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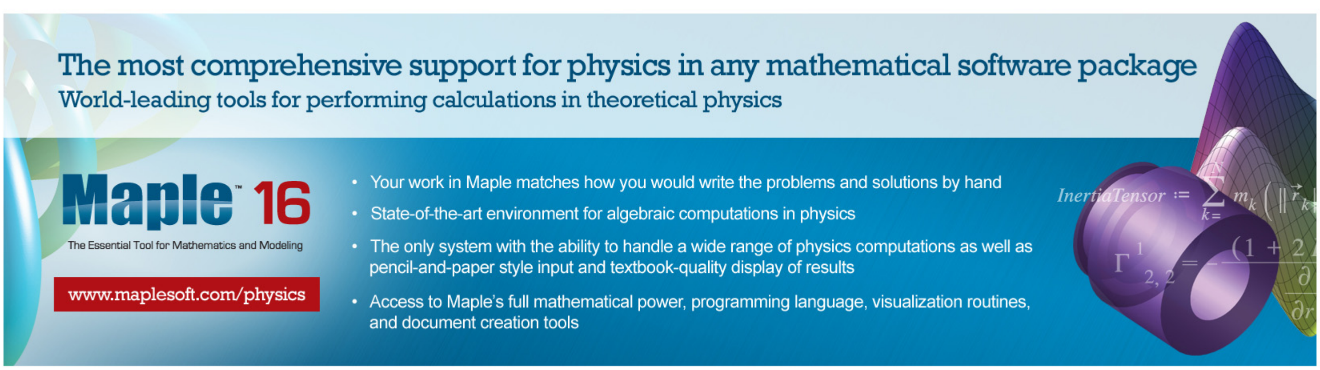

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On a geometrical formulation of the Nambu dynamics

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A geometrical formulation of the N-triplet Nambu dynamics is given. It enables one to treat some general issues of the latter, including integral invariants and canonical transformations in a simple way. A possible underlying geometrical structure is discussed.

I. INTRODUCTION

In Ref. 1 Nambu presented an interesting modification of the classical Hamiltonian dynamics. In his approach a three-dimensional phase space with the coordinates $\mathbf{r} \equiv (x,y,z)$ is introduced ("Nambu triplet"), with the dynamics given by the system of first-order differential equations

$$\dot{\mathbf{r}} = \nabla H \times \nabla G, \tag{1}$$

$H(\mathbf{r}),G(\mathbf{r})$ being arbitrary functions on phase space. Equivalently, one can write

$$\dot{f} = \frac{\partial(f,H,G)}{\partial(x,y,z)} + \frac{\partial f}{\partial t} \tag{2}$$

for any function on the "extended" phase space $f(\mathbf{r},t)$. Thus the time evolution is governed by two "Hamiltonians," $H,G,$ as opposed to a single one, H, in the Hamiltonian case. He showed that the Euler equations for a free rigid rotator take just this form if one identifies (x,y,z) with the body-system projections of the angular momentum and H and G with the two conserved quantities available in this case, *viz.*, the energy and the square of the angular momentum. He demonstrated that the dynamical systems (see Ref. 2 for some other examples) given by (1) share some important properties with the Hamiltonian ones, in spite of being obviously non-Hamiltonian (the dimension of the phase space is odd). In particular, the Liouville theorem holds (such "Liouville" systems were studied later in Ref. [3]). He also proposed the natural generalization to more "triplets":

$$\dot{\mathbf{r}}_a = \nabla_a H \times \nabla_a G, \quad a=1,\ldots,N, \tag{3}$$

or, equivalently,

$$\dot{f} = \sum_{a=1}^{N} \frac{\partial(f,H,G)}{\partial(x_a,y_a,z_a)} + \frac{\partial f}{\partial t}, \quad f=f(\mathbf{r}_1,\ldots,\mathbf{r}_N,t). \tag{4}$$

Later on, several different aspects of the dynamics were studied by many authors.[4]

In this paper, we focus our attention on a geometrical formulation of Eqs. (3), since it enables one to study the more advanced issues of Nambu dynamics in a simple way.

In Ref. 5 (published almost immediately after Ref. 1), the first attempt in this direction was made. As is well known,[6] the geometrical version of the Hamiltonian equations reads (for the autonomous system)

$$\Gamma \lrcorner\, \omega_s = -dH, \tag{5}$$

where ω_s is symplectic form and Γ is the dynamical vector field. The standard form

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a} \tag{6}$$

is obtained using the canonical coordinates q^a,p_a (guaranteed via the Darboux theorem) in which

$$\omega_s = dp_1 \wedge dq^1 + \cdots + dp_N \wedge dq^N. \tag{7}$$

In full analogy, the three-form

$$\omega = dx_1^1 \wedge dx_1^2 \wedge dx_1^3 + \cdots + dx_N^1 \wedge dx_N^2 \wedge dx_N^3 \tag{8}$$

is introduced [it is closed and has maximum rank just as does (7)] in Ref. 5 and the equation

$$\Gamma \lrcorner\, \omega = dH \wedge dG \tag{9}$$

is claimed to be equivalent to (3) and thus to represent its geometrical version.

The first hint that it cannot be the case comes from the observation (made in Ref. 5) that the Liouville theorem does not hold for $N>1$ for (9). Indeed, although

$$\pounds_\Gamma \omega = 0 \tag{10}$$

is a simple consequence of (9), the $3N$-form $\omega \wedge \cdots \wedge \omega$ [the only candidate for a volume form constructed from the elements occurring in (9)] vanishes identically (because of the properties of $\wedge$). This means that all but the lowest "Poincaré–Cartan" integral invariants vanish identically. In particular, there is no Γ-invariant volume

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form for (9); i.e., the Liouville theorem does not hold (for $N>1$) for (9).

This is, however, in contradiction to the opposite assertion [concerning, however, (3)] in Ref. 1, and one can simply verify by "traditional" means that the dynamical field in (3) actually is divergenceless with respect to the natural volume form [cf. (27)]

$$\Omega\equiv dx_1^1\wedge\cdots\wedge dx_N^3. \tag{11}$$

Indeed, if $\boldsymbol{\Gamma}\equiv\boldsymbol{\Gamma}_1+\cdots+\boldsymbol{\Gamma}_N$, $\boldsymbol{\Gamma}_a\equiv\boldsymbol{\nabla}_aH\times\boldsymbol{\nabla}_aG$, then

$$\operatorname{div}_\Omega\boldsymbol{\Gamma}=\boldsymbol{\nabla}_1\cdot\boldsymbol{\Gamma}_1+\cdots+\boldsymbol{\nabla}_N\cdot\boldsymbol{\Gamma}_N=0, \tag{12}$$

and the Liouville theorem does hold [cf. (29)] in agreement with Ref. 1. This indicates that (9) cannot represent the geometrical version of (3) for $N>1$ and it has to be changed (in the next section, we show that it is even inconsistent).

II. A GEOMETRICAL FORMULATION OF EQ. (3)

The three-form ω at a fixed point P can be interpreted as a linear map:

$$\omega_P: T_P\to\Lambda^2T_P^*,\quad v\mapsto v\lrcorner\,\omega_P \tag{13}$$

(T_P is the tangent space in P, $\Lambda^2T_P^*$ is the space of two-forms in P). Since $\dim T_P<\dim\Lambda^2T_P^*$ (for $N>1$), the two-form field $\Gamma\lrcorner\,\omega$ in (9) is not "general" and cannot be therefore equated to $dH\wedge dG$ without some additional restrictions on H and G. There are, however, no restrictions on H,G in (3), or, in the coordinates used in (8), $\Gamma\lrcorner\,\omega$ does not contain products $dx_a^i\wedge dx_b^j$ if $a\neq b$, while $dH\wedge dG$, in general, does. This shows that (9) is not only unequivalent to (3) for more than one triplet but is then even inconsistent for general H,G.

Let us define "partial" (ath triplet) exterior derivative operators d_a, $a=1,\dots,N$, operating in the following way: If

$$\sigma=\sigma_{j_1\cdots j_p}dx^{j_1}\wedge,\quad\cdots\wedge dx^{j_p}\equiv\sigma_J\,dx^J$$

is any p-form, then let (the summation convention with respect to i from 1–3 is understood but is in abeyance with respect to the index "a" in what follows)

$$d_a\sigma:=\frac{\partial\sigma_J}{\partial x_a^i}dx_a^i\wedge dx^J. \tag{14}$$

Then,

$$d=d_1+\cdots+d_N \tag{15}$$

and

$$d_a\,d_b=-d_bd_a; \tag{16}$$

in particular,

$$d_a^2=0. \tag{17}$$

In addition, let us define N three-forms as

$$\omega_a:=dx_a^1\wedge dx_a^2\wedge dx_a^3. \tag{18}$$

Then, a simple calculation shows that

$$\Gamma\lrcorner\,\omega_a=d_aH\wedge d_aG,\quad a=1,\dots,N \tag{19}$$

is equivalent to (3) and therefore represents a geometrical formulation of the N-triplet Nambu equations.

III. SOME PROPERTIES OF THE NAMBU DYNAMICS

In this section we use the formulation (19) of the Nambu dynamics to analyze some of its more advanced issues.

A. Integral invariants

From the Hamiltonian equations, it follows immediately that

$$\begin{aligned}&\pounds_\Gamma\omega_s=0\\&\vdots\\&\pounds_\Gamma(\omega_s\wedge\cdots\wedge\omega_s)\equiv\pounds_\Gamma\omega_s^n=0,\end{aligned} \tag{20}$$

which results in the series of the Poincaré–Cartan (absolute) integral invariants:

$$I_{2k}\equiv\int_{M^{2k}}\omega_s^k,\quad k=1,\dots,n, \tag{21}$$

M^{2k} being any $2k$-dimensional domain in the phase space.

Let us investigate whether similar integral invariants occur in Nambu dynamics also. We have

$$\pounds_\Gamma\omega_a=\Gamma\lrcorner\,(d\omega_a)+d(\Gamma\lrcorner\,\omega_a)=d(d_aH\wedge d_aG)\neq0, \tag{22}$$

in general, (since $dd_a\neq0$) and similarly

$$\pounds_\Gamma\omega=\sum_{a=1}^N d(d_aH\wedge d_aG)\neq0 \tag{23}$$

[ω coincides with the three-form introduced in (8) and (9)]. This means, however, that neither

$$(I_3)_a:=\int_{M^3}\omega_a \tag{24}$$

nor

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$$I_3 := \int_{M^3} \omega \tag{25}$$

is invariant with respect to the time evolution in phase space. In the same manner, one verifies that none of the products $\omega_a \wedge \omega_b$, $\omega_a \wedge \omega_b \wedge \omega_c$,..., is Γ invariant except for the last one:

$$£_\Gamma(\omega_1 \wedge \cdots \wedge \omega_N) = 0. \tag{26}$$

If we therefore define the volume $3N$-form as

$$\Omega := \omega_1 \wedge \cdots \wedge \omega_N \equiv dx_1^1 \wedge \cdots \wedge dx_N^3 \tag{27}$$

[it is up to a constant multiple the only candidate which can be constructed from ω_a and happens to coincide with (11)], we have

$$£_\Gamma \Omega = 0 \tag{28}$$

equivalent to the Liouville theorem

$$I_{3N} := \int_{M^{3N}} \Omega = \text{invariant}. \tag{29}$$

The conclusion is then that no integral invariant exists but the highest one—the result just opposite to that given in Ref. 5 (and also different from Ref. 7, according to which even the complete series of the integral invariants $I_3,...,I_{3N}$ exists).

B. Canonical transformations

Let Φ_λ be the one-parameter group of coordinate transformations generated by the vector field U. Then (19) shows that they are canonical [preserving the structure of Eqs. (3)] iff they preserve the structure (18) of each ω_a separately; i.e., iff

$$£_U \omega_a = 0, \quad a = 1,...,N \tag{30}$$

holds. Equation (30) leads to

$$U = \sum_{a=1}^{N} U_a^i(x_a^1, x_a^1, x_a^3) \frac{\partial}{\partial x_a^i}, \quad \frac{\partial U_a^i}{\partial x_a^i} = 0 \tag{31}$$

(U_a^i does not depend on x_b^j for $b \neq a$).

This means that a general canonical transformation (homotopic to the identity) preserves the decomposition into the triplets (only the variables of the ath triplet enter the "new" ath triplet) and, in addition, it is volume preserving within each triplet:

$$x_a'^i = f_a^i(x_a^1, x_a^2, x_a^3), \quad J(x_a, x_a') = 1, \tag{32}$$

where f_a^i are arbitrary functions and J is the Jacobian of the transformation $x_a \to x'_a$. Since any permutation of complete triplets

$$x_a'^i = x_{\sigma(a)}^i \tag{33}$$

[σ—a permutation of $(1,...,N)$] is clearly a canonical transformation, too; its combination with (32) then represents the most general canonical transformation.

This is clearly a much more trivial situation than in the Hamiltonian dynamics, where a general canonical transformation thoroughly mixes up all the old coordinates (it does not preserve the canonical pairs). In particular, the time development is not a series of successive canonical transformations, since it does mix the coordinates of different triplets in general. These conclusions agree with the analysis performed in Ref. 1, but once again contradict.[7]

Remark: Recall that the integrals (21) are, in fact, invariant with respect to arbitrary canonical transformation in the Hamiltonian dynamics. Since the time development is a special canonical transformation, they are, in particular, invariant with respect to the time development.

In Nambu N-triplet dynamics, however, this connection breaks down, and one should therefore strictly distinguish between integral invariants of both types. Then, the results of Secs. III A and B show that, although there are a lot of "integral invariants with respect to the canonical transformations" [$(I_3)_a$, $(I_6)_{ab}$—the integral of $\omega_a \wedge \omega_b$,...,], only a single one (29) exists invariant with respect to time development.

IV. ON A GEOMETRY BEHIND (19)

The geometrical formulation (19) of Nambu N-triplet equations singles out a class of distinguished coordinates in which the decomposition into the triplets occurs and one can search for the intrinsic geometrical (coordinate independent) reason of this phenomenon.

Let us analyze some concrete physical system of this type, e.g., a coupled spin system mentioned in Ref. 1. Its phase space is the direct product of the phase space for a single rotator. The case of one rotator was studied in Ref. 8 and the coordinate-free description of the three-form ω ($\equiv \omega_1$) as well as the geometrical meaning of the necessary distinguished coordinates is given there. (One should realize that the rotator is, in fact, a Hamiltonian system, its Hamiltonian phase space is six dimensional, and the Euler dynamical equations represent only a half of the whole system of equations—the "kinematical" Euler equations are to be added to make the system complete). Therefore, the origin of the dimensionality ($\equiv 3N$) of the Nambu phase space as well as the decomposition of all $3N$ coordinates into N triplets is quite clear (ath triplet describes ath rotator).

However, one can as well forget the primary physical context and treat instead the dynamical system described by (19) in its own right, trying to find the underlying geometrical structure leading to (19) in appropriate dis-

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tinguished coordinates. This remains, in fact, an open problem, which can be formulated as follows: Find a tensor field such that it can live only on a $3N$-dimensional manifold and it singles out naturally "canonical" coordinates occurring in (15)–(19). A manifold M endowed with a tensor field of this type should be then the natural living space for N-triplet Nambu dynamics.

As an example, the symplectic form, as is well known, can live only on an even dimensional manifold and singles out the class of canonical coordinates (via Darboux theorem).

Another useful example is when the almost complex structure on a manifold [the (1,1) type tensor field C with the property $C^2 = -1$], which can also exist only on an even-dimensional manifold, induces the coordinates $z^i,\bar{z}^i$ as well as the decomposition [compare with (15)–(18)] of the exterior derivative operator:

$$d=\partial+\bar{\partial}, \tag{34}$$

$$\partial(\alpha_J\,dx^J):=\frac{\partial\alpha_J}{\partial z^i}\,dz^i\wedge dx^J, \tag{35}$$

$$\bar{\partial}(\alpha_J\,dx^J):=\frac{\partial\alpha_J}{\partial\bar{z}^i}\,d\bar{z}^i\wedge dx^J, \tag{36}$$

$$d^2=\partial^2=\bar{\partial}^2=0,\quad \partial\bar{\partial}=-\bar{\partial}\partial \tag{37}$$

[here, $x\equiv(z^1,...,z^n,\bar{z}^1,...,\bar{z}^n)$; note that, by introducing the notations $x_1^i\equiv z^i$, $x_2^i\equiv\bar{z}^i$, the formal coincidence of the corresponding formulas is obtained). Similarly, the well-known decomposition of a p-form as the sum of (k,l)-type $(k+l=p)$ forms invariant with respect to complex-analytical coordinate transformations (playing the role of the canonical transformations for C) has its evident counterpart in Nambu dynamics in the unique decomposition of a p-form into the sum of $(k_1,...,k_N)$-type forms, $k_1+\cdots+k_N=p$, invariant with respect to the coordinate transformations preserving triplets.

One might guess that the role of the tensor field mentioned above could be played by the maximum-rank closed three-form as opposed to the maximum-rank closed two-form ($\equiv$ symplectic form), which serves this purpose in the Hamiltonian dynamics. If a theorem (analogous to the Darboux theorem) leading to the canonical expression (8) (and restricting thus automaticaly the dimension of the manifold to $3N$) existed, then it would be possible to define the triplet-decomposition via this form, since one can readily verify that the canonical transformations preserving expression (8) for ω as a whole coincide with those for each ω_a separately (they do not, in particular, mix triplets).

Unfortunately, such generalization of the Darboux theorem does not exist since the forms with the properties mentioned above occur in dimensions different from $3N$, too. An elementary example provides the three-form $dx^1\wedge dx^2\wedge dx^3 + dx^3\wedge dx^4\wedge dx^5$ in $\mathbb{R}^5$—it is closed and has rank 5.

This means that the requirement of maximum rank and closure is not enough for a three-form to guarantee the canonical expression (8) and some of its additional characteristics are needed.

V. CONCLUSIONS

The aim of this paper was to give a geometrical formulation of N-triplet Nambu dynamics and to demonstrate its usefulness for obtaining some general results. We showed that Eq. (9) proposed in Ref. 5 is, in general, inconsistent for $N>1$ and leads to unacceptable consequences. It was replaced by Eq. (19). Two issues were studied then. First, the question of the integral invariants was analyzed. It was shown that only the highest one, I_{3N}, exists (with respect to the time development; see remark at the end of Sec. III B). This means, in particular, that the Liouville theorem does hold for any N. Second, the canonical transformations were studied, and the analysis confirmed the results of Ref. 1.

A formulation (19) singles out a class of distinguished coordinates—a decomposition into triplets is needed. The origin of this decomposition is clear for some known concrete systems (e.g., the coupled spin system). Some speculations were made, however, concerning a possible geometrical structure leading to (19) naturally. It was shown that a maximum-rank closed three-form is not enough to serve this purpose since a straightforward generalization of the Darboux theorem to such three-forms does not hold.

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