**Differential Geometry and Lie Groups for Physicists**
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(selected fragments, mostly footnotes)

The purpose of this chapter is to introduce the concept of a smooth manifold, including the ABCs of the technical side of its description. The main idea is to regard a manifold as being "glued-up" from *several* pieces, all of them being very simple (open domains in $\mathbb{R}^n$). The notions of a *chart* (local coordinates) and an *atlas* serve as essential formal tools in achieving this objective. In the introductory section we also briefly touch upon the concept of a topological space, but for the level of the knowledge in manifold theory we need in this book it will not be used later in any non-trivial way.

(From the didactic point of view our exposition leans heavily on recent scientific knowledge, for the most part on ethnological studies of Amazon Basin Indians. The studies proved convincingly that even those prodigious virtuosos of the art of survival within wild jungle conditions make do with only intuitive knowledge of smooth manifolds and the medicine-men were the only members within the tribe who were (here and there) able to declaim some formal definitions. The fact, to give an example, that the topological space underlying the smooth manifold should be *Hausdorff* was observed to be told to a tribe member just before their death and as eyewitnesses reported, when the medicine-man embarked on analyzing examples of *non-Hausdorff* spaces, the horrified individual preferred to leave his or her soul to God's hands as soon as possible.)

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An *integral curve* of a vector field $V$ is then the curve $\gamma$ on $M$, such that at each point of its image, the equivalence class $[\gamma]$ given by the curve, coincides with the class $V_P$, given by the value of the field $V$ in $P$. Put another way, from each point it reaches, it moves away exactly in the direction (as well as with the speed) dictated by[1] the vector $V_P$. All this may be written as a succinct geometrical equation

$$\dot{\gamma} = V \qquad \text{i.e.} \qquad \dot{\gamma}(P) = V(P)$$

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This isomorphism suggests using a standard mathematical trick - *identification* of the spaces $L$ and $(L^*)^*$, and, by analogy then, the $n$-th with the $(n-2)$-nd dual spaces. Only the first two members, $L$ and $L^*$, thus survive from the threateningly looking, potentially infinite chain of still higher and higher dual spaces. (This, in a moment, will result in the fact that we will make do with only *two kinds* of indices, "lower" and "upper", on general tensors.)[2] If a non-degenerate bilinear form were *added* to $L$, the situation would change significantly, since it *would be* possible already to identify $L$ with $L^*$ in a *canonical* way (via the "raising and lowering of indices" procedure, see (2.4.13)).

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Thus we have learned that $L$ induces an infinite number of further linear spaces - for each pair $(p, q)$ of nonnegative integers there is the $n^{p+q}$-dimensional space $T^p_q(L)$. (This means that if we envisage tensor spaces as a "tower", the tower dilates in the upward direction, like a pyramid does on a photograph snapped in Giza by a distrait yogi, forgetting he has just performed a headstand.)

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The result (2.4.7) shows that all tensors constitute an ($\infty$-dimensional non-commutative) *associative algebra* (Appendix A2), called the *tensor algebra* $T(L)$. As a linear space, it is a *direct sum* of all spaces $T^p_q(L)$

$$\begin{aligned} T(L) &:= \bigoplus_{r,s=0}^{\infty} T^r_s(L) \\ &\equiv T^0_0(L) \oplus T^1_0(L) \oplus T^0_1(L) \oplus T^2_0(L) \oplus T^1_1(L) \oplus T^0_2(L) \oplus \dots \end{aligned}$$

(up to infinity), i.e. an element from $T(L)$ may be regarded as a linear combination of tensors of all types $\binom{p}{q}$. Multiplication $\otimes$ is defined as a linear extension of the definition of $\otimes$ on *homogeneous terms* (terms with fixed

[1] like a well-disciplined hiker, always walking in the direction of arrows on destination signs and obediently following the instructions concerning time indications given there (how many minutes would he or she need to reach the next arrow)

[2] This step saves the huge number of higher dual spaces as well as various kinds of indices for future generations, so it can be regarded as highly satisfactory far-sighted behavior from an ecological point of view; one should not lavishly waste any non-renewable resources, including mathematical structures.

$\binom{p}{q}$)), i.e. according to the rule "everybody with everybody":[3]

$$(k + v + \alpha + \dots) \otimes (q + w + \beta + \dots) := k \otimes q + k \otimes w + k \otimes \beta + \dots + v \otimes q + v \otimes w + \dots$$

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$\boxed{2.4.8}$ Check that
*i*) the result is indeed a tensor (multilinearity)
*ii*) $C$ does not depend on the choice of the basis $e_a$ (when $e_a$ has been fixed, however, $e^a$ is to be the dual)
*iii*) in components the rule for $C$ looks like[4]

$$t^{\dots\dots}_{\dots\dots} \mapsto t^{\dots a.}_{.a\dots} \qquad \text{i.e.} \qquad \text{as a } \textit{summation} \text{ with respect to a pair of upper and lower indices}$$

*iv*) independence of a choice of basis results from the component formula, too
Hint: *ii*) (2.4.2); *iv*) (2.4.6) □

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The maps $\flat_g$ and $\sharp_g$ are known as *lowering* and *raising of indices* (with the help of $g$), respectively. The quantities $v_a, v^a$ are often called covariant and contravariant *components* of (the same) vector $v$. We will not adopt this nomenclature, however. We will always strictly discriminate between a *vector* $v = v^a e_a$ and a *covector* $v_a e^a$ (as being elements of $L$ and $L^*$) and interpret the operations of raising and lowering of indices as maps between two *different* spaces $L \leftrightarrow L^*$. Note, that the graphical expressions used for these maps originate from well-known musical symbols.[5]

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The metric tensor makes it possible to change the position of indices on higher rank tensors, too, for example

$$t^a_{bc} \mapsto t_{abc} := g_{ad}\ t^d_{bc} \qquad\qquad R^{ab}_{cd} \mapsto R_{abcd} := g_{ae}\ g_{bf}\ R^{ef}_{cd}$$

This belongs to basic exercises of *index gymnastics*.[6]

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$\boxed{2.4.14}$ Prove the validity of the exercise

$$t^{\dots a.}_{..a\dots} = t^{..a\dots}_{\dots a.}$$

Hint: do you intend to base your proof upon the fact that the total potential energy remains unchanged? [red herring] □

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Nearly all situations in geometry, as we will see in this text over and over again, are closely related to maps of manifolds $f : M \to N$; in particular, often $M = N$. It turns out that each mapping of *points* of manifolds automatically leads to a mapping of *tensors* in these points and (provided some restrictions are satisfied) also of tensor *fields* on $M$ or $N$. Some of them move in the same direction as the points under the action of $f$, that is from $M$ to $N$, while others *reverse the arrow* and move against the direction of $f$. A clear understanding of this transport of tensors serves as a ticket into a number of following chapters.[7]

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*vi*) if $\psi : N \to \mathbb{R}$ is a function then

$$(f_* \dot\gamma)\psi = \dot\gamma(f^* \psi)$$

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[3] The maximum promiscuity rule.

[4] Each contraction thus unloads a tensor by two indices. It breathes with fewer difficulties immediately (fewer indices = fewer worries), it feels like after a rejuvenation cure. This human aspect of the matter is reflected sensitively in German terminology, where the word *Verjüngung* (rejuvenescence) is used.

[5] Namely "flat" and "sharp". Thoughtful graduates of schools of music might recall, that *no g was present* on sharps and flats they had read in sheets of music - this is simply because the validity of *Euclidean* geometry is normally assumed in concert halls, so that musical flats and sharps are conventionally associated with *this Euclidean g* (and are not indicated explicitly).

[6] It should be performed, as is the case for arbitrary gymnastics, at an open window, never directly after a substantial meal.

[7] Fortunately, the price/value relation of this ticket is very favorable and since the penalty for fare dodgers is high, there is no point in trying to travel without paying.

This means that an independent (and equivalent) way *to define* $f_*$ is given by the formula[8]

$$(f_* V)\psi := V(f^*\psi)$$

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Let us have a closer look at how the induced metric tensor actually works. By the definition in (3.1.4), the scalar product of two vectors $V, W$ in the sense of $g$ on $M$ is

$$g(V,W) \equiv (f^*h)(V,W) := h(f_*V, f_*W)$$

We can see from this formula that if we use the induced metric tensor the result is the same as if we first transported the vectors $V, W$ onto $N$ and then performed the computation of the scalar product *in the sense of* $h$ there.[9]

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There is *no reason* for a *general* tensor field $A$, however, to be constant along the integral curves of a field $V$: the tensor $(\Phi_t^* A)(x)$, which has been transported into $x$ from the point $\Phi_t(x)$, in general depends on $t$. A convenient measure of this dependence (i.e. of Lie *non*-constancy = *non*-invariance with respect to $V$) is given by the object

$$\mathcal{L}_V A := \left.\frac{d}{dt}\right|_0 \Phi_t^* A$$

which is called the *Lie derivative* of a tensor field[10] $A$ along a vector field $V$. This derivative "palpates" the changes of tensor fields induced by a tiny Lie transport along $V$: first, the value of the field $A$ in the "slightly drained away" point $\Phi_\epsilon(x)$ is transported back into $x$ and then it is compared with the initial value of $A$ in $x$.

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As we will see in a while (4.3.4), the component expression of the Lie derivative of a general tensor field is a sum of several pieces, each one carrying a number of indices. The overall structure is given by a system of clear rules; the resulting expression looks, however, fairly complicated at first glance. All the properties of $\mathcal{L}_V$ may, in principle, be derived[11] from its component expression, but the use of simple *algebraic* properties of the Lie derivative (which may be ultimately traced back to the simple properties of the pull-back $\Phi_t^*$) turns out to be both more efficient and more instructive.

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$\boxed{4.3.4}$ Check that
*i*) the Lie derivative of the coordinate basis fields reads

$$\mathcal{L}_V dx^i = V^i_{\ ,j} dx^j \qquad \mathcal{L}_V \partial_i = -V^j_{\ ,i}\partial_j$$

*ii*) this results in the following component expression of the Lie derivative of an arbitrary rank tensor field

$$(\mathcal{L}_V A)^{i\ldots j}_{k\ldots l} = V^m A^{i\ldots j}_{k\ldots l,m} + V^m_{\ ,k} A^{i\ldots j}_{m\ldots l} + \cdots - V^j_{\ ,m} A^{i\ldots m}_{k\ldots l}$$

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[8]Whether you put the shoe on your foot or you put your foot in the shoe, the result is the same - your foot is in the shoe. It is thus possible to define the procedure both ways.

[9]The following analogy with computer networks could be helpful: $M$ and $N$ are computers "here" and "there", $h$ is a useful piece of software there (we are sitting here). We have to make a choice: either to run the program there (which might be inconvenient, if the work is to be done at the time when the network is overloaded), or first to download the software onto our disk ($f^*$ serves as, say, *ftp*), obtaining $(M, f^*h)$ ($\leftrightarrow$ our computer endowed with useful downloaded software), and then run the program (for performing scalar products and computing expressions containing them) conveniently at any time *here*.

[10]In Arnold's monograph [1] the Lie derivative is also mentioned under the well-turned name the *fisherman's derivative*: a fisherman stands in a river and differentiates tensor fields, floating around him. Unfortunately, the present-day status of the human environment makes this juicy bon mot barely intelligible to the younger generation. The lamentable quality of water causes tensor fields of higher ranks to simply not be able to survive in the overwhelming majority of rivers and the exciting stories narrated by our grandfathers on how they (when small boys) used "to guddle fifth-rank completely antisymmetric tensors in a spruit behind a village" may seem to be typical *fish stories*, today.

[11]It used to be done in this way in older textbooks. As an example, the walls of Altamira and Lascaux caverns have been reported to be densely covered by such fairly long component expressions. Let us remark, as a nice illustration of the inventiveness of the primeval hunters in masterful use of terrain irregularities, that in caves of calcite, limestone and dolomite they used stalactites for the location of upper indices, stalagmites for lower indices and stalagnates as the most convenient places for the contraction of a pair of indices.

i.e. there is the first term (flat amount), plus there is one term to be added for each index of the tensor (with + sign for lower index and - sign for upper one). These rules may be concisely summarized in the form of a table - recipe for cooking the house speciality $(\mathcal{L}_V A)^{i\dots j}_{k\dots l}$; compare with (15.2.7)

| | for preparation of $\mathcal{L}_W A$ |
|---|---|
| first put on the bottom of a pan | $WA^{\cdots}_{\cdots} \equiv W^m A^{\cdots}_{\dots,m}$ |
| plus for each $A^{\dots i \dots}$ add | $-W^i_{\ ,m} A^{\dots m \dots}$ |
| plus for each $A_{\dots i \dots}$ add | $+W^m_{\ ,i} A_{\dots m \dots}$ |

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As we will see later, the appropriate choice of a *frame field* $e_a(x)$ and a *coframe field* $e^a(x)$ may strongly simplify both reasoning and computation in various situations. Important examples are provided by *orthonormal* frame fields on Riemannian manifolds (see, for example, (15.6)) or *left-invariant* fields on Lie groups (11.1). In the general theory of relativity a frame field (appropriately chosen, most often orthonormal) is usually called a *tetrad field*[12] and a formalism working with components of tensors with respect to this kind of frame field is known as the *tetrad formalism* (see for example (15.6.20), (16.5) and (22.5)).

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The following exercises of the *index gymnastics*[13] will prove to be useful in what follows. $\boxed{5.2.6}$ Justify the legitimacy of the following steps ($\alpha, \beta, A, t$ being arbitrary indexed objects commuting one with another, such as the components of tensors)

*i)*

$$\alpha_{[a\dots b]}\beta^{a\dots b} = \alpha_{[a\dots b]}\beta^{[a\dots b]} = \alpha_{a\dots b}\beta^{[a\dots b]}$$

$$\alpha_{(a\dots b)}\beta^{a\dots b} = \alpha_{(a\dots b)}\beta^{(a\dots b)} = \alpha_{a\dots b}\beta^{(a\dots b)}$$

*ii)*

$$A^a_{[c}\dots A^b_{d]} = A^{[a}_{[c}\dots A^{b]}_{d]} = A^{[a}_{c}\dots A^{b]}_{d}$$

$$A^a_{(c}\dots A^b_{d)} = A^{(a}_{(c}\dots A^{b)}_{d)} = A^{(a}_{c}\dots A^{b)}_{d}$$

*iii)*

$$t^{\cdots}_{[\dots a\dots[\dots b\dots c\dots]\dots d\dots]} = t^{\cdots}_{[\dots a\dots b\dots c\dots d\dots]}$$

$$t^{\cdots}_{(\dots a\dots(\dots b\dots c\dots)\dots d\dots)} = t^{\cdots}_{(\dots a\dots b\dots c\dots d\dots)}$$

*iv)*

$$t^{\cdots}_{[\dots a\dots(\dots b\dots c\dots)\dots d\dots]} = t^{\cdots}_{(\dots a\dots[\dots b\dots c\dots]\dots d\dots)} = 0$$

where the round brackets represent complete *symmetrization* (all the terms on the right are to be summed *with plus sign* in the definition from (5.2.2)). The idea of *i)* - *iii)* is to recognize typical situations, in which some (anti)symmetrizations may be omitted (or conversely added formally), since they are ensured automatically by means of other (anti)symmetrizations; *iv)* says that a symmetrization, when performed *inside* an antisymmetrization (and vice versa), gives zero.

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We see that three out of four terms drop out (vanish). After some practice, such unlucky terms are immediately recognized and one displays directly the non-vanishing part of the result alone. Note that it is

[12] Since a space-time $(M, g)$ is a *four-dimensional* manifold; in general, the nomenclature *vielbein field* is widely used, i.e. a "manypod" or "manyvet field"; a frame in *three* dimensions resembles (with a bit of fantasy, no doubt a fairly useful instrument in the realm of mathematics as such) a *dreibein* $\equiv$ a tripod or a trivet, so that a tetrad is the same thing as a *vierbein*.

[13] They used to be fairly popular in those fitness centers in which both square and round brackets are installed.

the highly effective (and merciless) mechanism n.4 which bears full responsibility for the fact that so many (innocent and agreeable) terms are not allowed to survive.[14]

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$\boxed{5.2.10}$ Repeat the computation of the product $\alpha \wedge \beta$ (treated above) *in components* and convince yourself how cumbersome the component method is in comparison with the way presented above.
Hint: starting with the standard expressions (5.2.9) $\alpha = \alpha_a e^a$ and $\beta = \frac{1}{2}\beta_{ab} e^a \wedge e^b$, identify first the components $\alpha_a$, $\beta_{ab}$, then plug them into (5.2.5), computing thus $(\alpha \wedge \beta)_{abc}$ and finally reconstruct the whole form $\frac{1}{3!}(\alpha \wedge \beta)_{abc} e^a \wedge e^b \wedge e^c$; in the course of the computation, do your best to avoid (in spite of the temptation being increasingly hard to resist) shouting highly substandard words (all the more accurate, however), unworthy of a true lady (gentleman).

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Generalized Kronecker symbols $\delta^{a\dots b}_{c\dots d}$ ($p$-delta symbols) play a similar role in the machinery of forms as does the ordinary Kronecker delta symbol $\delta^a_b$ for vectors or covectors. In this section several useful identities involving $p$-deltas are derived.[15] Furthermore, we will learn how they are related to some other useful objects, like the Levi-Civita symbol and the determinant.

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So there is a freedom in a *single parameter* $\lambda$ in the formula for computation of the volume of a parallelepiped in $L$. This parameter may be fixed by ascribing a definite value of the volume to any *one particular* (non-degenerate) parallelepiped. In a "general" linear space (endowed with no additional structure, like a metric tensor), however, *all* (non-degenerate) parallelepipeds are *completely equivalent* (a parallelepiped is given by an $n$-tuple of vectors and all vectors are equivalent) and there is no reason for preferring some of them for the purpose of fixing the constant $\lambda$. Put another way, there is no *natural scale* of volumes. *All* the volume forms and, consequently, all the formulas for computation of volumes (i.e. with any choice of $\lambda$) based on them are equivalent. We can speak of a *ratio* of two volumes rather than of "the" volume itself.[16]

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We see that the orientations of the band $P'$, induced from $P$ via the channels $A = A'$ and $B = B'$ respectively, contradict one another. This actually means that we obtain *no consistent* global orientation on the *union* (on the whole Möbius band). One can prove that this is really an *inherent* problem of Möbius band itself.[17]

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[14] "Heterogeneity" turns out to be a strong evolutionary advantage within the population of exterior forms: $e^1 \wedge e^2 \wedge e^3$ survives, $e^1 \wedge e^1 \wedge e^3$ is not fit enough (its mortal sin being "repeating $e^1$"). Remarkably, five years on the Beagle (1831-1836) seemed to be not enough for young Charles Darwin to notice this simple example of how natural selection works (although, in those times, there was a flourishing colony of exterior forms living in the Galapagos, their multiplication being routine activity, well known to native people; nor did Alfred Russel Wallace use it in his independent speculations). It was observed only by a teacher of "Gymnasium" in Stettin (today's Szczecin in Poland), Hermann Grassmann, in 1844. Because of a lukewarm response to his work, however, he was so frustrated as to leave this battlefield and set his brain to understanding Sanskrit (where he was fairly successful, at last). The ideas of Grassmann were fully appreciated and then developed by Elie Cartan.

[15] A reader who suffers from index sickness might use a half tablet of an anti-indexicum or, preferably, skip this section completely.

[16] Intense and merciless advertisements, hammering us day after day, try to make us think that an individual has not the remotest chance of surviving without a credit card, wireless phone and a metric tensor. Some of us, however, never shared this opinion. John Lennon, as an example, expressed his visionary dreams about a life in a linear space with no metric tensor (a situation one nowadays can hardly imagine, indeed) in his famous composition *Imagine*. In the original version we might hear the courageous verse

| | |
|---|---|
| Imagine there's no metric | Imagine there's no countries |
| It isn't hard to do | It isn't hard to do |
| No way to measure angles | Nothing to kill or die for |
| No lengths of vectors too | And no religion too |

The time was, however, not ripe and people not mature enough to be able to accept such a far-reaching idea in those times; censorship (closely intertwined to the tensor lobby, of course) forced him (under pressure) to revise substantially the first strophe and the result is well known today: in the new innocent first strophe, which occurred at the shop counters and which we like to sing up to the present day, no reference to the metric tensor has remained at all.

[17] It is *unavoidable* by any trick like, say, some ingenious choice of coordinate patches and the structure of their overlaps; see (6.3.6)

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(Here we encountered the tiny tip of a huge iceberg on our voyage, the volume of its underwater part being, as is well known, much bigger than that of its visible part. Unfortunately majority of the iceberg will remain under the surface until the end of the book. What we are speaking about is a close relation between the differential geometry and the topology of manifolds. We see that global topological properties of manifolds may, as an example, *obstruct* the introduction of some particular geometrical structures (here the orientation or, equivalently (6.3.5), a volume form). Similar "topological conditions" are imposed by several other celebrities of the geometrical heaven, like spinor fields or a metric tensor with Lorentzian signature (the latter cannot be globally defined on the ordinary sphere $S^2$ !). They might be more modest and follow the example of such a reputable and useful quantity as the "ordinary" (positive definite) metric tensor undoubtedly is: without any idle talk it gladly allows to be defined, when nicely asked, on an *arbitrary* manifold.)

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Manifolds like the Möbius band are said to be *non-orientable manifolds.*[18]

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The formal[19] linear combinations

$$c = c_i s_p^i \qquad\qquad c_i \in \mathbb{R},\ s_p^i = i\text{-th } p\text{-simplex}$$

of $p$-simplices are called *p-chains* and the corresponding ($\infty$-dimensional) linear space is denoted $C_p$.

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Let us also mention how the expressions mentioned above may be transcribed from the noble hieroglyphic writing into the demotic writing used by common people. Common people use the notation

$$\omega_g \leftrightarrow d\Omega \equiv \sqrt{|g|}\ d^n x \qquad\qquad \omega_{\hat{g}} \leftrightarrow dS \qquad\qquad d\Sigma_i \leftrightarrow dS_i \leftrightarrow d\mathbf{S}$$

in which Gauss' theorem looks like

$$\int_D (\operatorname{div} V) d\Omega \equiv \int_D (\operatorname{div} V)\ \sqrt{|g|} d^n x = \oint_{\partial D} (\mathbf{V}.\mathbf{n}) dS \equiv \oint_{\partial D} \mathbf{V}.d\mathbf{S} \equiv \oint_{\partial D} V^i dS_i$$

the small circle put around the integral sign indicating that the integral is performed over a *closed* "surface" (the boundary[20] of the domain $D$). Again it is true (see the note in (6.3.11)) that in general *neither* $d\Omega$ *nor* $dS$ are exterior derivatives of anything else; this is nothing but the conventional way to write down such objects (here "d" is related to the conception of being "infinitesimal").

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Let us begin with the possibility of encoding scalar and vector fields into forms and vice versa. If we have an $n$-dimensional manifold endowed with a metric tensor and orientation, the canonical isomorphisms $\sharp \equiv \sharp_g$, $\flat \equiv \flat_g$ (the raising and lowering of indices) and $* \equiv *_{g,o}$ (the Hodge operator) are available. One can then identify the spaces of vector fields, 1-forms and $(n-1)$-forms, as well as the spaces of 0-forms and $n$-forms. This means that it is easy to encode scalar and vector fields into forms, but we are not able to express *forms of all degrees* in terms of scalar and vector fields (it is possible for the "marginal pairs" $0, 1, (n-1), n$, but it is not for forms of "inner" degrees $2, 3, \dots, (n-2)$). There exists an important exception, however, namely *three-dimensional* manifolds (the most interesting from the practical point of view being undoubtedly the simplest one, the good old Euclidean space $E^3$), where the "inner" degrees are simply missing.[21]

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[18] There are long lasting heated disputes among scientists as to whether non-orientability of a manifold is congenital, unalterable by upbringing at all, or results from an emotionless approach within babyhood (some claim even during the prenatal period, when being glued from trivial pieces).

[19] A linear space may be specified by enumerating the basis elements. If $\mathcal{A}$ is an apple and $\mathcal{P}$ is a pear, we may introduce the two-dimensional linear space of elements of the form $v = v^1\mathcal{A} + v^2\mathcal{P}$ (the apple and the pear constitute its basis). In the case under consideration, the basis consists of simplices.

[20] So that *it is not*, as some books mistakenly claim, the trendy jewelry known as (body) *piercing*

[21] And they are also missing of course on 1- and 2-dimensional manifolds; on these manifolds there thus exists a (simplified version of) "vector analysis", too (it may also be regarded as the vector analysis "diluted up to homeopathic concentrations"). After reading this section the interested reader can work up the details of the corresponding theory as a simple exercise by him(her)self.

This creates a "full tube" $\mathcal{U}$. It is bounded by $\sigma$ from the left and by $\Phi_\infty(\sigma)$ from the right, and the side faces are formed from the integral curves of the field $\xi$, emanating from the boundary $\partial\sigma$. Our aim now is to compare the integral of an arbitrary $p$-form $\alpha$ over the end image of the simplex with the integral over the initial simplex itself, i.e. to compare the integrals $\int_{\Phi_\infty(\sigma)} \alpha$ and $\int_\sigma \alpha$. Since both $\Phi_\infty(\sigma)$ and $\sigma$ are parts of the *boundary* of $\mathcal{U}$, both the integrals occur in *Stokes' theorem* written for the form $\alpha$ and the domain $\mathcal{U}$. In addition to the longed-for two integrals Stokes' theorem appends two more terms, one of them being a "volume" integral over $\mathcal{U}$ and also a "surface" integral over the "side faces" of the boundary $\partial\mathcal{U}$. So we are expected to be able to compute these two integrals. The key idea lies in the observation that the tube (as well as the side faces of its boundary) may be *put together from the infinitesimal slices*[22] of the thickness $dt$ (put together $= \int_0^\infty dt \dots$). Stokes' theorem thus yields the equation in which there are two integrals which we need plus two additional integrals which contain the procedure of putting together the slices $\int_0^\infty dt \dots$. The last crucial technical point is to realize that the slices may be actually regarded as the "coins" from the problem (7.6.11), so that we may profit from the "coin interpretation" of the interior product $i_V$.

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$\boxed{9.3.3}$ Check that
*i)* if all $p$-cocycles happen to be $p$-coboundaries ($Z^p = B^p$), there will be (for given $p$) only *a single* class (the class $[0]$).
*ii)* if there is a *non-trivial* $p$-cocycle $z$ (such that it is not a $p$-coboundary), then all of its (non-zero) multiples $\lambda z$ are also non-trivial; moreover, the multiples by *different* numbers being *inequivalent*.
Hint: *i)* if $z \in Z^p = B^p$, then $z = dw$ for some $w \in C^{p-1}$. Then $z = dw = 0 + dw \Rightarrow [z] = [0]$, and this is true[23] for *each* $z$; *ii)* $z \neq d(\dots) \;\Rightarrow\; \lambda z \neq d(\dots)'$; if $\lambda_1 z = \lambda_2 z + d(\dots)$, then $(\lambda_1 - \lambda_2)z = d(\dots) \Rightarrow z = d(\dots)'$, which is a contradiction.

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From the perspective of differential geometry a special class of groups turns out to be of particular interest, namely the *Lie* groups. They represent the objects in which their two distinct aspects peacefully co-exist in a happy symbiosis, shoulder to shoulder - algebraic (they are groups) and geometrical or differential-topological (they are smooth manifolds). These two aspects restrict one another,[24] but (as the world goes in a good partnership) they also immensely enrich one another - the richness of the *geometry* on Lie groups ultimately springs from the existence of the *algebraic* structure of groups.

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The examples we have analyzed here were very simple. The same result could be obtained (with devotion of more time and labor), however, for all the remaining groups we treated in the previous section. All of them happen to be smooth manifolds, moreover the sub*groups* of $GL(n,\mathbb{R})$ mentioned there actually turn out to be also sub*manifolds* of $GL(n,\mathbb{R})$. This is ample motivation for introducing a separate concept, which combines in itself the structure of a group with the structure of a smooth manifold. This is exactly what a *Lie group* is.

If two different structures are expected to live together peacefully in a common household, they have to agree on the terms of this coexistence; they have to be *compatible*. One of the parties involved, the *smooth* structure on a manifold, insists on meeting at any moment only maps which are smooth. Its imperative towards the group (the other party involved) thus consists in always bringing home only smooth maps (if any).[25] The group cannot imagine its life without *three* key maps (so that it will certainly bring them home); thus, first and foremost, the request of smoothness concerns these three maps.

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Lie groups should be mentioned as role models in all handbooks on the "art of living" (savoir vivre) - from the point of view of differential geometry they indeed live life to the full ("put the hammer down"). There are several canonical geometrical objects living on them, and some specific procedures may be performed only

[22] Just like a piece of ham (or rather a carrot for us vegetarians) may be cut into thin slices

[23] Herein the author would like to thank the Indians for inventing the concept of zero (as well as all nations, individuals and firms that have merit in putting it on the market). In this (as well as in numerous other) proof(s) it came in handy, indeed.

[24] We will see, for example (11.1.6), that a manifold, which yearns to become a Lie group, has to first vow that for all its life it will be *parallelizable* (a *global frame field* should exist on it, i.e. $n = \dim M$ nowhere vanishing vector fields, being moreover linearly independent at each point) and by far not all manifolds are disposed to bind themselves by oath to this. It turns out that on the common sphere $S^2$, to give an example, there is *not a single* nowhere vanishing vector field, so that it is not possible to introduce the structure of a Lie group on $S^2$.

[25] Some requirements of the group structure towards the manifold were mentioned in section (10.1).

on them. This richness of the Lie group as a manifold is due to the Lie group as a group, i.e. it is ultimately to be traced back to the symbiosis of its algebraic and differential-topological structures. Numerous constructions to be discussed in what follows are based on the fundamental concept of *(left)invariant field*. We will learn this stuff in the next section.

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We know from (11.2.1) that the structure constants $c^c_{ab}$ actually coincide with the coefficients of anholonomy of the left-invariant frame field $e_a$ so that they carry information not only about the objects at the point $e$, but at least in some neighborhood of this point. Recall that a left-invariant field is uniquely "extended" from its prescribed value $E_a = e_a(e)$ at $e$ to points apart from $e$ by a left translation $L_g = m(g, \ .\ )$, which in turn depends on the composition law $m : G \times G \to G$ on a group $G$ (i.e. we get different left-invariant fields and consequently different structure constants for the same vectors $E_a$ ($\Rightarrow$ different Lie algebra), if we modify the composition law). We see then that a "genetic" information of vital importance about the *composition law on a group* (being "the heart" of the group) is encoded in a concentrated form in the structure constants (or, as a matter of fact, in the Lie algebra $\mathcal{G}$ itself)[26] even though they are formally expressed in terms of objects living in a single point (the unit element $e$) alone.

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$\boxed{11.3.3}$ Let $L_X$ be the left-invariant vector field on $G$ which is generated by a vector $X \in \mathcal{G}$. Show that
*i)* its integral curve $\gamma^X(t)$ starting from $e$ is a one-parameter subgroup

$$\gamma^X(t+s) = \gamma^X(t)\gamma^X(s) \qquad\qquad \gamma^X(0) = e$$

*ii)* if in turn $\gamma(t)$ is an arbitrary one-parameter subgroup, then it is necessarily the integral curve of the left-invariant[27] field $L_X$ with $X \equiv L_X(e) = \dot\gamma(0)$; the *complete* trajectory $\gamma(t)$ then turns out to be given by its *initial velocity* (in which direction and how fast does it rush forth), i.e by the tangent vector $\dot\gamma(0) = X$ in the starting point $e$.

==================================================================

Let $f : G \to H$ be a homomorphism of Lie groups. It turns out then that a homomorphism of the corresponding Lie algebras is automatically induced. It is known as the derived homomorphism. Let us start with a simple observation.[28]

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Numerous facts about *matrix* Lie groups and their Lie algebras may also be obtained with no use at all of the geometrical stuff like left-invariant fields, their integral curves etc. Here we try to "derive" this standard simplified machinery from the formalism adopted up to now and to work up convenient and quick algorithms,[29] which we then apply for analyzing some particular common Lie groups mentioned in section (10.1).

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$\boxed{11.7.14}$ Let $\mathcal{G} = sl(2, \mathbb{R})$. Show that

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[26]Some analogy may be found with the relation between some foodstuffs (a Lie group) and their "*powdered*" versions (its Lie algebra); the powdered form represents a "simplified" ("compressed" ,"zipped") version of the original one, preserving essential (total according to advertisement) part of the properties of the original.

[27]It came to our knowledge (as top-secret information; don't spread, please!) that the same curve leaves nothing to its fate ("a curve never can tell") and that it systematically prepares for the time, when in all textbooks throughout the country authors will give attention mainly to *right-invariant* objects. In secrecy, completely unwitnessed even now it is *at the same time* the integral curve of the *right-invariant field* $R_X$ *as well*, which any investigative journalist may easily convince himself/herself of by a computation. Less investigative individuals are recommended to consult (11.4.7).

[28]This observation is always to be performed under a soft red light in a thoroughly blacked-out room. In the literature one can also find the recommendations to prepare in advance some hemp yarn, a magnifying (or reducing) glass, a diode and a bit of lip-salve. However, the rich and long-lasting personal experience of the author of this book shows that one can always somehow get along without the facilities listed above and they might be used equally well in the course of some other amazing observation.

[29]A natural and valid objection might be raised as to why we actually started with a "complicated" geometrical exposition, when there is a "simple" matrix formalism available on the market. We can allege *ad defendendum* before the Court of Conscience that 1. after ten chapters we have got chummy with differential geometry in so far as we can understand its reasoning with equal ease as we understand matrix multiplication and 2. some issues look more complicated from the perspective of matrix fundamentalism or they need fairly non-ecological (highly paper-consuming) computations.

$i$) a general element of the Lie algebra has the form[30]

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad\qquad a, b, c \in \mathbb{R}$$

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As we already mentioned at the beginning of section 10.1, groups always occur as groups of transformations of something, through their *action* on a set (usually endowed with some additional structure). Thus there exists a rule which assigns to each element $g$ of a group a transformation $L_g$ of some set $M$. A study of group theory thus naturally incorporates[31] besides knowledge of the groups themselves also the question of where and how a given group may act.

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Both the adjoint representation Ad of a group $G$ as well as its derived representation ad of the Lie algebra $\mathcal{G}$ are frequently encountered in various applications. The group does not worry too much about finding a vector space $V$ to carry the representation. It simply uses its own Lie algebra to do this. So $(V, \rho, \rho')$ becomes in this particular case $(\mathcal{G}, \mathrm{Ad}\,, \mathrm{ad}\,)$.

While this may be regarded as an admirably economical behavior of $G$ (instead of two structures $\mathcal{G}, V$ to be paid from the taxpayer's money a single one makes do), it might at the same time make harder a bit to grasp the stuff quickly (one should be always careful to understand clearly whether a given $X \in \mathcal{G}$ stands in the role of the Lie algebra element to be represented or in the role of an element of the carrier space $V \equiv \mathcal{G}$

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$\boxed{12.4.11}$ Let $(\rho_1, V_1)$ and $(\rho_2, V_2)$ be two representations of the same group $G$. Check that
$i$) by the prescription[32]

$$(\rho_1 \otimes \rho_2)(g) := \rho_1(g) \otimes \rho_2(g)$$

(the right-hand side, i.e. the operator of the structure $A \otimes B$, being in the sense of Appendix A1) one indeed defines a representation of the group $G$

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Each *equivariant* map $A : (V_1, \rho_1) \to (V_2, \rho_2)$ provides us with a wand which enables us to reach the wishful thinking of whole generations of alchemists, namely a "transmutation of a quantity of type $\rho_1$ into a quantity of type $\rho_2$". It is enough to pretend deep concentration for a while, to mutter mysteriously abracadabra and (not forget) at the same time to assign to a vector $v_1 \in V_1$ in an unobtrusive way[33] the vector $Av_1 =: v_2 \in V_2$, since the action of $g$ then gives

$$v_1 \mapsto \rho_1(g)v_1 \qquad \Rightarrow \qquad v_2 \equiv Av_1 \mapsto A(\rho_1(g)v_1) = \rho_2(g)(Av_1) \equiv \rho_2(g)v_2$$

so that the effect of the equivariant map $A$ consists in a loss of cultural heritage of $V_1$ and the complete assimilation to the novel milieu of $V_2$. Now we are going to learn that the same thing may also be achieved with the help of an *invariant* element in the space $V_1^* \otimes V_2$.

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Hint: $\eta(Ax, Ax) = \eta(x, x)$ according to (10.1.5); the equation $x = Ax_0$ (for $x$ from the sphere of radius $r$) fixes the last column[34] of the matrix $A$: $A_{in} = r^{-1}x_i$ (it is normalized to unity); complete arbitrarily to a right-handed orthonormal basis in $E^n$ and locate the basis vectors as columns of the matrix $A$. The matrix $A$ belongs to $SO(n)$ and it sends $x_0$ to $x$.

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[30] A word of caution for beginners relativists is in order: $c$ in this expression has *nothing to do* with the velocity of light in vacuum.

[31] Especially in those despicable cases when we put our mind to this stuff with, from the very beginning, the view of using our knowledge somewhere.

[32] Note that the construction of the direct sum of representations $\rho_1 \oplus \rho_2$ may be obtained amazingly simply (and surprisingly no literature mentions this fact!) from the direct product $\rho_1 \otimes \rho_2$ by the well-known operation "turning a bulb by the angle $\pm\pi/4$" (its iteration is usually applied when the bulb $\otimes$ is blown and we replace it by a new one).

[33] The words *in an unobtrusive way* should be emphasized. Sometimes small children in the audience succeed in seeing through the trick and then they shout (not making any scruple of us) "ha ha, he applied the equivariant map $A : (\rho_1, V_1) \to (\rho_2, V_2)$!"

[34] A watchful reader of hints might feel this reasoning to be familiar.

Now we have at our disposal a tool for a simple construction of the class of homogeneous spaces of a group $G$ - it is enough to find its subgroups $H$. Our pleasure, being already far from negligible, grows to a true rapture when we learn that (up to isomorphism) there are actually no other homogeneous spaces at all except this class so that this construction exhausts as a matter of fact *all* homogeneous spaces.[35]

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$\boxed{13.2.13}$ Check that the homomorphisms on the left give the isomorphisms on the right:[36]

$$\begin{array}{rcccr} GL(n,\mathbb{R}) \to GL(1,\mathbb{R}) & & A \mapsto \det A & \Rightarrow & GL(n,\mathbb{R})/SL(n,\mathbb{R}) = GL(1,\mathbb{R}) \\ GL(1,\mathbb{C}) \to GL(1,\mathbb{R}) & & z \mapsto |z| & \Rightarrow & GL(1,\mathbb{C})/U(1) = GL_+(1,\mathbb{R}) \\ GL(1,\mathbb{C}) \to GL(1,\mathbb{C}) & & z \mapsto z^2 & \Rightarrow & GL(1,\mathbb{C})/\mathbb{Z}_2 = GL(1,\mathbb{C}) \\ SU(n)\times U(1) \to U(n) & & (A, e^{i\alpha}) \mapsto e^{i\alpha}A & \Rightarrow & SU(n)\times U(1)/\mathbb{Z}_n = U(n) \\ GL_+(1,\mathbb{R}) \to (\mathbb{R},+) & & x \mapsto \ln x & \Rightarrow & GL_+(1,\mathbb{R}) = (\mathbb{R},+) \end{array}$$

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$\boxed{12.4.14}$ Consider tensors of the type $\binom{p}{q}$ in a linear space $L$. Check that
*i)* in the formula for the transformation of components $t \equiv t^{a\dots b}_{c\dots d}$ under the change of a basis $e_a \mapsto A^b_a e_b$ in $L$ a representation $\rho$ of the group $GL(n,\mathbb{R})$ occurs[37]

$$t(eA) = \rho(A^{-1})t(e)$$

==================================================================

We know from problem (13.3.1) that any representation of the "covered" group "is" automatically also a representation of the covering group, but the opposite is not true in general. In order to soften the mental trauma which this unpleasant result causes to the covered[38] groups $\tilde{G}$, modern mental hygiene introduced the concept of a *multi-valued representations*. How does it work?

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As already mentioned in the hint to (13.3.1), the reason why the converse assignment $\rho \mapsto \tilde{\rho}$ is problematic is that the map $f$ cannot be inverted, since it is not injective. Neither the canonical choice of one of the pre-images in general exists. If there is no distinguished choice of a pre-image, the most fair decision is to take *all* the pre-images. (The other equally fair decision is to accept *no* pre-image, i.e. to take a conservative stand that there is ("we are sorry") no representation, there's an end of it.)

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This construction combines the two directions of generalization of the representation on $\mathcal{F}(M)$ (mentioned before (13.4.11)). We came to $\mathcal{T}^r_s(M,V)$, so that we generalized functions to arbitrary tensor fields and at the same time the $\mathbb{R}$-valued objects to the $V$-valued ones. Let us mention two simple examples illustrating the objects introduced above. We will see from them that actually the tensor fields of type $\hat{\rho}$ are by no means rare and dangerous beasts living in virgin forests far from here, but rather they are fairly frequent, useful and good-natured pets living in our immediate environment.

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$\boxed{14.3.4}$ Let $\mathcal{D} \subset M$ be a $2k$-dimensional domain on $(M,\omega)$, $\zeta_f$ a Hamiltonian field, $\Phi_t$ its flow and $\Phi_t(\mathcal{D})$ the image of $\mathcal{D}$ with respect to the flow. Define the expressions (the *Poincaré-Cartan integral invariants*)

$$I^k \equiv I^k[\mathcal{D}] := \int_{\mathcal{D}} \underbrace{\omega \wedge \dots \wedge \omega}_{k \text{ entries}}$$

---

[35]Including the homogeneous spaces, which any missions from other planets, "solar systems" or even other galaxies will carry sometime in the future (with a view to investigating it in laboratories under the microscope). Sometimes the strength of our slender earthly mathematics indeed takes the breath away.

[36]The last isomorphism (given by the logarithm) is the heart of how the *slide rule* functions, some years ago an essential piece of equipment for any true engineer. It converts a product into a sum, the latter then being realized mechanically. Rulers based on the remaining isomorphisms still await a producer.

[37]A *right* action is present in this formula due to the argument $A^{-1}$ in $\rho$ (13.1.1); this is all *right*, since also $e \mapsto eA$ is a right action.

[38]According to (13.3.1) the groups $\tilde{G}$ readily and voluntarily lend *all* of their representations to their covering groups $G$. Fairly often, however, there are serious problems with the reciprocity.

Prove that they indeed deserve their name; in particular[39] prove that

$$I^k[\Phi_t(\mathcal{D})] = I^k[\mathcal{D}]$$

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In the case of acceleration one is namely to shift the vector $\mathbf{v}(t+\varepsilon)$ from the point $\mathbf{r}(t+\varepsilon)$ back to the point $\mathbf{r}(t)$ (thus obtaining $\mathbf{v}_{\parallel}(t)$) and only this vector may be compared with the vector $\mathbf{v}(t)$. So it is nothing but the trick from (15.1.1), the role of the isomorphism $B$ being played by an appropriate shift. Everything is so clear *here* that one might even be abashed at why an issue should be made of all this.[40]

Now, let us try to repeat the same procedure on a different manifold, for example on the sphere $S^2$. Imagine that an ambitious technocratic ideal was accomplished at last - throughout the Earth, first all the irregularities were straightened out by bulldozers (they were, one should admit, both impractical and unaesthetic) and then the whole surface of the Earth was nicely covered by a neat asphalt. If we now roll a ball along such a smooth surface,[41] it has to roll, according to the laws of mechanics, uniformly along a straight line, since the only force available is the gravitational force, directed everywhere downwards. This force constrains the ball to remain on the two-dimensional *surface* of the Earth (it keeps the ball from flying away along a "truly" straight-line trajectory and escaping into space); the ball gets accustomed to this status quo and it does not regard it as a restriction.[42] It considers pragmatically the sphere $S^2$ to be its living space and it does not care whether the sphere actually is or is not a subset of any larger ambient space. Since the projection of the gravitational force to the plane which is tangent to the sphere always vanishes, the ball feels[43] *no force* acting on it and it thus has no reason to change its velocity (neither length nor direction); it therefore moves with vanishing acceleration along a straight line. Note, however, that from the point of view of the ambient space $E^3$ this is by no means an ordinary straight line, but rather it is a circle (with maximum possible radius), which encircles the whole Earth. The uniform motion along this circle which arises by the iteration of the (infinitesimal) shifts of the velocity vector is the straightest possible motion on the sphere. The shift of the velocity vector keeping its length as well as direction unaltered *in the sense of the sphere* is, as we see from the resulting trajectory, something considerably different from the same procedure performed *in the sense of* $E^3$ - from the point of view of $E^3$, in the course of the shifts the vector also continually *rotates* a bit in order to remain tangent to the sphere.

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The concept of a linear connection is very important in physics, although its presence is fairly obscure in many applications (like in acceleration in elementary mechanics).

$\boxed{15.1.3}$ Estimate (or evaluate exactly) the fraction

$$f = lc/a$$

where $a$ denotes the number of people on Earth who understand what is the acceleration (including the formula which enables one to compute it) and $lc$ denotes the number of people on Earth who are aware that the linear connection is used in this formula.
Hint: ask all of them and then divide the two numbers; (1.1.1) - (22.5.12)

However, there are also disciplines like the general theory of relativity, in which the linear connection lies at the very heart of the mathematical formulation, being explicitly present in the fundamental equations of the theory.

$\boxed{15.1.4}$ Estimate (or evaluate exactly) the fraction

$$f = lc/gr$$

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[39] For the proof of the "Poincaré-Cartan" part use an appropriate source on the history of mathematics; the term "integral" is clear; here we only concentrate on the word "invariant", namely invariant *with respect to the flow* $\Phi_t$.

[40] Mathematical physics is sometimes blamed for "making an issue" of quite "simple" things. There is a perfect consensus in that this blame is indeed legitimate in $p$ percent of concrete cases, a bit less concord takes place in the numerical value of the number $p$. Extensive research (based on elaborate questionnaires) revealed that the distribution of $p$ over the world population is actually uniform, bounded by the values $p = 0$ and $p = 100$.

[41] and we ensure zero air resistance and a couple of similar technical details

[42] A confidential information from one such ball; for reasons of protection of privacy it has no wish to make either its center or radius public.

[43] ibid.

where $gr$ denotes the number of people on Earth who understand elements of the general relativity (including the basic formulas) and $lc$ denotes the number of people on Earth who are aware that the linear connection is used in these formulas.
Hint: see the hint in (15.1.3)

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and eventually

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})$$

so that the requirement of being metric and symmetric indeed leads to the unique result for the Christoffel symbols of the connection. This distinguished linear connection on a Riemannian manifold is usually called the *Riemann connection* or the *Levi-Civita connection*; we will therefore use the abbreviation *RLC connection*[44]

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Let us have a look at some practical manipulations with the coordinate expressions containing covariant derivatives.[45]

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$\boxed{15.6.12}$ Solve the system (15.6.10) for $(T^2, g)$ = the two-dimensional torus with the metric induced from the embedding into $E^3$ (3.2.2). Compute its Gaussian curvature, Ricci tensor and the scalar curvature and check that

$$R_{abcd} = \frac{\sin\psi}{b(a + b\sin\psi)}\epsilon_{ab}\epsilon_{cd} \qquad R_{ab} = \frac{\sin\psi}{b(a + b\sin\psi)}\delta_{ab} \qquad R(x) = \frac{2\sin\psi}{b(a + b\sin\psi)} \equiv 2K(x)$$

so that the scalar curvature is no longer constant, rather it depends on the coordinate $\psi$ (in particular it *vanishes* on the two circles, where the torus touches the slices of bread when eaten for the lunch, it is positive on the part seen by the consumer from outside and negative on the part which is not visible).

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Since we know that the solution is unique, we try to find the *simplest possible* forms satisfying all the equations: e.g. the second equation suggests that (maybe) $\sigma^2_3 \sim e^3$ ($e^3$ is missing on the right-hand side) and $\sigma^1_2 = -e^2$ (there might be a term $\sim e^1$ there, but we try the simplest ansatz first[46] ); the result $\omega^1_2 = -d\vartheta$ gives $\nabla_{e_2} e_2 = \omega^1_2(e_2)e_1 = -(1/r)\partial_r$, at the same time it should be $(1/r^2)\Gamma^i_{\vartheta\vartheta}\partial_i$, from where we get $\Gamma^r_{\vartheta\vartheta} = -r$ and (the remaining) $\Gamma^i_{\vartheta\vartheta} = 0$, which is in agreement with (15.3.5).

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We encountered the concept of (tensor of) torsion in section devoted to the RLC connection (15.3.3), where we learned that the requirement of *vanishing* torsion leads in combination with metricity to a unique (= RLC) connection. So in this particular connection the torsion is *by definition* completely "disabled". On the other hand, exactly this particular connection is by far the most frequent linear connection met by common people (say, in general relativity). This results in that the torsion mostly remains hidden in the shadow of its much more popular sibling, the curvature.[47] The torsion must appreciate then (even be touched to the heart) knowing that we did not forget about it. In this section we will learn in which geometrical situation the (non-vanishing) torsion manifests its presence. Namely it turns out that it causes "disclosure of a geodesic parallelogram".

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Consider as a manifold the two-dimensional sphere $S^2$ with both the north and south poles removed, endowed with the common "round" metric tensor. If it is as big as the surface of the Earth, it may easily

[44] Its role in the analysis of RLC circuits in electronics still remains obscure.

[45] This may be regarded as a continuation of the exercises of the *index gymnastics* from (2.4.14) and (5.2.6) (see footnote 50), which is made possible by the addition of further popular gymnastic apparatus, the semicolon.

[46] Recall the "Ockham's razor" (law of parsimony) principle which advises us: Pluralitas non est ponenda sine necessitate, i.e. plurality should not be posited without necessity". Fortunately, there is no "necessity" for "positing plurality", here.

[47] As scientists recently discovered (under microscopes, I expect) this spectacular astronomical phenomenon was already pretty well known to Mayan civilization. Mayan astronomers compiled precise tables of positions for the Moon, Venus, Curvature and Torsion and were able to predict with astonishing accuracy torsion eclipses (caused by the curvature; their prediction namely stated that it always happens).

happen we actually do not recognize it is a sphere (it took some time for mankind, too) and we believe we walk on a Euclidean plane. Then it is natural to perform the parallel transport of vectors as follows:

First, we measure the length of the vector to be transported and arrange the *length* to be *the same* after the transport. Then the only issue which remains is its direction. In order to fix the direction we use a compass and measure the azimuth of the initial vector (i.e. the angle clockwise from due north; this does not work at the poles, but recall they were removed from the manifold at the very beginning with wondrous foresight). We then prescribe *the same azimuth* to the transported vector. If we believe we walk on a Euclidean plane (endowed with a distinguished "north" direction) we have a clear conscience we did our best to realize parallel transport in the *most common* intuitive sense.[48]

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A standard machinery developed in courses on the special theory of relativity consists of a mathematics of four-vectors and four-tensors in Minkowski space. In this language one achieves explicit Lorentz covariance of all equations. It turns out, for example, that from a four-dimensional perspective the electric and magnetic fields turn out to be parts of a single object, the tensor of the electromagnetic field with components $F_{\mu\nu}$. This tensor is, however, not "general", but rather, *antisymmetric*

$$F_{\mu\nu} = -F_{\nu\mu}$$

This fact is a clear signal for us "to switch over to another channel" in contemplation on this stuff, the new "channel" being the language of *differential forms*. And if we call to mind that in the four-tensor formalism the Maxwell equations look like

$$F^{\mu\nu}{}_{,\nu} = j^\mu \qquad\qquad F_{[\mu\nu,\rho]} = 0$$

then our experienced eye[49] readily reports that it noticed the component expression of *codifferential and differential of a two-form $F$* on the left-hand sides of the equations. Thus the fundamental equations of the electromagnetic theory may be written in terms of fundamental objects and operations of the theory of differential forms.

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Finally, let us have a look at the last two operators needed, the codifferential $\delta$ and the Laplace-deRham operator $\Delta$. Since they are only composed from operators which we already mastered, the only thing we are to do is to bring the parts together. As it is commonly done, we will denote the Laplace-deRham operator for the case $E^{1,3}$ by $\Box$ rather than $\Delta$ (one should keep in mind, however, that this operator in general acts on forms, not only on functions; in particular on functions it is also called the *d'Alembert operator* or the *wave operator*).[50]

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Each statics is thus governed by a pair of equations. One of them is inhomogeneous, where the sources of the fields stand on the right-hand side, the other is homogeneous, where no sources occur. Yet the two equations themselves differ a bit at first sight. It turns out, however, that they become as similar as two peas in a pod,[51] when written it terms of differential forms.

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When the variation of an action functional is performed, its linear increment (the first variation) has the structure of a volume integral, in which the expression under the integral sign depends linearly on the varied argument; the factor standing by this variation is (by definition) just the variational derivative. For example the result[52]

$$S[A+\epsilon\alpha] = S[A] - \langle \delta F + j, \epsilon\alpha\rangle + \cdots \equiv S[A] - \epsilon \int_{\mathcal{D}} \alpha^\mu (\delta F + j)_\mu d^4x$$

is rewritten as

$$\frac{\delta S[A]}{\delta A^\mu(x)} = -(\delta F + j)_\mu(x)$$

---

[48] This technique can be safely used at the scale of a town, say; as a preparation the reader is invited to use it at a copy-book scale.

[49] The left eye for right-handers and the right eye for left-handers (recall the well known crossing of neural pathways).

[50] Exceptional foresight was definitely not the strong point of people who introduced this notation long ago; they seem to have missed the elementary fact that the same symbol will denote in this book the end of a problem.

[51] Recently scientists found that actually the equations are similar as one oak leaf to another.

[52] In books on physics the quantity $\epsilon\alpha$ is often written as $\delta A$ (increment = variation of $A$), so that $S[A+\delta A] = S[A] - \int_{\mathcal{D}} \delta A^\mu(\delta F + j)_\mu d^4x$.

$\boxed{16.3.3}$ It is already an open secret that the salary which letters receive for their performance in mathematics and physics are scandalously poor indeed. We should not be surprised then to hear that many of them try to earn a little extra so that they put a signature to a contract for more than a single role ($\Rightarrow$ more salaries). Neither are they discouraged enough in awkward situations when they have to perform *two* roles in a *single* equation! Find out where in this equation $\delta$ performs the role of a *variation* and where it denotes[53] the *codifferential*.

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The reason why the field equations look exactly like this and, as a matter of fact, how the gravitational field itself is actually encoded into metric tensor $g_{ab}$, is discussed at length in textbooks on the general theory of relativity.[54]

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Hint: *i)* if $g = \beta(\tau)d\tau \otimes \beta(\tau)d\tau \equiv e^1 \otimes e^1$, then $\omega_g = e^1$ and $((g, f^*h)_g + 1/2)\omega_g$ reduces to $(1/2)(h_{ab}\dot{y}^a\dot{y}^b/\beta + \beta)d\tau$; *ii)* the "function" $\beta(\tau)$ is actually a *component of a 1-form*[55] $e^1$ with respect to the coordinate basis $d\tau$; then the transformation rule under a change of the coordinate follows immediately ($\beta d\tau \overset{!}{=} \beta' d\tau'$ should hold)

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The case of a *two*-dimensional manifold $M$ (two-dimensional minimal surfaces in $(N, h)$) turns out to be of particular interest for two groups of the civilian population.

The first group is represented by small children, who used to be fascinated by *soap bubbles* while playing in a bathtub. The surface tension forces the bubble to take a form with minimum area under the given additional conditions; these conditions may be realized, say, by a wire rim, to which the boundary of the bubble should be attached or by the pressure of the air enclosed by the bubble (if it is attached to nothing and hovers in the form of $S^2$ feather-light in the air). Some children continue with this fascination until adulthood, they write complicated papers and (complicated) monographs, in which they do not hesitate to attack the (complicated) problems of the theory of soap bubbles using "heavy artillery" of differential geometry and algebraic topology.

Other children continuously diffuse in full age from being fascinated by soap bubbles to being (even more) fascinated by a *string theory*; this theory tries to reach the ambitious goal of explaining all the physics in the universe from a minimal number of first principles. Instead of considering a *world-line* $y^a(\tau)$ of a point particle it introduces a *world-sheet* $y^a(\tau, \sigma)$ of a (one-dimensional) string.

The reader anxious to learn more about strings is recommended to read, just to start somewhere (best this very day!), several thousands of papers, waiting patiently in the electronic preprint library at the site http://arxiv.org/ .

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$\boxed{17.6.6}$ It turns out that the canonical 1-form $\theta$ on $T^*M$ may be regarded as the "Platonic eternal Idea" of a differential form on $M$ in the following sense: let $\alpha$ be a 1-form on $\mathcal{O} \subset M$ and let $\sigma : \mathcal{O} \to T^*M$ be the corresponding section of the cotangent bundle $\tau : T^*M \to M$ (17.2.6). Check that

$$\sigma^*\theta = \alpha$$

so that *any* differential form on $M$ may be viewed as a result of an appropriate pull-back of "the 1-form $\theta$" on $T^*M$. The 1-form $\theta$, living in the "real world of eternal Ideas" $T^*M$, is then "the Platonic Idea of a differential form" whereas $\alpha$, living in the "apparent world of material objects" $M$ is its "immersion in the material world".

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$\boxed{18.2.1}$ Find out when (Hermann Ludwig Ferdinand von) Helmholtz lived and estimate then how long the Helmholtz' criterion[56] of the existence of a Lagrangian for a given second-order ordinary differential equations has been known.
Hint: Appendix C or Google

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[53]Note that it is even a *juvenile* $\delta$. Evidently the problem has already developed insomuch that although the adult $\Delta$ by no means idles and denotes everything possible, it is still not enough to reasonably support a family.

[54]And we will not walk in their shoes.

[55]According to the general definition $e^a = e^a_\mu dx^\mu$ the function $\beta$ is often called the *vielbein field* ($e^1 = e^1_\tau(\tau)d\tau \equiv \beta d\tau$), although a defence of the word *viel* (many) in this (*one*-dimensional) case might be hard work even for an experienced lawyer.

[56]For the sake of incompleteness we do not mention an explicit form of the criterion here.

The nomenclature *vertical* distribution is fairly clear from the conventions concerning drawing of pictures of the type (19.4.4): fibers use to be drawn vertically. Now why is the distribution given by the connection called horizontal? If we took an opinion poll about what precise meaning people actually associate with the word horizontal, we would probably learn sort of "flat", "level" or maybe "contrary (perhaps perpendicular?) to vertical". In more official sources we read[57] *horizontal* = "flat or level; parallel to the ground or to the bottom or top edge of something", or[58] we first learn that *horizon* is "line at which the earth or sea and sky seem to meet" and then *horizontal* = "parallel to the horizon; flat or level" and finally[59] "parallel to, in the plane of, or operating in a plane parallel to the horizon or to a base line". And, by the way, *horizon* = the apparent junction of earth and sky, *level* = having no part higher than another: conforming to the curvature of the liquid parts of the earth's surface."

What part of this piece of wisdom concerns *our* notion of horizontality? If we look at the figure in problem (19.4.4), we can see that the horizontal vector $v^h$ is (on purpose) not displayed as being horizontal in the sense of "liquid parts of the earth's surface", since those "parts" (say the a surface of a lake) used to be *perpendicular* to the truly "vertical" direction (given by, say, a plumb line at rest). If we, however, adopt the definition which refers to a "line at which the earth ... and sky seem to meet" and by "earth" we understand the beautiful scenery of a national park with a marvellous chain of mountains afar, then the horizon need not be necessarily "flat" or "level" and the vector $v^h$ actually may be tangent to the horizon. So in this broader sense "horizontality" need not necessarily mean orthogonality with respect to the vertical direction (not all of us happen to be mariners), but rather *complementarity* to the latter. By this we mean that the vertical *plus* horizontal is already enough to produce *any* direction whatsoever. (Actually *any* direction, which is *not vertical*, may be declared to be horizontal - it suffices to find a place on the mountains afar with a slope just steep enough).

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In this section we first convince ourselves that the similarity naturally extends from the *linear* connection to the *general* connection as well and then we start to contemplate about what all this resemblance means. The contemplation will result in the joyful conclusion[60] that the formalism of the gauge fields and the theory of connections actually speak "about the same thing in different words".

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Hint: $\rightarrow:$ a global trivialization $\psi : P \rightarrow M \times G$ exists; a section is $\sigma : m \mapsto \psi^{-1}(m, e)$; $\leftarrow:$ if a section $\sigma$ exists, in each fiber we get a distinguished point[61] ($\sigma(m)$ over $m$). All the points in the fiber may be now related to $\sigma(m)$: they are associated with a unique group element $g$ such that $p = R_g\sigma(m) \equiv \sigma(m)g$; a global trivialization is $p \mapsto (m \equiv \pi(p), g)$

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The 1-form $\theta$ is clearly independent of the connection, it enjoys each day of life on $LM$, there being any connection or not[62]

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A useful vocabulary to be used in order to relate the connection theory with the gauge theory then reads

| | | |
|---|---|---|
| connection form $\omega$ | $\leftrightarrow$ | gauge potential $\mathcal{A}$ |
| curvature form $\Omega$ | $\leftrightarrow$ | gauge field strength $\mathcal{F}$ |
| function $\Phi$ of type $\rho$ | $\leftrightarrow$ | matter field $\phi$ of type $\rho$ |
| choice of a section $\sigma$ | $\leftrightarrow$ | gauge fixing |

(In the spirit of problem (17.6.6), the objects on the left live in the "real world of eternal Ideas" $P$ whereas the

[57]Cambridge Advanced Learner's Dictionary, Cambridge University Press, 2003.

[58]A.S.Hornby: Oxford Advanced Learner's Dictionary of Current English, Oxford University Press, 1974.

[59]Merriam-Webster's Collegiate Dictionary.

[60]A lot of our brain capacity is saved for other interesting facts if the same thing does not need to be stored twice.

[61]We already feel that we won: Archimedes (would) have moved the Earth, as soon he (would) have been given a fixed point, we (indeed) move forward with the proof, since we do (indeed) have a distinguished point.

[62]and it has lived there since long ago, when the connection was neither at the drawing board of Evolution. Several of the world's top natural history museums pride themselves on few intact components (mostly $\theta^1$) found in Palaeozoic layers (at those times $\theta$ fed on trilobites).

objects on the right live in the "apparent world of material objects" $M$. Each object on the right represents a particular "immersion in the material world" of the corresponding object on the left.)

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*i*) if $\sigma$ is the section which corresponds to a frame field $e_a$ in a domain $\mathcal{U} \subset M$, then

$$\sigma^* \theta^a = e^a$$

(so that in the spirit of (17.6.6) the canonical 1-form $\theta$ on $LM$ may be regarded as "Platonic eternal Idea" of a coframe field on $M$).

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We began a whole group of chapters treating connections by Chapter 19, introducing $LM$ and describing the linear connection in terms of this manifold. Then we extended the concept of the connection to an arbitrary principal bundle $\pi : P \to M$ and treated various facts already in the general setting. Clearly everything which is said about the general case is also true for the particular case $\pi : LM \to M$. Nevertheless, because of the exceptional importance as well as some specific features of the good old frame bundle it is worth returning there for a while.[63]

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$\boxed{22.1.8}$ Verify[64] that all the real Clifford algebras may be summarized in the following concise table, generalizing the table from problem (22.1.7):

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathbb{R}(2^l)$ | $\mathbb{R}(2^l) \oplus \mathbb{R}(2^l)$ | $\mathbb{R}(2^l)$ | $\mathbb{C}(2^l)$ | $\mathbb{H}(2^{l-1})$ | $\mathbb{H}(2^{l-1}) \oplus \mathbb{H}(2^{l-1})$ | $\mathbb{H}(2^{l-1})$ | $\mathbb{C}(2^l)$ |

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This bundle (which is also known as the *spin frame bundle*) is often identified with the *spin structure* on $M$. In analogy to the case of the restriction, neither is the existence of the prolongation of a given bundle guaranteed; there may exist various topological "obstructions". In particular, the spin structure may not exist for a given (pseudo)-Riemannian manifold. We will not treat the details of the issue,[65] but instead we will simply assume that *our* manifold $(M, g)$ *admits* a spin structure.

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It may not have escaped your notice that the last formula (which contains the coefficients of anholonomy $c^a_{bc}(x)$) actually does not need any objects *explicitly* characterizing the *connection*; it is just enough to evaluate (all) mutual commutators of the frame field (being sort of a simple homework from (quasi) quantum mechanics).[66]

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A subspace $\mathcal{B} \subset \mathcal{A}$ is a *subalgebra* if it (also) happens to be closed with respect to the multiplication and a subalgebra $\mathcal{I}$ is an *ideal* (left, right, two-sided) if the multiplication of an arbitrary element $a \in \mathcal{A}$ by an element $i \in \mathcal{I}$ (from the left, from the right, both) results in an element in $\mathcal{I}$ (for example, for the left ideal $ia = i' \in \mathcal{I}$ for any $a \in \mathcal{A}$).[67] Given a two-sided ideal $\mathcal{I}$ in the algebra $\mathcal{A}$, we may introduce the multiplication into the factor space $\mathcal{A}/\mathcal{I}$ by means of representatives and obtain the *factor-algebra* ($[a][b] := [ab]$; for other choice of representatives we get $[a+i][b+i'] = [ab + ai' + ib + ii'] = [ab]$, if $\mathcal{I}$ is a two-sided ideal).

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[63] We know from detective stories that culprits like to return to the scene of the crime; perhaps we have already committed enough on $LM$ in Chapter 19 so as to return there for a while.

[64] This problem is especially recommended for lifelong prisoners and shipwrecked persons living on desert islands to help pass the time.

[65] For the convenience of the reader who intended to actively join with nonchalance a debate of experts at an evening party, we mention that it is advisable to remember that the object which acts as an obstruction to introducing the spin structure is "the second Stiefel-Whitney class" $w_2(M)$ of the manifold $M$. Just after saying this we recommend to leave the group of experts as soon of possible under the guise of, say, tasting "that marvellous cake".

[66] This kind of computation is recommended to be performed, in order to save time, parallel to watching the evening news (except for breaking news coverage, leading often to sign errors), weather forecast or financial reports.

[67] If elements of the ideal $\mathcal{I}$ are regarded as carriers of the gene of *idealism*, then the offspring from mating (multiplication) of an idealist with any other element of $\mathcal{A}$ (including realists, pragmatists and so on) consists again only of idealists.