

On a variational principle for the Nambu dynamics

Marián Fecko

Citation: *J. Math. Phys.* **33**, 930 (1992); doi: 10.1063/1.529745

View online: <http://dx.doi.org/10.1063/1.529745>

View Table of Contents: <http://jmp.aip.org/resource/1/JMAPAQ/v33/i3>

Published by the [American Institute of Physics](#).

Related Articles

Classical and quantum probabilities as truth values
J. Math. Phys. **53**, 032101 (2012)

Time-independent description of rapidly driven systems in the presence of friction: Multiple scale perturbation approach
Chaos **22**, 013131 (2012)

Limitations on cloning in classical mechanics
J. Math. Phys. **53**, 012902 (2012)

Equivalence problem for the orthogonal webs on the 3-sphere
J. Math. Phys. **52**, 053509 (2011)

A Foucault's pendulum design
Rev. Sci. Instrum. **81**, 115102 (2010)

Additional information on *J. Math. Phys.*

Journal Homepage: <http://jmp.aip.org/>

Journal Information: http://jmp.aip.org/about/about_the_journal

Top downloads: http://jmp.aip.org/features/most_downloaded

Information for Authors: <http://jmp.aip.org/authors>

ADVERTISEMENT

The most comprehensive support for physics in any mathematical software package
World-leading tools for performing calculations in theoretical physics

Maple 16
The Essential Tool for Mathematics and Modeling
www.maplesoft.com/physics

- Your work in Maple matches how you would write the problems and solutions by hand
- State-of-the-art environment for algebraic computations in physics
- The only system with the ability to handle a wide range of physics computations as well as pencil-and-paper style input and textbook-quality display of results
- Access to Maple's full mathematical power, programming language, visualization routines, and document creation tools

$$\text{InertiaTensor} := \sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \mathbf{1} - \vec{r}_k \vec{r}_k^T \right)$$
$$\Gamma_{2,2}^1 = \frac{(1 + 2r)}{\partial r}$$

On a variational principle for the Nambu dynamics

Marián Fecko

Katedra teoretickej fyziky, Matematicko-fyzikálna fakulta UK, Mlynská dolina F2, 84215 Bratislava, Czechoslovakia

(Received 17 May 1991; accepted for publication 4 October 1991)

A variational principle for the Nambu dynamics is analyzed. Since the equations of motion single out a distinguished two-form rather than a one-form, the usual construction of the action $S[\gamma]$ as an integral of a one-form along the curve γ on the extended phase space has to be modified.

I. INTRODUCTION

In our previous paper¹ we discussed a geometrical formulation of Nambu dynamics (Ref. 2; see also references in Ref. 1), i.e., a dynamical system in 3 N -dimensional phase space M with local coordinates x_a^i , $i = 1, 2, 3$; $a = 1, \dots, N$, in which the time evolution is given by

$$\dot{r}_a = \nabla_a H \times \nabla_a G, \quad a = 1, \dots, N, \quad (1)$$

where $r_a \equiv (x_a^1, x_a^2, x_a^3)$, and H, G are arbitrary functions on the phase space M . In what follows we restrict ourselves to the basic "one triplet" case, $N = 1$, i.e., to the equation

$$\dot{r} = \nabla H \times \nabla G \quad (2)$$

and analyze the question of whether it can or cannot be derived from some variational principle.

In Ref. 3 the action integral

$$S \equiv \int_{t_1}^{t_2} L(\mathbf{r}, \dot{\mathbf{r}}) dt, \quad L \equiv H(\nabla G \cdot \dot{\mathbf{r}}) \quad (3)$$

was proposed. The Lagrangian is linear in velocities (and therefore singular, leading to a constrained Hamiltonian system) since (2) is a system of first-order equations. The corresponding Lagrange equations

$$\dot{\mathbf{r}} \times (\nabla H \times \nabla G) = 0 \quad (4)$$

are equivalent, however, to

$$\dot{\mathbf{r}} = f(\mathbf{r})(\nabla H \times \nabla G), \quad (5)$$

with arbitrary function $f(\mathbf{r})$, not only $f \equiv 1$ as is the case in (2). This means that not all extremals of the action [solutions of (4)] represent the solutions of the primary equation (2) but, in general, a reparametrization is needed to obtain a solution of (2) from a given extremal of (3). In other words the solutions of the variational problem given by (3) coincide with the solutions of (2) as paths but do not in general coincide as curves. Recall that such asymmetry between the extremals of the action

integral and the solutions of the dynamical equations is absent in the standard (nonsingular) Lagrangian dynamics as well as in the Hamiltonian one.

In what follows we analyze the problem of the variational principle, making use of the geometrical formulation of (2) given in Ref. 1 (or as well in Ref. 4 since they are identical for $N = 1$).

II. A DISTINGUISHED TWO-FORM χ ASSOCIATED WITH EQ. (2)

According to Refs. 1 and 4, the geometrical version of (2) reads (for the autonomous systems)

$$\Gamma_0 \lrcorner \omega = dH \wedge dG, \quad (6)$$

where ω is a nondegenerate (and automatically closed on the grounds of dimensionality) three-form on M ($\dim M = 3$) and Γ_0 is the dynamical vector field on M representing the time development. Recall that in the Hamiltonian case, one has to work on the extended phase space $M \times R^1[t]$ to either incorporate a general time-dependent Hamilton function situation or even to formulate the variational principle for the autonomous systems as well. Having this in mind, (6) is to be replaced by

$$\Gamma \lrcorner (\omega - dH \wedge dG \wedge dt) = 0, \quad (7)$$

where $\Gamma \equiv \Gamma_0 + \partial_t$ represents the dynamical vector field on $M \times R^1[t]$. This is clearly the counterpart of

$$\Gamma \lrcorner (\omega_s - dH \wedge dt) \equiv \Gamma \lrcorner d(\theta_s - H dt) = 0, \quad (8)$$

valid for the Hamiltonian dynamics ($\omega_s \equiv d\theta_s$ —the symplectic form). Guided by (8) we introduce a two-form θ such that (at least locally)

$$\omega = d\theta \quad (9)$$

and rewrite (7) as

$$\Gamma \lrcorner d\chi \equiv \Gamma \lrcorner d(\theta - H dG \wedge dt) = 0. \quad (10)$$

Thus the geometrical formulation of the Nambu dynamics reveals that the latter singles out the distinguished two-form $\chi \equiv \theta - H dG \wedge dt$ on the extended phase space $M \times R^1[t]$ just as the Hamiltonian dynamics singles out the one-form $\theta_s - H dt$. However, a distinguished one-form is needed for a straightforward construction of the action (it is a line integral). Thus we come to the conclusion that there is a substantial difference (from the point of view of a variational principle) between the two classes of the dynamical systems in question: the Hamiltonian (as well as Lagrangian) dynamics offers a natural candidate to the role of the action one-form, *viz.*, the Cartan one-form $\theta_s - H dt$ (and a similar one in the Lagrangian case). Nambu dynamics, on the contrary, offers no such one-form but instead the two-form $\chi \equiv \theta - H dG \wedge dt$. This fact is the source of some difficulties connected with the formulation of a variational principle for the Nambu dynamics.

III. A CONSTRUCTION OF THE ACTION INTEGRAL

The action integral is to assign a number $S[\gamma]$ to a given curve γ on $M \times R^1[t]$. Only the distinguished two-form χ is, however, available. There are in principle two ways out of this inconsonance: (a) We can construct some distinguished two-dimensional surface Σ_γ from γ and then integrate χ over Σ_γ , or (b) we can construct some distinguished one-form α from χ and integrate it along γ . In what follows, we show that both these procedures lead to the same result (for appropriate choices of Σ_γ and α).

A. The action as a surface integral

Let us define

$$S_1[\gamma] := \int_{\Sigma_\gamma} \chi, \tag{11}$$

where Σ_γ is a two-dimensional surface associated in some natural way with the given curve γ . There are two criteria for its choice: First, only the geometrical structure available is to be used for its construction and, second, the variation of (11) has to lead to the equation “as close as possible” to (10). The first criterion implies that, in fact, only the product structure (plus the natural geometry of $R^1[t]$) on $M \times R^1[t]$ can be utilized (e.g., since γ is not closed, no “minimal surface philosophy” is viable, etc.). This means that Σ_γ is to be taken as a cylinder over the projection of γ to $M \times t_1$. Its more precise shape is dictated by the needs of the second criterion (see below) and it is possible to choose it in the following way: Let x^i, t represent local coordinates on $M \times R^1[t]$ and let $(x^i(\tau), \tau)$ be a coordinate expression of γ . Then the coordinate expression of Σ_γ is

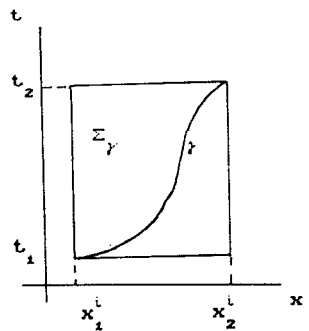


FIG. 1. A surface Σ_γ associated with γ .

$$x^i(\tau, \sigma) = x^i(\tau), \quad t(\tau, \sigma) = \sigma, \tag{12}$$

where $(\tau, \sigma) \in I^2 \equiv \langle t_1, t_2 \rangle \times \langle t_1, t_2 \rangle$ (see Fig. 1).

Now, let us turn to the evaluation of the change of the action integral caused by a small variation of the curve γ . Let U , be a deformation field (defined in some neighborhood of γ) and Φ_λ the corresponding flow. Then a deformation $\gamma \rightarrow \gamma_\epsilon \equiv \Phi_\epsilon(\gamma)$ leads to a uniquely defined deformation $\Sigma_\gamma \rightarrow \Phi_\epsilon(\Sigma_\gamma)$, inducing thus some flow $\dot{\Phi}_\lambda$ and its vector field \dot{U} defined on some neighborhood of Σ_γ . The change of the action integral is

$$\begin{aligned} S_1[\gamma_\epsilon] &= \int_{\dot{\Phi}_\epsilon(\Sigma_\gamma)} \chi \\ &= \int_{\Sigma_\gamma} \dot{\Phi}_\epsilon^* \chi \\ &= \int_{\Sigma_\gamma} \chi + \epsilon \int_{\Sigma_\gamma} \mathcal{L} \dot{U} \chi + o(\epsilon^2) \\ &= S_1[\gamma] + \epsilon \int_{\Sigma_\gamma} \mathcal{L} \dot{U} \chi + o(\epsilon^2). \end{aligned} \tag{13}$$

Thus the extremal γ is to satisfy

$$\int_{\Sigma_\gamma} \mathcal{L} \dot{U} \chi = 0. \tag{14}$$

Now,

$$\begin{aligned} \int_{\Sigma_\gamma} \mathcal{L} \dot{U} \chi &= \int_{\Sigma_\gamma} \dot{U} \lrcorner d\chi + \int_{\Sigma_\gamma} d(\dot{U} \lrcorner \chi) \\ &= \int_{\Sigma_\gamma} \dot{U} \lrcorner d\chi + \int_{\partial \Sigma_\gamma} \dot{U} \lrcorner \chi. \end{aligned} \tag{15}$$

If we restrict the class of possible deformations of γ by the condition

$$\int_{\partial \Sigma_\gamma} \dot{U} \lrcorner \chi = 0 \tag{16}$$

(this can be achieved, as a simple calculation shows, by keeping the endpoints of γ fixed in the course of a variation), then

$$\int_{\Sigma_\gamma} \dot{U} \lrcorner d\chi = 0 \tag{17}$$

is to be fulfilled for arbitrary \dot{U} [subjected to (16)] or

$$\dot{U} \lrcorner d\chi \Big|_{\Sigma_\gamma} = 0. \tag{18}$$

The vectors $\dot{\gamma}, \partial_t$ form the basis of the vectors tangent to Σ_γ (more precisely $\dot{\gamma}$ transported from the curve γ to arbitrary point of Σ_γ using the action of \mathbb{R}^1 on $M \times \mathbb{R}^1[t]$), so that (18) can be written as

$$d\chi(\dot{\gamma}, U, \partial_t) = 0, \tag{19}$$

or, equivalently,

$$(\dot{\gamma} \lrcorner d\chi)(\partial_\sigma, \partial_t) = 0. \tag{20}$$

A comparison of (20) with the equation

$$\dot{\gamma} \lrcorner d\chi = 0 \tag{21}$$

[valid for any solution of (2), see (10)] shows that, although each solution of (2) is an extremal of the action integral (11), the converse is not the case in general [the two-form $\dot{\gamma} \lrcorner d\chi$ vanishes identically for the solutions of (2), but only on some special arguments for the extremals of (11)].

Let us translate (11) into the coordinate language. If (12) is interpreted as $j: I^2 \rightarrow \Sigma_\gamma$, then

$$S_1[\gamma] = \int_{I^2} j^* \chi = \int_{t_1}^{t_2} d\tau \int_{x_1}^{x_2} d\sigma H(x(\tau)) G_{,i}(x(\tau)) \dot{x}^i(\tau) = (t_2 - t_1) S. \tag{22}$$

Thus it coincides (up to irrelevant constant multiple) with the action S given by (3).

Remark: If the surface Σ_γ is replaced by Σ_γ^ϵ given by

$$x^i(\tau, \sigma) = x^i(\tau), \quad t(\tau, \sigma) = \tau + \sigma, \tag{23}$$

where $(\tau, \sigma) \in (t_1, t_2) \times (0, \epsilon)$ (see Fig. 2), then a simple calculation shows that

$$S'_{1\epsilon}[\gamma] = \frac{1}{\epsilon} \int_{\Sigma_\gamma^\epsilon} \chi = S_1[\gamma] \tag{24}$$

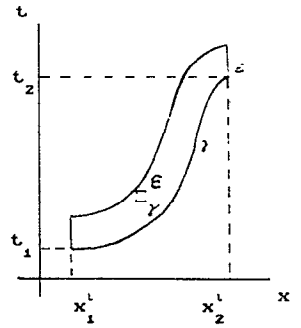


FIG. 2. A surface Σ_γ^ϵ associated with γ .

for any $\epsilon > 0$ and the action can be defined as well as the limit

$$S'_1[\gamma] \equiv \lim_{\epsilon \rightarrow 0^+} S'_{1\epsilon}[\gamma]. \tag{25}$$

Here, the integration is performed over the infinitesimally narrow strip obtained by the translation of γ in the t direction. This version of the action is directly connected to the one discussed in the following section.

B. The action as a line integral

One has to assign a one-form α to the two-form χ in some natural way (in the sense described in the previous section). The structure of $M \times \mathbb{R}^1[t]$ offers the distinguished vector field ∂_t ; consequently, one obtains the one-form

$$\alpha \equiv \partial_t \lrcorner \chi = H dG, \tag{26}$$

and therefore the action integral

$$S_2[\gamma] := \int_\gamma (\partial_t \lrcorner \chi) = \int_\gamma H dG. \tag{27}$$

Now,

$$S_2[\gamma_\epsilon] = \int_{\Phi_\epsilon(\gamma)} (\partial_t \lrcorner \chi) = S_2[\gamma] + \epsilon \int_\gamma \mathcal{L}_U (\partial_t \lrcorner \chi) + o(\epsilon^2). \tag{28}$$

If the variations of γ are restricted by

$$\int_{\partial\gamma} U \lrcorner (\partial_t \lrcorner \chi) \equiv \chi(\partial_\sigma U) \Big|_{\gamma(t_1)}^{\gamma(t_2)} = H(UG) \Big|_{\gamma(t_1)}^{\gamma(t_2)} = 0 \tag{29}$$

(it is achieved by keeping the endpoints of γ fixed, too), then

$$\int_{\gamma} U \lrcorner d(\partial_t \lrcorner \chi) = 0 \tag{30}$$

is to be fulfilled, or [since $U = U^i(x)\partial_i$, U^i arbitrary]

$$(\dot{\gamma} \lrcorner d\chi)(\partial_i \partial_t) = 0, \tag{31}$$

which is identical with (20). In coordinates, (24) gives

$$S_2[\gamma] = \int_{\gamma} H dG = \int_{t_1}^{t_2} H(x(\tau)) G_{,A}(x(\tau)) \dot{x}^i(\tau) \equiv S. \tag{32}$$

Once more, the action S given by (3) is the result.

IV. SUMMARY AND CONCLUSIONS

A variational principle for the one-triplet Nambu dynamics is analyzed, making use of the geometrical formulation of the latter. Equation (10) singles out the distinguished two-form $\chi \equiv \theta - H dG \wedge dt$ which is the counterpart of the Cartan one-form $\theta_s - H dt$ in the Hamiltonian dynamics. Since, however, it is a two-form, the action cannot be defined as the integral of χ along the curve γ (as is the case in the Hamiltonian dynamics). In order to make the integration possible, one has either to construct some one-form α from χ and integrate it along γ or to associate some two-dimensional surface Σ_{γ} with the curve γ and integrate χ over Σ_{γ} . Both of these possi-

bilities are studied and the resulting action integrals $S_1[\gamma]$ (or $S'_1[\gamma]$) and $S_2[\gamma]$ given by (11), (12), (25), and (27) turn out to be, in fact, identical. Moreover, they happen to coincide with the action S given by (3), which was proposed in Ref. 3.

The inevitable feature of the action for the Nambu dynamics is its reparametrization (in M) invariance: All the curves in M that coincide with some solution of (2) as paths represent the extremals of S . In terms of the extended phase space $M \times R^1[t]$, it means that the action S does not single out the extremal vector field Γ [given by (7)] but rather a two-dimensional distribution Δ , which spans on Γ and ∂_t . Indeed, if $\dot{\gamma} \in \Delta$, then

$$\dot{\gamma} = a\Gamma + b\partial_t \tag{33}$$

for some a, b , and

$$(\dot{\gamma} \lrcorner d\chi)(\partial_t) = 0. \tag{34}$$

This is, however, equivalent to (20), expressing just the fact that γ is an extremal of $S_1 \simeq S$. The distribution Δ is integrable (Γ and ∂_t commute) and Σ_{γ} is its integral submanifold.

¹M. Fecko, J. Math. Phys. 33, 926–929 (1992).

²Y. Nambu, Phys. Rev. D 7, 2405 (1973).

³F. Bayen and M. Flato, Phys. Rev. D 11, 3049 (1975).

⁴F. B. Estabrook, Phys. Rev. D 8, 2740 (1973).