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“Falling cat” connections and the momentum map

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We consider a standard symplectic dynamics on TM generated by a natural Lagrangian L. The Lagrangian is assumed to be invariant with respect to the action TR_g of a Lie group G lifted from the free and proper action R_g of G on M. It is shown that under these conditions a connection on principal bundle $\pi:M \to M/G$ can be constructed based on the momentum map corresponding to the action TR_g. A simple explicit formula for the connection form is given. For the special case of the standard action of $G=\mathrm{SO}(3)$ on $M=\mathbb{R}^3\times\cdots\times\mathbb{R}^3$ corresponding to a rigid rotation of an N-particle system the formula obtained earlier by Guichardet and Shapere and Wilczek is reproduced. © *1995 American Institute of Physics.*

I. INTRODUCTION

In their remarkable articles Guichardet[1] and Shapere and Wilczek[2] pointed out that the phenomenon of reorientation of deformable bodies (molecules represented by N point masses in Ref. 1 or cats, divers, astronauts, etc., in Ref. 2) in space, for a long time to be known in the case of cats to originate physically in the angular momentum conservation, lends itself to a simple and powerful description within the framework of the theory of connections (gauge structures). Namely, they showed that in the center-of-mass system ($\mathbf{P}=\mathbf{0}$) the condition of vanishing of the total angular momentum ($\mathbf{L}=\mathbf{0}$) can be rephrased in terms of the SO(3)-connection in the principle bundle $\pi:M \to M/\mathrm{SO}(3)$, where M is the configuration space of the deformable body ($\mathbb{R}^{3N}$ minus some forbidden configurations in Ref. 1 or “the space of located shapes” in Ref. 2), where SO(3) acts by rigid rotations (without deformation). In more detail the trajectories fulfilling $\mathbf{L}=\mathbf{0}$ represent the *horizontal* curves in the sense of the connection [“vibrational curves” in Ref. 1 as opposed to purely rotational ones given by (in general time-dependent) rigid rotations].

In what follows we try to understand the origin of the connection within the standard framework[3] of Lagrangian mechanics on TM.

It is known that the central object providing the link between the symmetries and conserved quantities in symplectic dynamics is the *momentum map*.[4,5] Now both $\mathbf{P}$ and $\mathbf{L}$ result (being linear in velocities) from the symmetries of a rather special type, namely, those *lifted* to TM from M. That is why the situation under consideration is the following: we have a Lagrangian system (TM,L) with appropriate action of a Lie group G lifted from the configuration space M. Then we show how one can construct (under some restrictions on the Lagrangian L) a connection in the principal bundle $\pi:M \to M/G$. This connection happens to coincide with the one in Refs. 1 and 2 in the case treated there, i.e., for $G=\mathrm{SO}(3)$, M being the configuration space of an N-particle system.

The organization of the article is the following. In Sec. II (as well as in Appendix A) the relevant facts concerning the momentum map within the context mentioned above are collected. The construction of the connection itself is described in Sec. III; the general properties of the latter are discussed in Sec. IV. Several examples, including completely elementary ones as well as the N-particle system, are given in Sec. V. Some technicalities are treated in the Appendices.

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II. THE MOMENTUM MAP FOR THE LIFTED ACTION TR_g

Let

$$R_g : M \to M \tag{1}$$

be a right action of a Lie group G on a manifold M. Then the tangent map

$$TR_g : TM \to TM$$

is a right action of G on TM. Let $L : TM \to \mathbb{R}$ be a G-invariant Lagrangian, i.e.,

$$L \circ TR_g \equiv (TR_g)^* L = L \tag{2}$$

for all $g \in G$. The (exact) symplectic form on TM is given by (Ref. 3; see Appendix A)

$$\omega_L = d\theta_L = dS(dL),$$

where (1,1)-type tensor field S on TM (almost tangent structure≡vertical endomorphism) is a lift of the identity tensor on M ($S = I^{\uparrow}$; in canonical local coordinates x^i, v^i on TM, $S = dx^i \otimes \partial/\partial v^i$ or $S = dx^i \otimes \partial/\partial \dot{x}^i$ if the notation $v^i \equiv \dot{x}^i$ is used). Since ω_L is to be maximum rank two-form, the condition

$$\det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0$$

must be fulfilled (nondegenerate Lagrangian).

Let $a \in \mathscr{G}$ (the Lie algebra of G), X_a the corresponding fundamental field of the action R_g on M. Then the fundamental field of the lifted action TR_g is the *complete lift* $\tilde{X}_a$ [in coordinates if $V = V^i \partial_i$ on M then $\tilde{V} = V^i \partial_i + V^i{}_{,j} v^j (\partial/\partial v^i)$ on TM]. Now

$$\mathscr{L}_{\tilde{X}_a} \theta_L = (\mathscr{L}_{\tilde{X}_a} S)(dL) + S(d\tilde{X}_a L) = \theta_{\tilde{X}_a L}$$

($\mathscr{L}_{\tilde{V}} S = 0$ for any V). In the case of invariant Lagrangian Eq. (2) gives

$$\tilde{X}_a L = 0, \tag{3}$$

i.e.,

$$\mathscr{L}_{\tilde{X}_a} \theta_L = 0.$$

Then

$$i_{\tilde{X}_a} d\theta_L + d i_{\tilde{X}_a} \theta_L = 0$$

or

$$i_{\tilde{X}_a} \omega_L = -dP_a$$

($\Rightarrow \tilde{X}_a$ is the Hamiltonian field generated by P_a) where $P_a : TM \to \mathbb{R}$ is defined by

$$P_a := \langle \theta_L, \tilde{X}_a \rangle = S(dL, \tilde{X}_a) = S(\tilde{X}_a) L = X_a^{\uparrow} L \tag{4}$$

($X_a^{\uparrow}$ is a *vertical lift* of X_a). Since P_a depends linearly on $a \in \mathscr{G}$, the *momentum map* associated with the (exact symplectic) action TR_g on TM,

$$P:TM \to \mathscr{G}^*,$$

can be introduced by

$$\langle P(v),a\rangle_0 := P_a(v), \quad v \in TM,$$

where $\langle .,.\rangle_0$ is the evaluation map (canonical pairing) for $\mathscr{G}$ and its dual $\mathscr{G}^*$. Fixing a basis E_α, $\alpha=1,...,\dim \mathscr{G}$ in $\mathscr{G}$ and the dual one E^α in $\mathscr{G}^*$ one can write

$$P = P_\alpha E^\alpha,$$

where

$$P_\alpha \equiv P_{E_\alpha}:TM \to \mathbb{R}$$

are the components of P with respect to E^α.

One verifies easily the important (equivariance) property of P

$$(TR_g)^* P = \mathrm{Ad}_g^* \, P$$

or in components

$$(TR_g)^* P_\alpha = (\mathrm{Ad}_g^*)_\alpha^\beta P_\beta,$$

where $\mathrm{Ad}_g^*:\mathscr{G}^* \to \mathscr{G}^*$ is the coadjoint action of G on $\mathscr{G}^*$. If $\Omega^k(\mathscr{M},\rho)$ denotes the space of k-forms of type ρ on G-space $\mathscr{M}$ [i.e., V-valued k-forms on $\mathscr{M}$ obeying $R_g^*\sigma = \rho(g^{-1})\sigma$, ρ being a representation of G in V], we see that

$$P \in \Omega^0(TM,\mathrm{Ad}^*); \tag{5}$$

it is $\mathscr{G}^*$-valued 0-form of type Ad^* on TM. Thus a right action (1) of G on M which is a symmetry of a nondegenerate Lagrangian L [in the sense of Eq. (2)] leads automatically to the existence of Eq. (5).

III. THE CONSTRUCTION OF A CONNECTION FORM

Let R_g be the action (1). In order to obtain a principal G-bundle

$$\pi:M \to M/G \tag{6}$$

the action is to be in addition free (all isotropy groups trivial) and proper [the map $(g,x) \mapsto (x,R_g x)$ is proper; i.e., inverse images of compact sets are compact]. A connection form on Eq. (6) is $A \in \Omega^1(M,\mathrm{Ad})$ such that

$$\langle A,X_a\rangle = a \tag{7}$$

holds for all $a \in \mathscr{G}$. Thus $P \in \Omega^0(TM,\mathrm{Ad}^*)$ is available whereas we need $A \in \Omega^1(M,\mathrm{Ad})$. These two objects are different, but fortunately "not too much" and one can quite easily obtain some A from P.

First there is a bijection between one-forms on M and functions on TM "linear in velocities," viz.,

$$\sigma(v) := \langle \tilde{\sigma},v\rangle_{\pi_M(v)}$$

$[\sigma \in \Omega^0(TM), \tilde{\sigma} \in \Omega^1(M)]$, or in coordinates

$$\sigma_i v^i \leftrightarrow \sigma_i \, dx^i.$$

Then if our P was linear in velocities, one could associate with it $\tilde{P} \in \Omega^1(M,\mathrm{Ad}^*)$ by

$$\langle \tilde{P}, v \rangle_{\pi_M(v)} := P(v)$$

(the fact that $\tilde{P}$ really remains to be Ad* type is easily verified). The demand of linearity in velocities of P_α restricts the form of Lagrangian: according to Eq. (4)

$$P_\alpha(v) = X_\alpha^\uparrow L = X_\alpha^i(x) \frac{\partial L(x,v)}{\partial v^i}.$$

If this is to be of the form $P_{\alpha i}(x) v^i$, the Lagrangian has to be *natural*, i.e.,

$$L(x,v) = \tfrac{1}{2} g_{ij}(x) v^i v^j - U(x) \tag{8}$$

(a standard Lagrangian for potential system with time-independent holonomic constraints). Then explicitly

$$P_\alpha(v) = (X_\alpha^\uparrow L)(v) = X_\alpha^i(x) g_{ij}(x) v^j = (\flat_g X_\alpha)_i(x) v^i$$

and

$$\tilde{P}_\alpha = (\flat_g X_\alpha)_i(x) dx^i = \flat_g X_\alpha,$$

where $\flat_g$ is the "lowering index" operator (by means of the metric tensor g on M given by the kinetic energy term in L) from vector to covector fields (the metric tensor g is denoted by the same letter as the group element $g \in G$; the proper meaning of g is, however, always clear from the context). One also verifies that [see Eqs. (3) and (8)]

$$\mathscr{L}_{X_a} g = 0,$$

i.e., G acts on (M,g) as a group of isometries (X_a are the Killing vectors).

The next step is a "correction" of Ad* type to Ad type (needed for A). This can be done by composition with a map $\hat{h}: \mathscr{G} \to \mathscr{G}^*$ induced by some Ad-invariant nondegenerate bilinear form h on $\mathscr{G}$ (see Appendix B). Then

$$\hat{A} := \hat{h}^{-1} \circ \tilde{P} \in \Omega^1(M,\mathrm{Ad});$$

i.e., $\hat{A}$ is already type Ad $\mathscr{G}$-valued one-form on M.

Finally one has to check whether Eq. (7) is fulfilled. We have

$$\langle \hat{A}, X_\alpha \rangle = \hat{h}^{-1} \langle \tilde{P}, X_\alpha \rangle = \langle \tilde{P}_\beta, X_\alpha \rangle \hat{h}^{-1}(E^\beta) = \langle \flat_g X_\beta, X_\alpha \rangle h^{\beta\gamma} E_\gamma = g(X_\alpha, X_\beta) h^{\beta\gamma} E_\gamma = C_\alpha^\gamma(x) E_\gamma,$$

where

$$\begin{aligned} C_\alpha^\gamma &:= g_{\alpha\beta}(x) h^{\beta\gamma}, \\ g_{\alpha\beta}(x) &:= g(X_\alpha, X_\beta). \end{aligned} \tag{9}$$

Thus

$$\langle \hat{A}, X_a \rangle = C(x)(a),$$

where

$$C(x):\mathscr{G} \to \mathscr{G}, \quad E_\alpha \mapsto C_\alpha^\beta E_\beta .$$

According to Appendix C the $\mathscr{G}$-valued one-form

$$A:=C^{-1}\circ\hat{A}=C^{-1}\circ\hat{h}^{-1}\circ\tilde{P} \tag{10}$$

already has all the necessary properties of a connection form, i.e.,

$$R_g^* A=\mathrm{Ad}_{g^{-1}}\,A,$$

$$\langle A,X_a\rangle=a$$

and defines thus a connection on $\pi:M \to M/G$. Explicitly we have

$$A=C^{-1}\circ\hat{h}^{-1}(\tilde{P}_\alpha E^\alpha)=\tilde{P}_\alpha h^{\alpha\beta}C^{-1}(E_\beta)=\tilde{P}_\alpha(h^{\alpha\beta}h_{\beta\mu}g^{\mu\nu})E_\nu=(g^{\alpha\beta}\tilde{P}_\beta)E_\alpha ,$$

where $g^{\alpha\beta}(x)$ is the inverse to $g_{\alpha\beta}(x)$ defined in Eq. (9). Thus it turns out to be given by a surprisingly simple expression, viz.

$$A=A^\alpha E_\alpha=(g^{\alpha\beta}\tilde{P}_\beta)E_\alpha=g^{\alpha\beta}(\flat_g X_\beta)E_\alpha . \tag{11}$$

Note: notice that the bilinear form $h_{\alpha\beta}$ was present on the scene only temporarily and it dropped out from the resulting formula (and thus one does not need it in fact for the construction of A).

IV. SOME PROPERTIES OF THE CONNECTION GIVEN BY *A*

Let $\gamma:\mathbb{R}\to M$ be a curve on M representing some motion of the system under consideration. What does it mean in physical terms if it is purely horizontal (i.e., represents a parallel translation in the sense of A)? According to Eq. (11) we have

$$\langle A,\dot{\gamma}\rangle=0 \Rightarrow \langle\tilde{P}_\alpha ,\dot{\gamma}\rangle=0$$

or

$$P_\alpha(\hat{\gamma}(t))=0,$$

where $\hat{\gamma}$ is the natural lift of γ to TM [$(x^i(t),\dot{x}^i(t))$ in coordinates]. Thus a horizontal curve is such a motion of the system that all conserved quantities P_α have all the time zero value (remember $\mathbf{P}=\mathbf{0}$, $\mathbf{L}=\mathbf{0}$ in Sec. I).

Now let $W\in \mathrm{Hor}_x\,M$ be any horizontal vector. Then

$$0=\langle A,W\rangle=g^{\alpha\beta}\langle\tilde{P}_\beta ,W\rangle E_\alpha=g^{\alpha\beta}g(X_\beta ,W)E_\alpha$$

or

$$g(X_\alpha ,W)=0$$

for all α. But X_α just span the vertical subspace so that

$$\mathrm{Ver}_x\,M\perp \mathrm{Hor}_x\,M.$$

Thus the horizontal subspace is simply the orthogonal complement of the vertical one with respect to the scalar product in T_xM given by the kinetic energy metric tensor. Note that this serves as the *definition* of the connection (it gives it uniquely) in Ref. 1 [the special case of $G=SO(3)$, etc., is discussed in more detail in Sec. V C]. In the approach presented here it came as its *property*.

V. EXAMPLES

We illustrate the construction of the connection form A on three examples, the first two being completely elementary and the last one being that discussed in Refs. 1 and 2.

A. A point mass on a board

Let us have a (one dimensional) board of mass m_1 lying on the surface of the water and denote x the distance of its left end from some reference point on the surface. Let ξ denote the distance of a point mass m_2 from the left end of the board. The Lagrangian of the system reads

$$L(x,\xi,\dot{x},\dot{\xi})=\tfrac{1}{2}m_1\dot{x}^2+\tfrac{1}{2}m_2(\dot{x}+\dot{\xi})^2-U(\xi)$$

(interaction of the point mass m_2 with the board depends only on their relative position). The translational invariance of the system means that there is the action of $G\equiv\mathbb{R}$, on the configuration space $M[x,\xi]$ given by

$$R_b:(x,\xi)\mapsto(x+b,\xi),\quad b\in\mathbb{R}\equiv G$$

(the ''unlocated shape'' is given by the position of m_2 with respect to the board, i.e., by ξ) such that L is invariant with respect to its lift

$$TR_b:(x,\xi,\dot{x},\dot{\xi})\mapsto(x+b,\xi,\dot{x},\dot{\xi}).$$

Now

$$X_1=\partial_x,\quad g_{11}\equiv g(X_1,X_1)=m_1+m_2,$$

$$A=A^1E_1=g^{11}(\flat_g X_1)E_1=\frac{1}{m_1+m_2}\,((m_1+m_2)dx+m_2\,d\xi)=dx+\frac{m_2}{m_1+m_2}\,d\xi$$

(one can take $E_1=1$ since $\mathscr{G}=\mathbb{R}$). The curve $\gamma\leftrightarrow(x(t),\xi(t))$ is horizontal if $\langle A,\dot{\gamma}\rangle\equiv\langle A,\dot{x}\partial_x+\dot{\xi}\partial_\xi\rangle=0$, i.e., if

$$\dot{x}(t)+\frac{m_2}{m_1+m_2}\,\dot{\xi}(t)=0$$

or

$$m_1\dot{x}(t)+m_2(\dot{x}(t)+\dot{\xi}(t))=0,$$

which is just the vanishing of the total (linear) momentum of the system.

B. A point mass on a gramophone disc

Let us have a gramophone disc (its moment of inertia with respect of the axis being I) and a point mass m on it. If the angle α measures the orientation of the disc with respect to the outer space and r, φ are the polar coordinates of the point mass m with respect to the disc, the Lagrangian of the system is

$$L(r,\varphi,\alpha,\dot{r},\dot{\varphi},\dot{\alpha})=\tfrac{1}{2}I\dot{\alpha}^2+\tfrac{1}{2}m(\dot{r}^2+r^2(\dot{\alpha}+\dot{\varphi})^2)-U(r,\varphi)$$

(interaction of the point mass m with the disc depends only on their relative position). The rotational invariance of the system means that there is the action of $G \equiv \mathrm{SO}(2)$ on the configuration space $M[r,\varphi,\alpha]$ given by

$$R_\beta:(r,\varphi,\alpha) \mapsto (r,\varphi,\alpha+\beta)$$

(the "unlocated shape" is given by the position of m with respect to the disc, i.e., by r,φ) such that L is invariant with respect to its lift

$$TR_\beta:(r,\varphi,\alpha,\dot{r},\dot{\varphi},\dot{\alpha}) \mapsto (r,\varphi,\alpha+\beta,\dot{r},\dot{\varphi},\dot{\alpha}).$$

Now

$$X_1=\partial_\alpha, \quad g_{11}\equiv g(X_1,X_1)=I+mr^2,$$

$$A=A^1E_1=g^{11}(\flat_g X_1)E_1=\frac{1}{I+mr^2}\,((I+mr^2)d\alpha+mr^2\;d\varphi)=d\alpha+\frac{mr^2}{I+mr^2}\,d\varphi$$

(one can take $E_1=1$ since $\mathscr{G}=\mathbb{R}$ as in the previous example). The curve $\gamma \leftrightarrow (r(t),\varphi(t),\alpha(t))$ is horizontal if $\langle A,\dot{\gamma}\rangle \equiv \langle A,\dot{r}\partial_r+\dot{\varphi}\partial_\varphi+\dot{\alpha}\partial_\alpha\rangle=0$, i.e., if

$$\dot{\alpha}(t)+\frac{mr^2}{I+mr^2}\,\dot{\varphi}(t)=0$$

or

$$I\dot{\alpha}(t)+mr^2(\dot{\alpha}(t)+\dot{\varphi}(t))=0,$$

which is just the vanishing of the total angular momentum of the system.

If $\sigma(t) \leftrightarrow (r(t),\varphi(t))$ is a curve in the space of unlocated shapes M/G, the resulting curve in M is $\gamma(t)=\sigma^h(t)=$the horizontal lift of $\sigma(t)$, given by $(r(t),\varphi(t),\alpha(t))$, where

$$\alpha(t)=\alpha(0)+\int_0^t\left(-\dot{\varphi}(s)\,\frac{mr^2(s)}{I+mr^2(s)}\right)ds.$$

In particular, the holonomy [the angle corresponding to the element of SO(2)] for the *closed* path (loop) $\sigma(0)=\sigma(1)$ is

$$\beta=\alpha(1)-\alpha(0)=-\int_0^1\left(\frac{mr^2(s)}{I+mr^2(s)}\right)\dot{\varphi}(s)ds.$$

If, for example, the point goes round the disc once counterclockwise at constant distance $r_0(r(t)=r_0,\varphi(t)=2\pi t)$, the net rotation of the disc is

$$\beta_0=-2\pi\,\frac{I_0}{I+I_0}, \quad I_0\equiv mr_0^2$$

(clockwise). Clearly α does not change for radial motion (formally since $A_r^1=0$). There is nonzero curvature in this example being given explicitly by

$$F=DA=dA=\left(\frac{mr^2}{I+mr^2}\right)'\;dr\wedge d\varphi\equiv\frac{1}{2}\,F_{r\varphi}^1\;dr\wedge d\varphi.$$

C. *N*-particle system

Let $\mathbf{r}_a$, $a=1,\ldots,N$ denote the radius vector of the ath particle, x^i_a its ith component $(i=1,2,3)$, and m_a its mass. There is a natural action of the Euclidean group $G=E(3)$ on the configuration space of the N-particle system, consisting in rigid rotations and translations

$$\mathbf{r}_a \mapsto \mathbf{r}_a B+\mathbf{b}, \qquad B\in \mathrm{SO}(3).$$

We will treat the rotations and the translations separately. The standard summation convention is adopted in what follows, i.e., the sum is implicit for pairs of equal indices, otherwise the symbol of the sum is written explicitly.

The *translational* subgroup acts by

$$x^i_a \mapsto x^i_a+b^i.$$

If $\mathscr{E}_i$ is the standard basis of the Lie algebra $(\equiv \mathbb{R}^3)$, i.e., $(\mathscr{E}_i)^j=\delta^j_i$, then the corresponding fundamental field is

$$X_i\equiv X_{\mathscr{E}_i}=\sum_a \partial^a_i\equiv(\boldsymbol{\nabla}_1+\cdots+\boldsymbol{\nabla}_N)_i$$

$(\partial^a_i\equiv\partial/\partial x^i_a)$. The kinetic energy is

$$T=\frac{1}{2}\sum_a m_a\dot{x}^k_a\dot{x}^k_a$$

so that the metric tensor reads

$$g=\sum_a m_a\, dx^k_a\otimes dx^k_a.$$

Then

$$g(X_i,X_j)=m\,\delta_{ij}$$

$(m\equiv\Sigma_a m_a$ is the total mass). Since

$$\tilde{P}_i=\flat_g X_i=m_a\, dx^i_a$$

we have the translational part of the connection

$$A_{\mathrm{tr}}=A^i_{\mathrm{tr}}\mathscr{E}_i=\frac{1}{m}\,\delta_{ij}\tilde{P}_j\mathscr{E}_i=\frac{m_a\, dx^i_a}{m}\,\mathscr{E}_i.$$

The *rotational* subgroup acts by

$$x^i_a \mapsto x^j_a B^i_j, \qquad B\in\mathrm{SO}(3).$$

If E_i is the standard basis of the Lie algebra so(3), i.e., $(E_i)^k_j=-\epsilon_{ijk}$, the corresponding fundamental field is

$$X_i\equiv X_{E_i}=-\epsilon_{ijk}x^j_a\partial^a_k\equiv-(\mathbf{r}_a\times\boldsymbol{\nabla}_a)_i.$$

Then

$$g(X_i,X_j)=\sum_a (\delta_{ij}\mathbf{r}_a^2-x_a^i x_a^j)=I_{ij},$$

where $I_{ij}(\mathbf{r}_1,...,\mathbf{r}_N)$ is the inertia tensor of the configuration. Since

$$\tilde{P}_i=\flat_g X_i=-\epsilon_{ijk}\sum_a m_a x_a^j\, dx_a^k\equiv-\left(\sum m_a\mathbf{r}_a\times d\mathbf{r}_a\right)_i$$

we have the rotational part (the one computed in Refs. 1 and 2) of the connection

$$A_{\mathrm{rot}}=A_{\mathrm{rot}}^i E_i=I^{ij}\tilde{P}_j E_i=-I^{ij}\left(\sum_a m_a\mathbf{r}_a\times d\mathbf{r}_a\right)_j E_i$$

(I^{ij} being the inverse matrix to I_{ij}). Putting both parts together the total (translational and rotational) connection form reads

$$A=A_{\mathrm{tr}}+A_{\mathrm{rot}}=\frac{m_a\, dx_a^i}{m}\mathscr{E}_i+\left(-I^{ij}(\mathbf{r}_1,\ldots,\mathbf{r}_N)\left(\sum_a m_a\mathbf{r}_a\times d\mathbf{r}_a\right)_j\right)E_i\equiv\frac{\tilde{p}^i}{m}\mathscr{E}_i-I^{ij}\tilde{L}_j E_i$$

($\tilde{p}^i,\tilde{L}_i$ being the total linear and angular momentum one-forms, respectively, on M). Let $\gamma(t)\leftrightarrow\mathbf{r}_a(t)$ be some motion of the system, now. Then it is horizontal provided that $\langle A,\dot{\gamma}\rangle=0$, i.e.,

$$\frac{m_a\dot{x}_a^i(t)}{m}\mathscr{E}_i-\sum_a I^{ij}(\mathbf{r}_1(t),\ldots,\mathbf{r}_N(t))m_a(\mathbf{r}_a(t)\times\dot{\mathbf{r}}_a(t))_j E_i=0$$

or

$$m_a\dot{\mathbf{r}}_a(t)\equiv\mathbf{P}(t)=\mathbf{0},$$

$$\sum_a m_a\mathbf{r}_a\times\dot{\mathbf{r}}_a\equiv\mathbf{L}=\mathbf{0}.$$

Thus horizontal motion is such that the total (linear) momentum $\mathbf{P}$ as well as the total angular momentum $\mathbf{L}$ of the system vanish.

VI. CONCLUSIONS AND SUMMARY

In this article we show that (under some restrictions mentioned in Sec. III) given a natural Lagrangian system (TM,L) with symmetry G lifted from the configuration space M a connection in principle bundle $\pi:M\rightarrow M/G$ can be constructed. The connection form A is given by a remarkably simple explicit formula (11). It generalizes "angular momentum equals zero"[6] connection from Refs. 1 and 2, corresponding to the group $G=\mathrm{SO}(3)$. The construction of A makes use of the momentum map of the associated exact symplectic action of G on TM, making the link between the connection and conserved quantities explicit. A calculation shows that the vertical and horizontal subspaces are mutually orthogonal, which was used as the definition in Ref. 1.

APPENDIX A: SOME USEFUL FACTS CONCERNING THE *TM* GEOMETRY

Here we collect some more details on the constructions and objects on TM, used in the main text (see Ref. 3).

If $w \in T_xM$, its *vertical lift* to $v \in TM(\pi_M(v)=x)$ is the tangent vector in $t=0$ to the curve $t \mapsto v+tw$. The vector field (on TM) obtained in such a way from the vector field V on M is denoted by $V^{\uparrow}$. In canonical coordinates (x^i,v^i) on TM

$$V \equiv V^i(x)\partial_i \mapsto V^{\uparrow} \equiv V^i(x)\,\frac{\partial}{\partial v^i}.$$

Let V be a vector field on M, and let us denote its local flow Φ_t. Then the generator of the local flow $T\Phi_t$ on TM is by definition the *complete lift* $\tilde{V}$ of V. In coordinates

$$V \equiv V^i(x)\partial_i \mapsto \tilde{V} \equiv V^i(x)\,\frac{\partial}{\partial x^i} + V^i{}_{,j}(x)v^j\,\frac{\partial}{\partial v^i}.$$

If $w \in T_vTM$, then the map

$$S_v : T_vTM \to T_vTM,\ w \mapsto (\pi_* w)^{\uparrow}$$

(the lift being to v) is linear, giving rise to the (1,1)-tensor in T_vTM. This pointwise construction defines a (1,1)-tensor field S on TM (almost tangent structure≡vertical endomorphism), in coordinates $S=dx^i \otimes \partial/\partial v^i$. Its properties used in the main text are (easily verified in coordinates)

$$\mathscr{L}_{\tilde{V}} S = 0,$$

$$S(\tilde{V}) = V^{\uparrow}.$$

If B is (1,1)-tensor field on M, then its lift to TM is defined by $B^{\uparrow}(w):=(B(\pi_* w))^{\uparrow}$. Then $S=I^{\uparrow}$ (I being the *unit* tensor field on M).

APPENDIX B: THE CHANGE OF Ad* TO Ad VIA $\hat{h}^{-1}$

Let

$$h : \mathscr{G} \times \mathscr{G} \to \mathbb{R}$$

be nondegenerate bilinear form on $\mathscr{G}$. It defines the map

$$\hat{h} : \mathscr{G} \to \mathscr{G}^*$$

by $(a,b \in \mathscr{G})$

$$\langle \hat{h}(a), b\rangle_0 := h(a,b)$$

$(E_\alpha \mapsto h_{\alpha\beta}E^\beta)$. If h is Ad-invariant, i.e.,

$$h(\mathrm{Ad}_g\,a, \mathrm{Ad}_g\,b) = h(a,b),$$

then $\hat{h}$ satisfies

$$\mathrm{Ad}_g^* \circ \hat{h} = \hat{h} \circ \mathrm{Ad}_{g^{-1}}.$$

Therefore

$$R_g^*(\hat{h}^{-1} \circ \tilde{P}) = \hat{h}^{-1} \circ R_g^* \tilde{P} = \hat{h}^{-1} \circ \mathrm{Ad}_g^* \tilde{P} = \mathrm{Ad}_{g^{-1}}(\hat{h}^{-1} \circ \tilde{P});$$

i.e., if $\tilde{P} \in \Omega^1(M,\mathrm{Ad}^*)$, then $\hat{A} \equiv \hat{h}^{-1} \circ \tilde{P} \in \Omega^1(M,\mathrm{Ad})$.

APPENDIX C: TRANSFORMATION OF THE CONNECTION FORM INTO THE “CANONICAL” FORM

Let $\pi:P \to M$ be a principal bundle and let $\bar{A}\in\Omega^1(P,\mathrm{Ad})$ define the connection by $\mathrm{Hor}_p\ P:=\mathrm{Ker}\ \bar{A}_p$. By definition $\langle \bar{A}_p, X_a\rangle \in \mathscr{G}$, depending linearly on $a\in\mathscr{G}$.

Then

$$\langle \bar{A}_p, X_a\rangle = C(p)(a), \tag{C1}$$

where

$$C(p):\mathscr{G}\to\mathscr{G}$$

is invertible (lest some X_a be horizontal). From

$$R_g^*\bar{A}=\mathrm{Ad}_{g^{-1}}\ \bar{A}$$

and Eq. (C1) one obtains

$$C(pg)=\mathrm{Ad}_{g^{-1}}\circ C(p)\circ \mathrm{Ad}_g$$

and therefore

$$A_p := C^{-1}(p)\circ \bar{A}_p \tag{C2}$$

already has the standard properties

$$\begin{aligned} R_g^* A &= \mathrm{Ad}_{g^{-1}}\ A, \\ \langle A, X_a\rangle &= a. \end{aligned} \tag{C3}$$

This shows that although the standard requirement (C3) on connection form can be modified to a more general one (C1), it can be always simplified back to the ‘‘canonical’’ choice (C3) via (C2).

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