Gauge-potential approach to the kinematics of a moving car

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Summary. — The kinematics of the motion of a car is reformulated in terms of the theory of gauge potentials. $E(2)$-gauge structure originates in the no-slipping contact of the car with a road.

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1. – Introduction

The physically most important field where the mathematical theory of connections ($\leftrightarrow$ gauge potentials $\leftrightarrow$ Yang-Mills potentials) is used with great success is undoubtedly the theory of elementary particles. Since, however, the concepts involved are rather abstract and (especially for a newcomer in the field) mixed with a number of other (equally abstract) ones, one should appreciate to find out that gauge potentials can be used in «much more mundane, but in return more readily visualized, context» [1], too, *viz* in the context of classical mechanics [1-4] or hydrodynamics [5]. A nice example of this sort is given in [1] (cf. also [2]). It was shown there that the natural kinematical framework for computing the net rotation of a (deformable) body due to a sequence of deformations is the non-Abelian gauge structure over the space of shapes of the body.

In this paper we show that (and rather in detail how) the kinematics of a motion of a car on a road can be reformulated in terms of non-Abelian gauge potentials, too. The gauge group is $E(2)$, the Euclidean group of the translations and rotations of the 2-dimensional plane.

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It should be noted that the differential geometric treatment of the car's kinematics was given before in [6]. The new point here is the addition of the degree of freedom α (see sect. 2) which then makes it possible then to treat the problem in the language of connections.

Finally, let us mention a technical simplification made in computations. As is well known, the front (as well as rear) wheels of a car do not rotate with the same angular velocity in general (the device called *differential* is needed). When we speak about the angle α as being the angle measuring the orientation of the front wheel, the *average* angle is understood in fact. Or, equivalently, we compute everything as if the car were a *tricycle* (then α is the angle of *the* front wheel). The full account of the situation with two wheels can be done, of course, but it does not bring anything conceptually new.

2. – The configuration space of a car as a principal $E(2)$-bundle

Let P be the configuration space of a car. The coordinates $(\alpha, \beta, x, y, \varphi)$ are introduced according to fig. 1, 2, with the following meaning: (x, y) are the Cartesian coordinates of the centre of the front axle, φ is the angle between the x_1 axis and the tie rod («if that is the name of the thing connecting the front and rear axles» [6]; it measures the direction in which the car is headed), α measures the orientation of the front wheel with respect to the axle and β is the angle made by the front axle with the tie rod. Thus (x, y, φ) carry the information about the position of the tie rod alone in the (x_1, x_2)-plane irrespective of the «shape» of the car, whereas (α, β) encode the car's shape regardless of the position of the tie rod in the (x_1, x_2)-plane.

There is a natural action of the Euclidean group $E(2)$ on P, consisting in «rigid» motions (rotations and translations) of the car with no change of its shape, that is to say the motions of the tie rod keeping the shape fixed. This action $R_{\mathcal{B}}: P \to P$ (see appendix A for more technical details) results in the additional structure of the space P, *viz* the structure of a *principal fibre bundle* with the group $E(2)$. It is constructed as follows: two configurations $p, p' \in P$ are declared to be equivalent if they differ only by a rigid motion from $E(2)$, *i.e.* if there exists such $(B, b) \in E(2)$ that the action of (B, b) on p results in p', *i.e.* $R_{\mathcal{B}}p = p'$. We then define M as the factor-space $P/E(2)$,
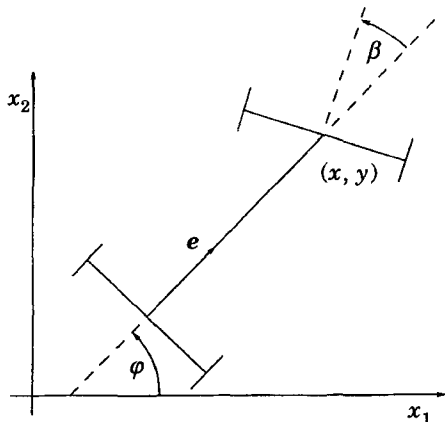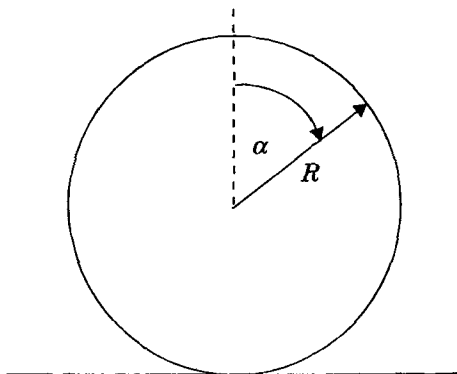

Fig. 1. – The coordinates x, y, φ, β.

Fig. 2. – The front wheel—the coordinate α.

i.e. the points of M are by definition the equivalence classes in P. There is a *projection* map

$$\pi \colon P \to M$$

sending the configuration p to its own equivalence class $[p] \equiv \pi(p) = m$, or in coordinates

$$\pi \colon (\alpha, \beta, x, y, \varphi) \mapsto (\alpha, \beta).$$

Thus π extracts from the complete configuration the information about the shape of the car and «forgets» the position of the tie rod within the (x_1, x_2)-plane.

According to the terminology of [1,5], P is the space of «located shapes» whereas M is the space of «unlocated shapes».

If $m \in M$, the set $\pi^{-1}(m) \subset P$ (all those $p \in P$ which project to the fixed $m \in M$) is called the *fibre over m* and here it represents all configurations ($\equiv$ «located shapes») sharing the same («unlocated») shape. Any two fibres $\pi^{-1}(m)$, $\pi^{-1}(m')$ are mutually diffeomorphic (equally looking) and their abstract model, the *typical fibre*, is denoted by $\mathscr{E}$ (the space of the locations of the tie rod) in appendix A and happens to be diffeomorphic to the group $E(2)$ itself.

Notice that the knowledge of the configuration $p \in P$ is equivalent (globally) to the knowledge of the ordered pair $(m, e) \in M \times \mathscr{E}$. In other words, our *total space P* of the bundle is (diffeomorphic to) the product $M \times \mathscr{E}$ of the *base M* and the *typical fibre* $\mathscr{E}$

$$P = M \times \mathscr{E}$$

and the bundle projection π is realized as a projection π_1 on the first factor

$$\pi_1 \colon M \times \mathscr{E} \to M, \qquad\qquad\qquad (m, e) \mapsto m.$$

This means that our bundle is *trivial* (in general this is the case only locally).

The *section* of the bundle $\pi \colon P \to M$ (the *fixation of the gauge*) is a map
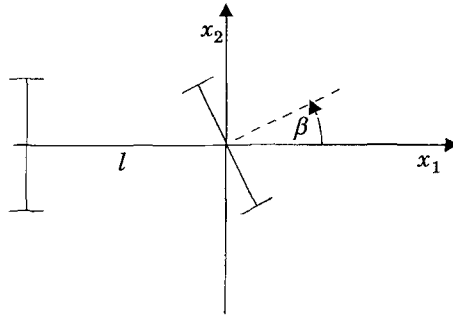
$$\sigma \colon M \to P$$

Fig. 3. – The gauge fixation σ.

obeying

$$\pi \circ \sigma = \text{identity on } M$$

($\sigma(m)$ is to be in the fibre over m). It helps to visualize the abstract shapes (elements of M) localizing each of them somewhere in the (x_1, x_2)-plane. The convenient (global) section is given in coordinates by

(2.1) $\sigma :(\alpha, \beta) \mapsto (\alpha, \beta, 0, 0, 0)$.

It realizes all shapes by means of the configurations with the tie rod situated at the x_1-axis to the left with respect to the origin (fig. 3). Notice that the coordinates (x, y, φ) are closely related (adapted) to this very section (in fact they are introduced just *with respect to this* section): the section defines (for all $m \in M$) the fiducial point $\sigma(m)$ in the fibre over m. This point is (by definition) labelled by the coordinates $(\alpha, \beta, 0, 0, 0)$. Then a general point p in the same fibre (with the same shape) acquires the coordinates $(\alpha, \beta, x, y, \varphi)$ if the element $(B, b) \in E(2)$ with

$$B = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}, \qquad\qquad b = (x, y),$$

is needed to obtain p from $\sigma(m)$ via the group action.

The useful possibility is to interpret the section (2.1) as the point of view of the *driver* (the driver's reference system): with respect to his axes x_1', x_2', the tie rod is clearly always at the origin and directed forward ($x = y = \varphi = 0$). Each other choice of a section (other gauge) corresponds to some different observer, which can, however, depend on the (unlocated) shape.

3. – The no-slipping contact with a road as a connection on $\pi: P \to M$

So far we have come to the conclusion that the 5-dimensional configuration space P of a car can be treated naturally as a total space of a (trivial) principal $E(2)$-bundle $\pi: P \to M$, $P \equiv M \times \mathscr{E}$. The motion of the car on a road ((x_1, x_2)-plane) is given by a curve $\gamma(t) \equiv (m(t), e(t))$ on $P \equiv M \times \mathscr{E}$. The essential point is, however, that it is only the projection $m(t) \equiv \pi \circ \gamma(t)$ which is under *direct* control of the driver ($\alpha(t)$: gas pedal, braces; $\beta(t)$: steering wheel). The driver governs directly the «motion» in the

space of shapes M (the base of the bundle), whereas what is really his goal is the change of the position of the tie rod, or, in other words, to move along the desired curve $e(t)$ in the typical fibre $\mathscr{E}$ of the bundle. The necessary «bridge» between M and $\mathscr{E}$ is given by a system of (anholonomic) *differential constraints* representing physically the condition of the *no-slipping contact* of the wheels with the road. In such a way the driver's activity represented as the curve $m(t)$ on M is transformed into the curve $e(t)$ on $\mathscr{E}$ or, equivalently, $\gamma(t) \equiv (m(t), e(t))$ on P. As we will see, the procedure of the reconstruction of the complete $\gamma(t)$ on P from its projection $m(t)$ on M is just the *horizontal lift* $m \mapsto m^{\mathrm{h}} \equiv \gamma$, where the structure necessary for it, *viz* the *connection* in the principal bundle $\pi: P \to M$ (gauge structure over M), enters the scene as a mathematical expression of the above-mentioned no-slipping contact of the car with the road, *i.e.* the constraints of contact can be interpreted in terms of the *connection form* on P.

In general a connection on a principal fibre bundle $\pi: P \to M$ with a group G is given [7] by a $\mathscr{G}$-valued ($\mathscr{G}$ being the Lie algebra of the group G) 1-form on P, a connection form. In our case it means the 3×3 *matrix* of 1-forms on P decomposable with respect to the basis e_0, e_1, e_2 of the Lie algebra $e(2)$ of the group $E(2)$ (see appendix B)

$$(3.1) \qquad \omega = \omega^a e_a = \omega^0 e_0 + \omega^1 e_1 + \omega^2 e_2 = \begin{pmatrix} 0 & \omega^0 & 0 \\ -\omega^0 & 0 & 0 \\ \omega^1 & \omega^2 & 0 \end{pmatrix},$$

where $\omega^0, \omega^1, \omega^2$ are 1-forms on P. Thus the condition of the horizontality

$$(3.2) \qquad \omega = 0, \qquad i.e. \qquad \omega^0 = \omega^1 = \omega^2 = 0,$$

represents just 3 independent relations between the differentials $\mathrm{d}\alpha$, $\mathrm{d}\beta$, $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}\varphi$ enabling one to express the infinitesimal changes δx, δy, $\delta\varphi$ of the coordinates of the rod in terms of the given changes $\delta\alpha$, $\delta\beta$ of the coordinates of the shape of the car.

Note: the equations $\omega^a = 0$ are not to be interpreted as 1-form identities on P but rather in the sense that the forms are *annihilated* (give zero) by the velocity ($\equiv$ tangent) vectors to the real ($\equiv$ obeying the constraints $\Rightarrow$ by definition horizontal) trajectories on P.

The computation of the explicit expression for the connection form is performed in appendix C. The result reads

$$(3.3) \qquad \begin{cases} \omega^0 = \mathrm{d}\varphi - \dfrac{R}{l}\sin\beta\,\mathrm{d}\alpha, \\[2mm] \omega^1 = \mathrm{d}x + y\omega^0 - R\cos(\beta+\varphi)\,\mathrm{d}\alpha, \\[2mm] \omega^2 = \mathrm{d}y - x\omega^0 - R\sin(\beta+\varphi)\,\mathrm{d}\alpha. \end{cases}$$

If one fixes the gauge by choosing the section σ (sect. 2), the *gauge potential* (in gauge

σ) is given as

$$(3.4) \quad \mathcal{A} := \sigma^* \omega = \mathcal{A}^a e_a = \mathcal{A}^0 e_0 + \mathcal{A}^1 e_1 + \mathcal{A}^2 e_2 = \begin{vmatrix} 0 & \mathcal{A}^0 & 0 \\ -\mathcal{A}^0 & 0 & 0 \\ \mathcal{A}^1 & \mathcal{A}^2 & 0 \end{vmatrix} =$$

$$= -\frac{R}{l} \begin{vmatrix} 0 & \sin\beta & 0 \\ -\sin\beta & 0 & 0 \\ l\cos\beta & l\sin\beta & 0 \end{vmatrix} d\alpha.$$

4. – Reconstruction of $\gamma(t)$ on P from $\pi(\gamma(t))$ on M as a horizontal lift

The driver's activity is represented by a curve $m(t) \equiv \pi(\gamma(t))$ on M (a sequence of shapes parametrized by time). The contact of the wheels with the road then results in a motion in the total configuration space P. According to the meaning of the connection as an object encoding all constraints of the contact, the resulting trajectory $\gamma(t)$ on P is the *horizontal lift* of the *curve* $m(t)$, *i.e.* the unique curve $m^h(t)$ on P enjoying the following two properties:

i) $\pi(m^h(t)) = m(t) \leftrightarrow m^h(t)$ is always exactly «over» $m(t)$,

ii) $\dot{m}^h \equiv$ its tangent (velocity) vector, is always horizontal, *i.e.* it annihilates ω^a, $a = 0, 1, 2$.

Let us express these conditions in coordinates. If

$$m(t) \leftrightarrow (\alpha(t), \beta(t))$$

is given, then its horizontal lift is

$$m^h(t) \leftrightarrow (\alpha(t), \beta(t), x(t), y(t), \varphi(t))$$

(the same α and β are there because of the condition i); x, y, φ are to be determined). Now

$$\dot{m}^h(t) = \dot{\alpha}(t)\,\partial_\alpha + \dot{\beta}(t)\,\partial_\beta + \dot{x}(t)\,\partial_x + \dot{y}(t)\,\partial_y + \dot{\varphi}(t)\,\partial_\varphi$$

and

$$\langle \omega^a, \dot{m}^h(t) \rangle = 0, \qquad\qquad a = 0, 1, 2,$$

gives

$$(4.1) \qquad \begin{cases} \dot{\varphi} = \dot{\alpha}\,\dfrac{R}{l}\sin\beta, \\[2mm] \dot{x} = \dot{\alpha}R\cos(\beta + \varphi), \\[1mm] \dot{y} = \dot{\alpha}R\sin(\beta + \varphi), \end{cases}$$

so that

(4.2) $$\dot{m}^{\mathrm{h}} = \dot{\alpha}(t)\, H_\alpha + \dot{\beta}(t)\, H_\beta\,,$$

where

(4.3) $$H_\alpha \equiv \partial_\alpha^{\mathrm{h}} := \partial_\alpha + R\cos(\beta + \varphi)\, \partial_x + R\sin(\beta + \varphi)\, \partial_y + \frac{R}{l}\sin\beta\, \partial_\varphi\,,$$

(4.4) $$H_\beta \equiv \partial_\beta^{\mathrm{h}} := \partial_\beta$$

are the *horizontal lifts* of the coordinate basis *vectors* on M.

The 1st-order linear autonomous system of equations (4.1), the *parallel-transport equations*, solves the reconstruction problem: given $\alpha(t)$, $\beta(t)$ for $t \in \langle t_i, t_f \rangle$ (sequence of shapes) and $(x(t_i), y(t_i), \varphi(t_i))$ (the initial position in the fibre over $m(t_i)$, *i.e.* the initial position of the car on the road), it provides the remaining information about the motion of the car, *viz* the sequence of the positions of the tie rod corresponding to the given sequence of shapes (an example—the motion with the fixed steering wheel—is computed in appendix D). The parallelly transported configuration is then, by definition, the configuration $m^{\mathrm{h}}(t_f)$. (Recall that according to the meaning of the connection here to follow the parallel-transport rule is the same thing as to be compatible with the constraints of the contact.)

Note that eqs. (4.1) are invariant (as is the case in general for the parallel-transport equations) with respect to reparametrization—the speed of the shape sequence is irrelevant, what matters is only the *path* corresponding to $m(t)$ rather than the *curve* $m(t)$ itself. Surprisingly, this rather subtle technical fact seems to be pretty well known intuitively to our wives when they prevent us to drive too quickly («you win nothing by it»).

5. – Parking cycles as a clever use of the curvature Ω of the connection

To get out of an extremely tight parking spot [6] a pure translation of the tie rod *perpendicular* to the latter, *i.e.* (infinitesimally)

(5.1) $$(x, y, \varphi) \mapsto (x - \varepsilon\sin\varphi,\ y + \varepsilon\cos\varphi,\ \varphi)$$

($\varepsilon \ll 1$), is strongly desirable lest we come to contact with the neighbouring car (and even much worse with the *owner* of the neighbouring car, then).

On the other hand, according to the results of sect. 4 only the motions generated by some horizontal lift $\dot{m}^{\mathrm{h}}$ are possible (allowed by the constraints), *i.e.* (cf. (4.2)-(4.4))

(5.2) $$(x, y, \varphi) \mapsto \left(x + \varepsilon\cos(\varphi + \beta),\ y + \varepsilon\sin(\varphi + \beta),\ \varphi + \frac{\varepsilon}{l}\sin\beta \right)$$

($\varepsilon = \dot{\alpha} R\, \delta t \ll 1$). In no special case (5.2) reduces to (5.1): (5.2) consists of both translation and *rotation* except for the case $\beta = 0$, when, however, the translation is just *along* the tie rod. Thus it seems that we are simply unlucky and we have to wait until the car in front of us leaves.

This conclusion is, however, too hasty, since we have not used yet the basic
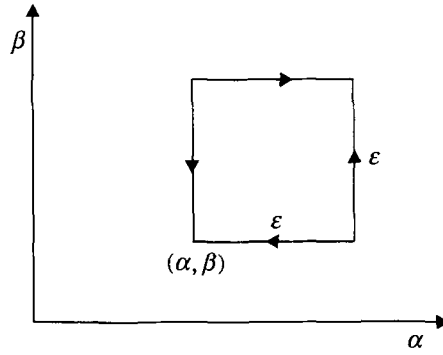
Fig. 4. – A simple infinitesimal parking cycle.

parking algorithm known to every driver, *viz* a *cycle* in the space M. Let us study for a moment the result of an infinitesimal cycle of the following structure:

    i) go forth ($\alpha \mapsto \alpha + \varepsilon$),

    ii) turn the steering wheel to the left ($\beta \mapsto \beta + \varepsilon$),

    iii) go (the same step) back ($\alpha + \varepsilon \mapsto \alpha$),

    iv) turn the steering wheel (the same angle) back to the right ($\beta + \varepsilon \mapsto \beta$)

(see fig. 4). Clearly we finish at (exactly) the same point *in* M; the *complete* configuration $p \equiv (m, e)$, however, changes: $p \equiv (m, e) \mapsto (m, e') \equiv p'$, *viz* up to the *second*-order terms in ε the result is (see below)

$$(5.3) \quad (\alpha, \beta, x, y, \varphi) \mapsto (\alpha, \beta, x - \varepsilon^2 R \sin(\varphi + \beta), y + \varepsilon^2 R \cos(\varphi + \beta), \varphi + \varepsilon^2 \frac{R}{l} \cos\beta).$$

Although this does not meet our requirements yet (rotation is present unless $\beta = \pi/2$; if β is $\pi/2$, the translation is once more *along* the tie rod), there is still something interesting here which will turn out to be the essential clue for the real solution of the parking problem. Namely, if one interprets (5.3) as a *direct* step from p to p' (and not as the *effective* one $\equiv$ the result of the cyclic motion described above), it is *forbidden* (it violates the constraints). This particular step is even forbidden in a «maximal way»—it is purely *vertical* (projects to the same point in M; both p and p' lie in the same fibre). Thus the fact of vital importance is that a cycle composed exclusively of allowed ($\equiv$ horizontal) steps can result in the (directly) forbidden motion ($\equiv$ non-vanishing vertical part). This means that, although we have come to the conclusion that (5.1) cannot be realized «directly» (as one step), there is still a real hope to produce it effectively—as a result of (maybe rather involved) cycle of allowed «simple steps».

    The most convenient tool for studying the effect of cycles is the language of vector fields (appendix E). The cycle i)-iv) above is just the infinitesimal cycle generated by H_α, H_β (on P; its projection to M is generated by ∂_α, ∂_β and the corresponding loop closes exactly since they commute); then the resulting motion (5.3) follows from the formula (see (E.1))

$$(5.4) \qquad\qquad \chi_{-\varepsilon}^{H_\beta} \circ \chi_{-\varepsilon}^{H_\alpha} \circ \chi_{\varepsilon}^{H_\beta} \circ \chi_{\varepsilon}^{H_\alpha} = \chi_{-\varepsilon^2}^{[H_\alpha, H_\beta]}$$

and the explicit computation of the needed commutator:

$$(5.5) \qquad [H_\alpha, H_\beta] = R \left\{ \sin(\varphi + \beta)\partial_x - \cos(\varphi + \beta)\partial_y - \frac{1}{l}\cos\beta\partial_\varphi \right\}.$$

Since (5.3) is not enough, we can try the «higher» (iterated) commutators. There are two of them to be computed and the results are

$$(5.6) \qquad [H_\alpha, [H_\alpha, H_\beta]] = \frac{R^2}{l}(\cos\varphi\partial_y - \sin\varphi\partial_x),$$

$$(5.7) \qquad [H_\beta, [H_\alpha, H_\beta]] = H_\alpha - \partial_\alpha \equiv \partial_\alpha^h - \partial_\alpha.$$

Both of these results deserve some attention. First, note that the r.h.s of (5.6) just generates the wanted motion (5.1)! In more detail, the identity (see (E.3))

$$(5.8) \qquad \chi_{-\varepsilon}^{H_\beta} \circ \chi_{-\varepsilon}^{H_\alpha} \circ \chi_{\varepsilon}^{H_\beta} \circ \chi_{\varepsilon}^{H_\alpha} \circ \chi_{\varepsilon^2}^{H_\alpha} \circ \chi_{-\varepsilon}^{H_\alpha} \circ \chi_{-\varepsilon}^{H_\beta} \circ \chi_{\varepsilon}^{H_\alpha} \circ \chi_{\varepsilon}^{H_\beta} \circ \chi_{\varepsilon^2}^{H_\alpha} = \chi_{-\varepsilon^4}^{[H_\alpha,[H_\alpha, H_\beta]]}$$

tells us that the iterated cycle standing on the l.h.s. of (5.8) (try to draw a picture by appropriately modifying fig. 4!) results in

$$(5.9) \qquad (x, y, \varphi) \mapsto \left(x + \varepsilon^4 \frac{R^2}{l}\sin\varphi, \ y - \varepsilon^4 \frac{R^2}{l}\cos\varphi, \ \varphi \right),$$

which is just the pure translation perpendicular to the tie rod. Note that this type of motion is *very* slow and laborious: it is necessary to perform ten «simple» steps (of the order ε or ε^2) to produce effectively a single step (which is of the order ε^4) in the «right» direction.

The same treatment applied to (5.7) leads to the identity

$$(5.10) \qquad \chi_{-\varepsilon}^{H_\beta} \circ \chi_{-\varepsilon}^{H_\alpha} \circ \chi_{\varepsilon}^{H_\beta} \circ \chi_{\varepsilon}^{H_\alpha} \circ \chi_{-\varepsilon^2}^{H_\beta} \circ \chi_{-\varepsilon}^{H_\alpha} \circ \chi_{-\varepsilon}^{H_\beta} \circ \chi_{\varepsilon}^{H_\alpha} \circ \chi_{\varepsilon}^{H_\beta} \circ \chi_{\varepsilon^2}^{H_\beta} = \chi_{-\varepsilon^4}^{[H_\beta,[H_\alpha, H_\beta]]},$$

which shows that performing the (iterated) cycle standing on the l.h.s. of (5.10) the car moves just as if the driver simply moved forth, but the front wheel did not rotate (no change of α at all: ideally slipping contact—ice on the road).

As we will see in appendix E, the possibility of producing «forbidden» motions by means of the cycles composed of «allowed» steps leans heavily on the fact that the *curvature* of the connection in question does not vanish (↔ the horizontal lifts of coordinate basis vectors do not commute). The (Lie-algebra–valued) *curvature* 2-*form*, which happens [7] to be the measure of this non-commutation, can be easily computed explicitly (using the formula (E.8)) here and the result reads

$$(5.11) \qquad \Omega \equiv \mathrm{hor}\, d\omega = \Omega^a e_a = \Omega^0 e_0 + \Omega^1 e_1 + \Omega^2 e_2 = \begin{pmatrix} 0 & \Omega^0 & 0 \\ -\Omega^0 & 0 & 0 \\ \Omega^1 & \Omega^2 & 0 \end{pmatrix},$$

where

(5.12)
$$
\begin{cases}
\Omega^0 = \dfrac{R}{l}\cos\beta\, \mathrm{d}\alpha \wedge \mathrm{d}\beta\,, \\[2ex]
\Omega^1 = \left\{ -R\sin(\beta+\varphi) + \dfrac{R}{l}\, y\cos\beta \right\} \mathrm{d}\alpha \wedge \mathrm{d}\beta\,, \\[2ex]
\Omega^2 = \left\{ R\cos(\beta+\varphi) - \dfrac{R}{l}\, x\cos\beta \right\} \mathrm{d}\alpha \wedge \mathrm{d}\beta\,.
\end{cases}
$$

If one fixes the gauge by choosing the section σ (sect. 2), the *field strength* (in gauge σ) is given as

(5.13) $\mathscr{F} := \sigma^*\Omega = \mathscr{F}^a e_a = \mathscr{F}^0 e_0 + \mathscr{F}^1 e_1 + \mathscr{F}^2 e_2 = \begin{pmatrix} 0 & \mathscr{F}^0 & 0 \\ -\mathscr{F}^0 & 0 & 0 \\ \mathscr{F}^1 & \mathscr{F}^2 & 0 \end{pmatrix} =$

$$
= \frac{R}{l} \begin{pmatrix} 0 & \cos\beta & 0 \\ -\cos\beta & 0 & 0 \\ \sin\beta & l\cos\beta & 0 \end{pmatrix} \mathrm{d}\alpha \wedge \mathrm{d}\beta\,.
$$

6. – Particle fields

The gauge potentials $\mathscr{A} \equiv \sigma^*\omega$ (and the field strengths $\mathscr{F} \equiv \sigma^*\Omega$) do not exhaust all the building blocks of the gauge theory of elementary particles—there are also *particle fields* there: particles interact via gauge fields (bosons).

In our model of the kinematics of a car we used only the «connection part of the theory» yet. The question arises whether there is an object here which is described mathematically by a particle field and whether some standard computation with it does make sense in this context.

If V is a vector space in which a representation ϱ acts, then [7] a particle field of type ϱ is a V-valued function on P which transforms according to the representation ϱ with respect to the action of G on P; in our model it means

(6.1) $\psi : P \to V$

such that

(6.2) $\psi(R_{\mathscr{B}}p) = \varrho(\mathscr{B}^{-1})\psi(p)\,.$

Here we give a simple example of such ψ. Let $V = \mathbf{R}^2$ and define the function ψ on P

(6.3) $\psi : (a, \beta, x, y, \varphi) \mapsto \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$

Then $\psi(p)$ just gives the components of the unit vector $\boldsymbol{e}$ fixed on the car and directed

along the tie rod (fig. 1). According to appendix A, the action of $E(2)$ has the explicit form

$$(6.4) \quad (\alpha, \beta, x, y, \varphi) \mapsto R_{\mathscr{B}}(\alpha, \beta, x, y, \varphi) \equiv$$

$$\equiv (\alpha, \beta, x \cos \Theta - y \sin \Theta + b_1, x \sin \Theta + y \cos \Theta + b_2, \varphi + \Theta)$$

and thus, if

$$(6.5) \qquad\qquad \psi(p) \leftrightarrow \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix},$$

then

$$(6.6) \quad \psi(R_{\mathscr{B}}p) \leftrightarrow \begin{pmatrix} \cos(\varphi + \Theta) \\ \sin(\varphi + \Theta) \end{pmatrix} = \begin{pmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} =: \varrho(\mathscr{B}^{-1}) \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix},$$

where the representation ϱ of $E(2)$ in $V \equiv \mathbf{R}^2$ is given by

$$(6.7) \qquad\qquad \varrho(\mathscr{B}) \equiv \varrho((B(\Theta), b)) = \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix}.$$

Thus our ψ is the particle field of type ϱ given by (6.7).

Let the motion in M be given by $m(t) \leftrightarrow (\alpha(t), \beta(t))$. Then the change δe of the vector e between t and $t + \delta t$ can be computed as

$$(6.8) \qquad\qquad \delta e \leftrightarrow \delta t \dot{m}^h \psi = \frac{\delta t \dot{\alpha} R}{l} \sin \beta \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}.$$

Since $\begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}$ is just the unit vector orthogonal to e, the net angle of rotation of e is

$$(6.9) \qquad\qquad \frac{\delta t \dot{\alpha} R}{l} \sin \beta \equiv \frac{\delta \alpha R}{l} \sin \beta,$$

which can be checked by inspection of fig. 1. The same angle can be computed within the gauge fixation σ, too, making use of the *covariant derivative* of

$$\Phi := \sigma^* \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow e_1,$$

viz

$$\delta \Phi = \delta t \nabla_{\dot{m}} (\sigma^* \psi) = \delta t \langle \sigma^* D\psi, \dot{m} \rangle = \frac{\delta \alpha R}{l} \sin \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow \frac{\delta \alpha R}{l} \sin \beta e_2$$

or

$$\delta \boldsymbol{e}_1 = \frac{\delta a R}{l} \sin \beta \boldsymbol{e}_2$$

in agreement with (6.9).

7. – Conclusions and comments

In this paper we have presented in some detail a gauge-theoretic approach to the kinematics of the motion of a car. It can serve as one more example of application of the ideas and techniques of the mathematics of gauge fields and related structures within the rather mundane context of the elementary (classical) mechanics (as opposed to their standard occurrence in «noble» = «fundamental» physics).

The formal scheme is the same here as in [1] or [2]: there is a «total» configuration space (P here ↔ the space of located shapes in [1] ↔ X in [2]) which happens to carry the structure of the total space of the principal fibre bundle. The group G acts there ($E(2)$ here ↔ $SO(3)$ in [1] ↔ $SO(d)$ in [2]) and the space of orbits of this action (M here ↔ the space of unlocated shapes in [1] ↔ $\bar{X}$ in [2]), the base of the bundle, represents the «directly controllable part» of the total configuration space. The connection in $\pi: P \to M$ provides the bridge linking the motions in these two spaces.

The main difference lies in the *physical origin* of the connection in question: here (and also in [4]) it encodes the constraints expressing the no-slipping (direct) *contact* of the car with the road, whereas in [1,2] and [3] it results from the conservation laws (of the linear as well as the angular momentum) in a «*nothing to push against*» situation, *i.e.* the constraints enter the problem dynamically.

APPENDIX A

The action of $E(2)$ on $\mathcal{E}$ and on P

Let $B \in SO(2)$, $b \equiv (b_1, b_2) \in \boldsymbol{R}^2$. Then one can define the transformation of the points $\chi \equiv (x_1, x_2) \in \boldsymbol{R}^2$ by the couple (B, b) by

(A.1) $$\chi \mapsto \chi B + b =: \mathcal{R}_{(B, b)} \chi .$$

Geometrically it represents the rotation by Θ around the origin $\left(\text{if } B = \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix} \right)$ followed by the translation by (b_1, b_2), *i.e.* the *Euclidean* transformation of χ by $(B, b) \in E(2)$. The rule (A.1) can be written in purely matrix form (which is advantageous for manipulations with the gauge potentials) using the following standard trick: let us associate the 3×3 matrix $\mathcal{B}$ and the row vector η with the couple (B, b) and the row vector χ, respectively, according to

(A.2) $$\mathcal{B} := \begin{pmatrix} B & 0 \\ b & 1 \end{pmatrix}, \qquad \eta := (\chi, 1) \equiv (x_1, x_2, 1).$$

Then the matrix multiplication of η by $\mathscr{B}$ gives

(A.3) $$\eta\mathscr{B} = \eta' = (\chi B + b, 1) \equiv (\mathscr{R}_{(B, b)}\chi, 1),$$

i.e. the rule (A.1) is reproduced from the matrix multiplication of the auxiliary quantities η and $\mathscr{B}$. The action given by (A.1) or (A.3) transforms the (x_1, x_2)-plane «rigidly», *i.e.* all distances are preserved ($\mathscr{R}_\mathscr{B} \equiv \mathscr{R}_{(B, b)}$ is an isometry). It then enables to define the action $\hat{R}_\mathscr{B}$ of $E(2)$ on the space $\mathscr{E}$ of the locations of the tie rod, by simply transforming both endpoints by $\mathscr{R}_\mathscr{B}$. If the coordinates (x, y, φ) are introduced to $\mathscr{E}$ according to fig. 1, one obtains

(A.4) $\quad (x, y, \varphi) \mapsto \hat{R}_\mathscr{B}(x, y, \varphi) \equiv$

$$\equiv (x\cos\Theta - y\sin\Theta + b_1, x\sin\Theta + y\cos\Theta + b_2, \varphi + \Theta).$$

Notice that the general position (x, y, φ) of the rod can be reached from the reference one $(0, 0, 0)$ (the rod being situated on the x_1-axis left to the origin) by means of the unique $\hat{R}_\mathscr{B}$:

(A.5) $\quad \hat{R}_\mathscr{B}(0, 0, 0) = (x, y, \varphi) \quad$ for $\quad B = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad b = (x, y).$

This means that the action $\hat{R}_\mathscr{B}$ is transitive and free and thus $\mathscr{E}$ is the «principal $E(2)$-space». (A.5) gives the diffeomorphism of $\mathscr{E}$ and the group $E(2)$ itself, too.

Finally the action $R_\mathscr{B}$ on $P = M \times \mathscr{E}$ is given by

$$(m, e) \mapsto (m, \hat{R}_\mathscr{B}e) =: R_\mathscr{B}(m, e)$$

or, in coordinates,

(A.6) $\quad (\alpha, \beta, x, y, \varphi) \mapsto R_\mathscr{B}(\alpha, \beta, x, y, \varphi) \equiv$

$$\equiv (\alpha, \beta, x\cos\Theta - y\sin\Theta + b_1, x\sin\Theta + y\cos\Theta + b_2, \varphi + \Theta).$$

APPENDIX B

The Lie algebra $e(2)$ of the group $E(2)$

According to appendix A, the group $E(2)$ can be realized by the matrices $\mathscr{B} = \begin{pmatrix} B & 0 \\ b & 1 \end{pmatrix}$, where $B \in SO(2)$. By definition, the Lie algebra $e(2)$ then consists of all 3×3 matrices $\mathscr{C}$ such that $1 + \varepsilon\mathscr{C} \equiv \mathscr{B}(\varepsilon) \in E(2)$ when the 2nd-order terms in ε are neglected. This leads to $\mathscr{C} = \begin{pmatrix} C & 0 \\ c & 0 \end{pmatrix}$ with the additional restriction (coming from $B^T B = 1$) $C^T = -C$, or, explicitly,

$$\mathscr{C} = \begin{pmatrix} 0 & \lambda_0 & 0 \\ -\lambda_0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 \end{pmatrix}, \qquad \lambda_0, \lambda_1, \lambda_2 \in \mathbf{R}.$$

The matrices

$$e_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

can then serve as the basis of $e(2)$; their commutation relations read

(B.1)
$$\begin{cases} [e_0, e_1] = -e_2, \\ [e_0, e_2] = +e_1, \\ [e_1, e_2] = 0 \end{cases}$$

and so the only non-zero structure constants are

(B.2)
$$c_{10}^2 = -c_{01}^2 = c_{02}^1 = -c_{20}^1 = 1.$$

APPENDIX C

A computation of the connection form ω

In general, a connection form can be written as follows:

(C.1)
$$\omega = \mathscr{B}^{-1}\overline{\omega}\mathscr{B} + \mathscr{B}^{-1}d\mathscr{B},$$

where $\overline{\omega} \equiv \overline{\omega}^a e_a$ is some (yet unknown) $e(2)$-valued 1-form *on* M and

(C.2)
$$\mathscr{B} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ x & y & 1 \end{pmatrix} \in E(2).$$

The form ω defines the horizontal directions (the relations between $d\alpha$, $d\beta$, dx, dy and $d\varphi$ as a result of the no-slipping contact of the wheels with the road) by the equations $\omega^a = 0$, $a = 0, 1, 2$. In particular, at the points of the section $\sigma(M) \subset P$, corresponding to the «standard» position of a car (*i.e.* for $x = y = \varphi = 0$; cf. sect. 2) one has $\mathscr{B} = 1 = \mathscr{B}^{-1}$ and

(C.3)
$$\omega_{\mathscr{B}=1} = \overline{\omega} + (d\mathscr{B})_{\mathscr{B}=1}.$$

Thus

(C.4)
$$\omega_{\mathscr{B}=1} = \begin{pmatrix} 0 & \overline{\omega}^0 & 0 \\ -\overline{\omega}^0 & 0 & 0 \\ \overline{\omega}^1 & \overline{\omega}^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & d\varphi & 0 \\ -d\varphi & 0 & 0 \\ dx & dy & 0 \end{pmatrix} =$$

$$= (d\varphi + \overline{\omega}^0)e_0 + (dx + \overline{\omega}^1)e_1 + (dy + \overline{\omega}^2)e_2.$$

The equation $\omega_{\mathscr{B}=1} = 0$ by definition singles out the horizontal directions for

$x = y = \varphi = 0$; one has

(C.5) $$d\varphi = -\overline{\omega}^0, \qquad dx = -\overline{\omega}^1, \qquad dy = -\overline{\omega}^2.$$

On the other hand, the no-slipping contact constraints for the standard position $x = y = \varphi = 0$ can be easily read out from fig. 3: if $\alpha \mapsto \alpha + \delta\alpha$ $(\delta\alpha \ll 1)$, then $(x, y) \equiv (0, 0) \mapsto (\delta\alpha R \cos\beta, \delta\alpha R \sin\beta) \equiv (\delta x, \delta y)$, $(-l, 0) \mapsto (-l + \delta\alpha R \cos\beta, 0) \Rightarrow$ $\Rightarrow \varphi \mapsto \varphi + \delta\alpha (R/l) \sin\beta$; if $\beta \mapsto \beta + \delta\beta$ $(\delta\beta \ll 1)$, then $(x, y, \varphi) \mapsto (x, y, \varphi)$. Thus

(C.6) $$d\varphi = \frac{R}{l} \sin\beta \, d\alpha, \qquad dx = R \cos\beta \, d\alpha, \qquad dy = R \sin\beta \, d\alpha.$$

A comparison with (C.5) gives

(C.7) $$\overline{\omega}^0 = -\frac{R}{l} \sin\beta \, d\alpha, \qquad \overline{\omega}^1 = -R \cos\beta \, d\alpha, \qquad \omega^2 = -R \sin\beta \, d\alpha,$$

(C.8) $$\overline{\omega} \equiv \overline{\omega}^a \, e_a = -\frac{R}{l} \begin{pmatrix} 0 & \sin\beta & 0 \\ -\sin\beta & 0 & 0 \\ l\cos\beta & l\sin\beta & 0 \end{pmatrix} d\alpha.$$

Inserting this into (C.1) leads finally to

$$\omega = \omega^0 e_0 + \omega^1 e_1 + \omega^2 e_2,$$

where

(C.9) $$\begin{cases} \omega^0 = d\varphi - \dfrac{R}{l} \sin\beta \, d\alpha, \\[2mm] \omega^1 = dx + y\omega^0 - R \cos(\beta + \varphi) \, d\alpha, \\[2mm] \omega^2 = dy - x\omega^0 - R \sin(\beta + \varphi) \, d\alpha. \end{cases}$$

Thus the differential constraints in *general* configuration are $(\omega^a = 0)$

(C.10) $$\begin{cases} d\varphi = \dfrac{R}{l} \sin\beta \, d\alpha, \\[2mm] dx = R \cos(\beta + \varphi) \, d\alpha, \\[2mm] dy = R \sin(\beta + \varphi) \, d\alpha. \end{cases}$$

Notice the absence of the differential $d\beta$ on the r.h.s.—it reflects the evident fact that turning the steering wheel alone results in no motion of the tie rod.

APPENDIX D

Motion of the car with fixed steering wheel

In the case of a fixed steering wheel $(\beta(t) = \beta_0 = \text{const})$ the parallel-transport

equations (4.1) read

$$\begin{cases} \varphi'(\alpha) = \dfrac{R}{l}\sin\beta_0, \\[2mm] x'(\alpha) = R\cos(\beta_0 + \varphi(\alpha)), \\[2mm] y'(\alpha) = R\sin(\beta_0 + \varphi(\alpha)) \end{cases}$$

(D.1)

$(\varphi'(\alpha) \equiv \mathrm{d}\varphi/\mathrm{d}\alpha, \ldots)$. They are easily solved. If $\beta_0 \neq 0$, then

$$\varphi(\alpha) = \varphi_0 + \alpha\frac{R}{l}\sin\beta_0,$$

$$x(\alpha) = x_0 + \frac{l}{\sin\beta_0}(\sin(\varphi(\alpha) + \beta_0)) - \sin(\varphi_0 + \beta_0)),$$

$$y(\alpha) = y_0 - \frac{l}{\sin\beta_0}(\cos(\varphi(\alpha) + \beta_0) - \cos(\varphi_0 + \beta_0))$$

and, consequently,

$$(x(\alpha) - x_c)^2 + (y(\alpha) - y_c)^2 = r_c^2,$$

where

$$r_c \equiv \frac{l}{\sin\beta_0},$$

$$x_c \equiv x_0 - r_c\sin(\varphi_0 + \beta_0),$$

$$y_c \equiv x_0 + r_c\cos(\varphi_0 + \beta_0).$$

Thus, as expected, the front wheel draws a *circle* with radius r_c and centre (x_c, y_c). If $\beta_0 = 0$, eqs. (D.1) give

$$\varphi(\alpha) = \varphi_0,$$

$$x(\alpha) = x_0 + \alpha R\cos\varphi_0,$$

$$y(\alpha) = y_0 + \alpha R\sin\varphi_0,$$

which is a *straight line* in the direction of the tie rod.

APPENDIX E

Commutators, infinitesimal cycles and the curvature

Let U, V be two vector fields on a manifold $\mathcal{M}$, $[U, V]$ their commutator (Lie bracket) and χ_t^U, χ_t^V and $\chi_t^{[U, V]}$ the corresponding flows (χ_t^U is the map $\mathcal{M} \to \mathcal{M}$ sending each point $x \in \mathcal{M}$ a (parameter) distance t along the integral curve of U; $\chi_{(t+s)}^U = \chi_t^U \circ \chi_s^U = \chi_s^U \circ \chi_t^U$) holds. Then a computation shows that up to the *second-*

order terms in $\varepsilon \ll 1$ the following important identity is valid:

(E.1)
$$\chi^V_{-\varepsilon} \circ \chi^U_{-\varepsilon} \circ \chi^V_{\varepsilon} \circ \chi^U_{\varepsilon} = \chi^{[U,V]}_{-\varepsilon^2}$$

or, equivalently,

(E.2)
$$\chi^{[U,V]}_{\varepsilon^2} \circ \chi^V_{-\varepsilon} \circ \chi^U_{-\varepsilon} \circ \chi^V_{\varepsilon} \circ \chi^U_{\varepsilon} = \text{identity on } \mathscr{M} .$$

From these formulae one deduces the standard interpretation of the commutator of two vector fields: the infinitesimal cycle generated by U and V (l.h.s. of (E.1)) does not end at the original point within the accuracy ε^2 (although it *does* within the accuracy ε), but rather one has to add a one-order-smaller step along $[U, V]$ to close the loop (l.h.s. of (E.2)). Now if V is itself a commutator, $V = [W, Z]$, the twofold use of (E.1) yields

(E.3)
$$\chi^Z_{-\varepsilon} \circ \chi^W_{-\varepsilon} \circ \chi^Z_{\varepsilon} \circ \chi^W_{\varepsilon} \circ \chi^U_{-\varepsilon^2} \circ \chi^W_{-\varepsilon} \circ \chi^Z_{-\varepsilon} \circ \chi^W_{\varepsilon} \circ \chi^Z_{\varepsilon} \circ \chi^U_{\varepsilon^2} = \chi^{[U,[W,Z]]}_{-\varepsilon^4} .$$

Thus the computation of «simple» ($[U, V]$) and iterated ($[U,[W, Z]]$) commutators tells us what is the result of a simple (4 steps) and iterated (4 steps, but two of them being themselves results of 4 steps, *i.e.* together 10 simple steps) cycles, respectively (the higher iterated commutators can be treated in the same way).

    All that was said until now is valid for any vector fields on any manifold. In the case when the vector fields in question are horizontal lifts, the resulting commutator can be expressed in terms of the *curvature* of the connection. For doing this we need first the concept of the *fundamental fields* of the action $R_{\mathscr{G}}$. By definition the field $\xi_{\mathscr{C}}$, $\mathscr{C} \equiv \mathscr{C}^a e_a \in e(2)$, generates the motion of any point p under the action of the one-parameter subgroup $\mathscr{B}(\lambda) = \exp[\lambda \mathscr{C}]$, *i.e.*, for $p(\lambda) := R_{\mathscr{B}(\lambda)}p$,

(E.4)
$$\xi_{\mathscr{C}}(p) := \dot{p}(0).$$

For the basis elements $e_0, e_1, e_2 \in e(2)$ we obtain explicitly

(E.5)
$$\begin{cases} \xi_0 \equiv \xi_{e_0} = -y\partial_x + x\partial_y + \partial_\varphi, \\ \xi_1 \equiv \xi_{e_1} = \partial_x, \\ \xi_2 \equiv \xi_{e_2} = \partial_y \end{cases}$$

and, in general,

(E.6)
$$\xi_{\mathscr{C}} \equiv \xi_{\mathscr{C}^a e_a} = \mathscr{C}^a \xi_{e_a} \equiv \mathscr{C}^a \xi_a .$$

These fields are purely *vertical* (directed along the fibre), since (by definition) the action is vertical (p and $R_{\mathscr{B}}p$ lie in the same fibre for all $p, \mathscr{B}$). Now the relevant formula for the commutator of H_α and H_β is

(E.7)
$$[H_\alpha, H_\beta] = \xi_{-\Omega(H_\alpha, H_\beta)} = -\Omega^a(H_\alpha, H_\beta)\xi_a ,$$

where the ($e(2)$-valued) *curvature 2-form* $\Omega = \Omega^a e_a$ is given by

(E.8)
$$\Omega^a = d\omega^a + \frac{1}{2}c^a_{bc}\omega^b \wedge \omega^c$$

(c^a_{bc} being the structure constants; they are computed in appendix B). The formula (E.7) shows that

i) $[H_\alpha, H_\beta]$ is non-zero if and only if Ω is non-zero,

ii) $[H_\alpha, H_\beta]$ is purely vertical $\Rightarrow$ the corresponding cycle generates «forbidden» motion.

The explicit form of Ω in our case is displayed in sect. 5 (see (5.11), (5.12)).

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