e^a e^b \mapsto \mathcal{E}^{ab}$	derived isomorphism spin $(p,q) \to so(p,q)$	22.2.7
$\gamma^a := \rho(e^a)$	γ-matrices	22.3.1
$\mathcal{D}\psi = d\psi + (1/4)\hat{\omega}_{ab}\gamma^a\gamma^b\psi$	exterior covariant derivative of a spinor field	22.5.1
$\chi^\alpha_\mu(x) dx^\mu E_\alpha \equiv \chi^\alpha_a(x) e^a(x) E_\alpha$	Rarita-Schwinger field	22.5
$\not{D} := i_{\not{\mathcal{E}}} \circ D \equiv \gamma^a i_{\mathcal{E}_a} D$	Dirac operator on SM	22.5.3
$\not{\mathcal{D}} = i_{\not{e}} \circ \mathcal{D} \equiv \gamma^a i_{e_a} \mathcal{D}$	Dirac operator on M	22.5.3
$\not{\mathcal{D}}\psi = \gamma^a e^\mu_a (\partial_\mu \psi + (1/4)\omega_{bc\mu}\gamma^b\gamma^c\psi)$	action of the Dirac operator on spinor fields	22.5.4
$\not{\mathcal{D}}\psi = \gamma^a e^\mu_a (\partial_\mu \psi + (1/2)\alpha_\mu \gamma_5 \psi)$	how it simplifies for two dimensional M	22.5.4
$\rho(u)^\alpha_\tau \rho(u^{-1})^\sigma_\beta A^a_b \gamma^{b\tau}{}_\sigma = \gamma^{a\alpha}{}_\beta$	γ-matrices are Spin (p,q)-invariant tensors	22.5.11

INDEX OF (frequently used) SYMBOLS

curvature form on the total space P	Ω	20.4.1
curvature forms (linear connection on M)	Ω^a_b	15.6.3
curvature operator	$R(U,V)$	15.5.1
Dirac operator	$\not{D}$	22.5.3
Dirac operator on the base M	$\not{\mathcal{D}}$	22.5.3
dual space	L^*	2.4.1
electric and magnetic fields	$\mathbf{E},\mathbf{B}$	9.2.9
Einstein tensor	G_{ab}	16.5.
Euclidean space	E^n	2.6
exterior algebra of L	ΛL^*	5.3
exterior derivative	d	6.2.5
exterior covariant derivative	D	20.3.5
exterior covariant derivative on the base	$\mathcal{D}$	21.2.4
exterior product of forms	$\alpha \wedge \beta$	5.4.1
exterior product of $\mathcal{G}$ and (W,ρ') valued forms	$\alpha \dot{\wedge} \beta$	20.4.5
exterior product of Lie algebra valued forms	$[\alpha \wedge \beta]$	11.2.6
frame bundle	$\pi : LM \to M$	19.1.1
flow generated by vector field V	Φ^V_t, Φ_t	2.3
fundamental field (generator of action)	ξ_X	13.4.3
gauge potential	A	21.2.4
gauge potential (represented)	$\mathcal{A}$	21.2.4
gauge field strength	F	21.2.4
gauge field strength (represented)	$\mathcal{F}$	21.2.4
general affine group	$GA(n,\mathbb{R})$	10.1.15
general linear group	$GL(n,\mathbb{R})$	10.1.3
Hamiltonian field generated by f	ζ_f	14.1.6
Hodge star operator	$*, *_g$	5.8.1
horizontal distribution	$\mathcal{D}^h$	19.4.3
horizontal lift of a curve γ	γ^h	20.3.2
horizontal lift of a vector v	v^h	20.3.1
horizontal p-forms of type ρ on P	$\bar{\Omega}^p(P,\rho)$	21.2
interior product	$i_v, v\lrcorner$	5.4.1
Laplace-deRham operator	Δ, Δ_g	8.3.3

Lie algebra of the group $G, H, \ldots$	$\mathcal{G}, \mathcal{H}, \ldots$	11.2.2
Lie derivative along vector field V	$\mathcal{L}_V$	4.2
Lie group	$G, H, \ldots$	10.2
metric volume form	ω_g	5.7.3
orthogonal group	$O(n), O(n, \mathbb{R})$	10.1.5
Poisson tensor	$\mathcal{P}$	14.1.1
Poisson bracket of f and g	$\{f, g\}$	14.1.1
pseudo-Euclidean space	$E^{r,s}$	2.6
pseudo-orthogonal group	$O(r, s)$	10.1.5
pull-back	f^*	3.1.4
push-forward	f_*	3.1.2
quaternions, complex and real numbers	$\mathbb{H}, \mathbb{C}, \mathbb{R}$	22.1.4
Ricci tensor	R_{ab}	15.5.
right and left action of a group	R_g, L_g	13.1
representation of a group	$\rho(g)$	13.1
scalar curvature	R	15.5
scalar product of forms in (L, g)	$(\alpha, \beta)_g$	5.8.4
scalar product of forms on (M, g)	$\langle \alpha, \beta \rangle$	8.3.1
scalar product of forms from $\bar{\Omega}^p(P, \rho)$	$\langle \alpha, \beta \rangle_h$	21.5.1
special orthogonal group	$SO(n), SO(n, \mathbb{R})$	10.1.8
special unitary group	$SU(n)$	10.1.12
spin bundle	$\pi : SM \to M$	22.4
structure constants with respect to E_i	c^i_{jk}	11.2.2
symplectic form	ω	14.1.4
symplectic group	$\mathrm{Sp}\,(m, \mathbb{R})$	10.1.6
tangent bundle	$\pi : TM \to M$	17.1.1
tangent map at the point $x \in M$	$T_x f$	3.1.2
tangent vector to the curve γ	$\dot{\gamma}$	2.2.2
tangent space at the point $x \in M$	$T_x M$	2.2.2
tensors of type (r, s) in L	$T^r_s(L)$	2.4.5
tensor fields of type (r, s) on M	$\mathcal{T}^r_s(M)$	2.5.2
torsion forms	T^a	15.6.3
two-form of the electromagnetic field	F	16.2.1
unitary group	$U(n)$	10.1.12

INDEX

D

E

F

I

J

K

L

M

N

T

U

V

W

Z