AIP Journal of Mathematical Physics

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Citation: J. Math. Phys. **29**, 1079 (1988); doi: 10.1063/1.527945
View online: http://dx.doi.org/10.1063/1.527945
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v29/i5
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A representation of SO($p+q,p+q$) on SO(p,q)

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(Received 7 August 1986; accepted for publication 22 December 1987)

A construction of a representation of SO($p+q,p+q$) by operators on SO($p+q$) is presented, connected with a relativistic top kinematics. In addition to a first-order differential operator part there is a multiplicative term containing a parameter λ in some of the generators. It is shown by the explicit evaluation of the corresponding Casimir invariants that the representation $(\lambda,...,\lambda)$, $\lambda = 0,\frac{1}{2},1,...$, is realized by this construction.

I. INTRODUCTION

Trying to formulate a relativistic theory of a top without enlarging its nonrelativistic configuration space [SO(3) group manifold], one is faced with a problem of realizing SO(3,1) on SO(3), which reduces (if standard generators of rotations are used for $\mathbf{s}$) to a realization of boost generators $\mathbf{N}$. The ansatz

$$N_i = B_{ik}s_k + b_i \tag{1}$$

leads to a system of equations

$$\begin{aligned}&\varepsilon_{jmn}B_{im}B_{nk} + \varepsilon_{kmn}B_{im}B_{jn} + \varepsilon_{imn}B_{jn}B_{mk} = \varepsilon_{ijk},\\ &(B_{im}\varepsilon_{jmn} - B_{jm}\varepsilon_{imn})b_n = 0,\end{aligned} \tag{2}$$

for unknown quantities B_{ik} and b_i.[1] The general solution is

$$B_{ik} = \varepsilon_{ikj}n_j + \alpha n_i n_k, \quad \alpha,\beta\in\mathbb{C},$$

$$b_i = \beta n_i, \quad \mathbf{n} = \text{unit vector},$$

i.e.,

$$\mathbf{N} = -\mathbf{n}\times\mathbf{s} + \alpha\mathbf{n}(\mathbf{n}\cdot\mathbf{s}) + \beta\mathbf{n}. \tag{3}$$

It is shown in Ref. 1 that the free parameters α,β are connected with Casimir invariants $c_1^{\mathrm{L}} = -\mathbf{s}\cdot\mathbf{N}$, $c_0^{\mathrm{L}} = -(\mathbf{s}^2 - \mathbf{N}^2)$ of the Lorentz group, i.e., one can use them for the fixation of the spin of a top. (The case of spin $\frac{1}{2}$ was studied before in Ref. 2; it corresponds to $\alpha = 0$, $\beta = \frac{1}{2}$.)

In this paper we present a generalization of these results. We descibe a systematic construction of the generators of SO($p+q,p+q$) acting on functions on SO($p+q$). As in the above-mentioned case, nondifferential (multiplicative) terms containing a free parameter λ occur in some of the generators. By explicit evaluation of the corresponding Casimir invariants we identify the representation with $(\lambda,...,\lambda)$.

II. CONSTRUCTION OF THE GENERATORS

Let $\mathbf{e}_\alpha$, $\alpha = 1,2,...,(p+q)\equiv N$, form an orthonormal right-handed system of vectors in $M^{p,q}$, i.e.,

$$\eta^{\alpha\beta}e_{\alpha i}e_{\beta j} = \eta_{ij}, \tag{4}$$

$$\eta^{ij}e_{\alpha i}e_{\beta j} = \eta_{\alpha\beta}, \tag{5}$$

$$\varepsilon^{i_1\cdots i_N}e_{\alpha_1 i_1}\cdots e_{\alpha_N i_N} = (-1)^q\varepsilon_{\alpha_1\cdots\alpha_N}, \tag{6}$$

$$\varepsilon^{\alpha_1\cdots\alpha_N}e_{\alpha_1 i_1}\cdots e_{\alpha_N i_N} = (-1)^q\varepsilon_{i_1\cdots i_N} \tag{7}$$

hold, where

$$\eta^{ij}\equiv\mathrm{diag}(\underbrace{1,...,1}_{p}, \underbrace{-1,...,-1}_{q}), \tag{8}$$

$$\eta^{\alpha\beta}\equiv\mathrm{diag}(\underbrace{1,...,1}_{p}, \underbrace{-1,...,-1}_{q}), \tag{9}$$

and $e_{\alpha i}$ is the ith component of $\mathbf{e}_\alpha$. Generators of SO(p,q) can be represented by the antisymmetric tensor $S_{ij} = -S_{ji}$, $i,j = 1,...,(p+q)$, obeying

$$[S_{ij},S_{kl}] = -(n_{k[i}S_{j]l} + S_{k[j}n_{i]l}), \tag{10}$$

$$[S_{ij},e_{\alpha k}] = e_{\alpha[i}\eta_{j]k}, \tag{11}$$

where

$$A_{[kl]}\equiv A_{kl} - A_{lk}. \tag{12}$$

[The $e_{\alpha k}$ are to be expressed as functions of the coordinates on SO(p,q) and the S_{ij} are first-order differential operators with respect to the latter.]

One can construct additional operators on SO(p,q) now, combining $e_{\alpha i}$ and the S_{ij}:

$$S_{\alpha i}\equiv\eta^{kl}e_{\alpha k}S_{li}\equiv e_\alpha{}^kS_{ki}, \tag{13}$$

$$S_{\alpha\beta}\equiv e_\alpha{}^ie_\beta{}^jS_{ij}\equiv e_{\beta j}S_\alpha{}^j, \tag{14}$$

i.e., transforming successively the vector components of the S_{ij} to the scalar ones by means of "vielbein" $e_{\alpha i}$. Then using (10), (11) we obtain

$$[S_{ij},S_{\alpha k}] = S_{\alpha[i}\eta_{j]k}, \tag{15}$$

$$[S_{ij},S_{\alpha\beta}] = 0, \tag{16}$$

$$[S_{\alpha i},S_{\beta j}] = \eta_{\alpha\beta}S_{ij} - \eta_{ij}S_{\alpha\beta}, \tag{17}$$

$$[S_{\alpha i},S_{\beta\gamma}] = \eta_{\alpha[\gamma}S_{\beta]i}, \tag{18}$$

$$[S_{\alpha\beta},S_{\gamma\delta}] = \eta_{\gamma[\alpha}S_{\beta]\delta} + S_{\gamma[\beta}\eta_{\alpha]\delta}, \tag{19}$$

from which we deduce that the $S_{\alpha i}$ form $N\equiv(p+q)$ vectors and that the $S_{\alpha\beta}$ are scalars, both with respect to "right" rotations (generated by the S_{ij}). We can also change our point of view and classify objects according to their transformational properties with respect to "left" rotations (generated by the $S_{\alpha\beta}$). Then the S_{ij} are "scalars" and the $S_{\alpha i}$ form $N\equiv(p+q)$ "vectors" for fixed i on each.

This situation, for the special case $p = 3$, $q = 0$, is to some extent familiar from the theory of the nonrelativistic quantum-mechanical top,[3] where the projections on the laboratory as well as on the body axes of the quantities in question are used [including a change of sign in (19) in comparison with (10)]. "Mixed" operators $S_{\alpha i}$, however, are not discussed there at all.

It is not difficult to determine the algebra generated by S_{ij}, $S_{\alpha i}$, $S_{\alpha\beta}$. In order to do this we switch to a more compact

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notation. Let us introduce an index $A \equiv (i,\alpha)$, $A = 1,...,2(p+q)$, $i = 1,...,N \equiv (p+q)$, $\alpha = N+1,...,2N$ (we changed the numeration of Greek indices). All generators form the components of a single object $S_{AB} = -S_{BA}$ now, and (10) and (15)–(19) read

$$[S_{AB},S_{CD}] = -(g_{C[A}S_{B]D} + S_{C[B}g_{A]D}), \tag{20}$$

where

$$g_{ij} \equiv \eta_{ij}, \tag{21}$$

$$g_{\alpha\beta} \equiv -\eta_{\alpha\beta}, \tag{22}$$

$$g_{\alpha i} \equiv g_{i\alpha} \equiv 0, \tag{23}$$

i.e.,

$$g_{AB} \equiv \mathrm{diag}(\underbrace{1,...,1}_{p}, \underbrace{-1,...,-1}_{q+p}, \underbrace{1,...,1}_{q}), \tag{24}$$

which reveals that the S_{AB} generate $\mathrm{SO}(p+q,p+q)$.

So far all generators S_{AB} were first-order differential operators on $\mathrm{SO}(p,q)$. Now we add a multiplicative term to $S_{\alpha i}$, introducing

$$\Sigma_{ij} \equiv S_{ij}, \tag{25}$$

$$\Sigma_{\alpha i} \equiv S_{\alpha i} + \lambda e_{\alpha i}, \tag{26}$$

$$\Sigma_{\alpha\beta} \equiv S_{\alpha\beta}, \tag{27}$$

where λ is a constant. Direct computation yields

$$[\Sigma_{AB},\Sigma_{CD}] = -(g_{C[A}\Sigma_{B]D} + \Sigma_{C[B}g_{A]D}) \tag{28}$$

independently of λ. That means that the Σ_{AB} form the generators of $\mathrm{SO}(p+q,p+q)$ as well.

Note that in the case $p=3$, $q=0$ discussed above we obtain—in addition to

$$s_i \equiv \tfrac{1}{2}\varepsilon_{ijk}S^{jk}, \tag{29}$$

$$s'_\alpha \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}S^{\beta\gamma}, \tag{30}$$

forming the $\mathrm{SO}(3)\otimes\mathrm{SO}(3)$ algebra of the laboratory and body projections of the angular momentum vector $\mathbf{s}$ of a top, respectively—three new vector operators Σ_α, $\alpha = 1,2,3$, where

$$\Sigma_{\alpha i} = e_\alpha{}^k S_{ki} + \lambda e_{\alpha i} = e_{\alpha k}\varepsilon_{kij}s_j + \lambda e_{\alpha i} \equiv (-\mathbf{e}_\alpha \times \mathbf{s} + \lambda \mathbf{e}_\alpha)_i, \tag{31}$$

closing together with s_i and s'_α to $\mathrm{SO}(3,3)$ and thus offering a possibility of the relativization of the description of a top. We notice that (3) is just of the form of (31). [According to Ref. 1, for finite-dimensional representations of a Lorentz group one has to choose $\alpha = 0$, $\beta = 0,\frac{1}{2},1,...$ in (3).]

III. SPECIFICATION OF THE REPRESENTATION

In this section we evaluate Casimir invariants formed from (25)–(27), which enable us to specify the representation realized by this construction.

Several papers deal with the problem of the explicit form and eigenvalues of independent Casimir operators of classical groups (e.g., see Refs. 4–8). It was established in Ref. 7 that $N \equiv (p+q)$ invariants are to be evaluated in our case, viz., $(N-1)$ scalar operators C_n, $n = 2,4,...,2(N-1)$, and a pseudoscalar operator C'_N, where

$$C_n \equiv \Sigma^{A_1}{}_{A_2}\Sigma^{A_2}{}_{A_3}\cdots\Sigma^{A_n}{}_{A_1} \equiv (\Sigma^n)^A{}_A, \tag{32}$$

$$C'_N \equiv \varepsilon_{A_1B_1\cdots A_NB_N}\Sigma^{A_1B_1}\cdots\Sigma^{A_NB_N}. \tag{33}$$

A. Evaluation of C_n

Evaluation of C_n is based on the identity [specific for the construction (25)–(27)]

$$(\Sigma^2)^{AB} \equiv \Sigma^{AC}\Sigma_C{}^B = \lambda(\lambda+N-1)g^{AB} + (N-1)\Sigma^{AB}, \tag{34}$$

proved in Appendix A. This makes it possible to express an arbitrary "power" of Σ by Σ itself and a constant:

$$(\Sigma^n)^{AB} = a(n)g^{AB} + b(n)\Sigma^{AB} \tag{35}$$

(a and b can depend on λ and N in general as well). Multiplying (35) by $\Sigma_B{}^C$ we obtain recurrence relations

$$\begin{aligned} a(n+1) &= a(2)b(n), \\ b(n+1) &= a(n) + b(2)b(n), \end{aligned} \tag{36}$$

or

$$\begin{pmatrix} a \\ b \end{pmatrix}_{n+1} = R\begin{pmatrix} a \\ b \end{pmatrix}_n, \tag{37}$$

where

$$R \equiv \begin{pmatrix} 0 & a(2) \\ 1 & b(2) \end{pmatrix} = \begin{pmatrix} 0 & \lambda(\lambda+N-1) \\ 1 & N-1 \end{pmatrix}, \tag{38}$$

so that

$$\begin{pmatrix} a \\ b \end{pmatrix}_n = R^{n-1}\begin{pmatrix} a \\ b \end{pmatrix}_1 = R^{n-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{39}$$

Evaluation of the necessary power of R gives

$$\begin{pmatrix} a \\ b \end{pmatrix}_n = \frac{1}{2\lambda+N-1}\begin{pmatrix} \lambda(\lambda+N-1)\{(\lambda+N-1)^{n-1} - (-\lambda)^{n-1}\} \\ (\lambda+N-1)^n - (-\lambda)^n \end{pmatrix}, \tag{40}$$

i.e.,

$$(\Sigma^n)^{AB} = \frac{\lambda(\lambda+N-1)\{(\lambda+N-1)^{n-1} - (-\lambda)^{n-1}\}}{2\lambda+N-1}g^{AB} + \frac{(\lambda+N-1)^n - (-\lambda)^n}{2\lambda+N-1}\Sigma^{AB}. \tag{41}$$

Then

$$C_n(\lambda,N) \equiv (\Sigma^n)^A{}_A = a(n)g^A_A = 2Na(n) = \frac{2N\lambda(\lambda+N-1)}{2\lambda+N-1}\{(\lambda+N-1)^{n-1} - (-\lambda)^{n-1}\}. \tag{42}$$

We observe that for each n, C_n is a number—not a differential operator, as is usually the case—when the harmonic analysis approach on corresponding homogeneous space is used. This means that the explicit construction of the functions on $\mathrm{SO}(p,q)$ in the space of which our representation of $\mathrm{SO}(p+q,p+q)$ is realized reduces to solving the eigenvalue equations for the generators of a Cartan subalgebra (e.g., $\Sigma_{12},...,\Sigma_{2N-1,2N}$), i.e., only first-order differential equations are to be solved.

B. Evaluation of C'_N

For evaluation of $C'_N(\lambda,N)$ it is useful to realize that (41) reveals Σ_{AB} to be the only antisymmetric second-rank tensor (and g_{AB} the only symmetric one as well) available.That means then that also

$$\varepsilon_{A_1B_1A_2B_2\cdots A_NB_N}\Sigma^{A_2B_2}\cdots\Sigma^{A_nB_N}\sim\Sigma_{A_1B_1} \tag{43}$$

holds, which can be readily verified explicitly for not too large N, e.g., the coefficient of proportionality is 2 for $N=2$, $8(1+\lambda)$ for $N=3$, etc. (see Appendix B). Multiplication of both sides of (43) by $\Sigma^{A_1B_1}$ and taking into account (42) leads to the conclusion that C'_N does not contain differential operators, but reduces instead to the multiplication by a number, too (as was the case for all C_n). Thus we have to extract just this nondifferential part of (33), ignoring the differential terms completely (they cancel). We can introduce formally a symbol MULT, which when applied to any combined differential-multiplicative operator leaves its multiplicative part only. The above-mentioned conclusion can be written then as

$$C'_N(\lambda,N)=\mathrm{MULT}\ C'_N(\lambda,N). \tag{44}$$

Nonvanishing components of a Levi-Civita tensor come from the cases where all indices are mutually different; in particular, an equal number of Latin and Greek indices should occur. The only distribution of indices which survives under the MULT symbol is one in which a pair of Latin and Greek indices stands on each Σ. [In the opposite case there is at least one Σ with both indices Latin, and it can be shifted to the right-hand side of the expression (33) giving a differential operator.] Taking into account two possibilities for the order of indices on each Σ ($i\alpha$ and αi) we can write

$$\mathrm{MULT}\ C'_N(\lambda,N)=2^N\mathrm{MULT}\ \varepsilon_{i_1\alpha_1\cdots i_N\alpha_N}\Sigma^{i_1\alpha_1}\cdots\Sigma^{i_N\alpha_N}, \tag{45}$$

and, using the result of Appendix C,

$$\begin{aligned}\mathrm{MULT}\ C'_N(\lambda,N)&=(-1)^{N(N-1)/2}2^N\varepsilon^{i_1\cdots i_N}\varepsilon^{\alpha_1\cdots\alpha_N}\\&\quad\times(S_{\alpha_1i_1}+\lambda e_{\alpha_1i_1})\cdots(S_{\alpha_Ni_N}+\lambda e_{\alpha_Ni_N}).\end{aligned} \tag{46}$$

With the help of the commutator

$$\begin{aligned}&\varepsilon^{i_1\cdots i_N}\varepsilon^{\alpha_1\cdots\alpha_N}[S_{\alpha_li_l},e_{\alpha_{p_1}i_{p_1}}\cdots e_{\alpha_{p_k}i_{p_k}}]\\&\quad=k\varepsilon^{i_1\cdots i_N}\varepsilon^{\alpha_1\cdots\alpha_N}e_{\alpha_li_l}e_{\alpha_{p_1}i_{p_1}}\cdots e_{\alpha_{p_k}i_{p_k}}\end{aligned} \tag{47}$$

[for any $k,l,p_m=1,...,(N-1)$, $m=1,...,k$], we can perform the MULT operation in (46) explicitly and obtain a polynomial of order N in λ. Let us determine the coefficient standing by λ^r, $r=1,...,N$. It comes from all cases in which r "e" 's and $(N-r)$ "S" 's are chosen from the brackets in (46) ($e_{\alpha_Ni_N}$ should always come from the last one). Then the desired coefficient is

$$(-1)^{N(N-1)/2}2^NN!\sum_{p_1,...,p_{N-r-1}=1}^{N-1}\prod_{l=1}^{N-r-1}p_l \quad (\text{all } p_i \text{ different}), \tag{48}$$

where also [see (6),(7)]

$$\varepsilon^{i_1\cdots i_N}\varepsilon^{\alpha_1\cdots\alpha_N}e_{\alpha_1i_1}\cdots e_{\alpha_Mi_M}=N! \tag{49}$$

was used. One can compare it with the coefficient standing by λ^r in the expression

$$(\lambda+N-1)\cdots(\lambda+1)\lambda, \tag{50}$$

which, taking the numbers from $(N-r-1)$ brackets and λ from the rest, is just

$$\sum_{p_1,...,p_{N-r-1}=1}^{N-1}\prod_{l=1}^{N-r-1}p_l \quad (\text{all } p_i \text{ different}), \tag{51}$$

so that

$$\begin{aligned}&\mathrm{MULT}\ \varepsilon^{i_1\cdots i_N}\varepsilon^{\alpha_1\cdots\alpha_N}(S_{\alpha_1i_1}+\lambda e_{\alpha_1i_1})\cdots(S_{\alpha_Ni_N}+\lambda e_{\alpha_Ni_N})\\&\quad=N!\lambda(\lambda+1)\cdots(\lambda+N-1),\end{aligned} \tag{52}$$

and

$$\begin{aligned}&C'_N(\lambda,N)\\&\quad=(-1)^{N(N-1)/2}2^NN!\lambda(\lambda+1)\cdots(\lambda+N-1).\end{aligned} \tag{53}$$

C. Comparison with the general results: Conclusion

Let us study $C'_N(\lambda,N)$ first. It is known[7] that its value is

$$\begin{aligned}C'_N(m_1,...m_N)&=(-1)^{N(N-1)/2}2^NN!(m_1+N-1)\\&\quad\times\cdots(m_{N-1}+1)m_N\end{aligned} \tag{54}$$

for the representation $(m_1,...,m_N)$. Our construction thus corresponds to the case $(\lambda,...,\lambda)$. However, the invariants C_n, $n=2,4,...,2(N-1)$, are to be compared, too. For the evaluation of C_n for $(\lambda,...,\lambda)$ we make use of the results of Ref. 8, where the generating function for Casimir invariants of all classical groups was derived. In the case of interest to us this function reads

$$\begin{aligned}G(z)&=2N\frac{1+(\lambda-N)z}{(1+\lambda z)(1-Nz)}\\&\quad+\frac{2N\lambda(\lambda-1)}{(1+\lambda z)(1-Nz)[1-(\lambda+N-1)z]}\end{aligned} \tag{55}$$

and

$$G(z)=\sum_{n=0}^{\infty}C_nz^n, \tag{56}$$

so that

$$C_n=G^{(n)}(0)/n!. \tag{57}$$

The explicit calculation yields

$$C_n=\frac{2N\lambda(\lambda+N-1)}{2\lambda+N-1}\{(\lambda+N-1)^{n-1}-(-\lambda)^{n-1}\}, \tag{58}$$

in agreement with (42); that makes the identification with $(\lambda,...,\lambda)$ complete.

ACKNOWLEDGMENT

I would like to thank Professor M. Petráš for turning my attention to this problem and for permanent interest in the course of my work.

APPENDIX A: PROOF OF (34)

The proof is to be done for all four possible cases of pairs of indices, (ij), $(i\alpha)$, (αi), $(\alpha\beta)$. We restrict ourselves to the first case here; the rest can be done in the same way. We have

$$\begin{aligned}\Sigma_{iA}\Sigma^{A}{}_{j} &\equiv g^{AB}\Sigma_{iA}\Sigma_{Bj} = g^{kl}\Sigma_{lj} + g^{\alpha\beta}\Sigma_{i\alpha}\Sigma_{\beta j} = \eta^{kl}S_{ik}S_{ij} - \eta^{\alpha\beta}\Sigma_{i\alpha}\Sigma_{\beta j} = S_{ik}S^{k}{}_{j} + (e_{\alpha}{}^{m}S_{mi} + \lambda e_{\alpha i})(e^{\alpha n}S_{nj} + \lambda e^{\alpha}{}_{j})\\ &= S_{ik}S^{k}{}_{j} + e_{\alpha}{}^{m}e^{\alpha n}S_{mi}S_{nj} + e^{\alpha m}[S_{mi},e_{\alpha n}]S^{n}{}_{j} + \lambda e^{\alpha m}[S_{mi},e_{\alpha j}] + \lambda e_{\alpha}{}^{m}e^{\alpha}{}_{j}S_{mi} + \lambda e_{\alpha i}e^{\alpha n}S_{nj} + \lambda^{2}e_{\alpha i}e^{\alpha}{}_{j}\\ &= (N-1)S_{ij} + \lambda(\lambda+N-1)\eta_{ij} \equiv \lambda(\lambda+N-1)g_{ij} + (N-1)\Sigma_{ij}.\end{aligned}$$

(Notice that two "unpleasant" second-order terms cancel each other.) One should be careful when contracting Greek indices as to whether $g^{\alpha\beta} = -\eta^{\alpha\beta}$ [e.g., (45)] or $\eta^{\alpha\beta}$ is understood implicitly there.

APPENDIX B: (43) FOR $N=3$

We are to find the proportionality coefficient $\gamma(\lambda)$ in

$$\varepsilon_{ABCDEF}\Sigma^{CD}\Sigma^{EF} = \gamma(\lambda)\Sigma_{AB}.$$

Let us compute the $\alpha\beta$ component of the left-hand side:

$$\begin{aligned}\varepsilon_{\alpha\beta ABCD}\Sigma^{AB}\Sigma^{CD} &= 2\varepsilon_{\alpha\beta\gamma ijk}\Sigma^{\gamma i}\Sigma^{jk} + 2\varepsilon_{\alpha\beta jk\gamma i}\Sigma^{jk}\Sigma^{\gamma i} = -4\varepsilon_{ijk\alpha\beta\gamma}g^{\gamma A}\Sigma_{A}{}^{i}\Sigma_{jk} = 4\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}(e^{\gamma i}S_{l}{}^{i} + \lambda_{l}{}^{\gamma i})S^{jk}\\ &= 4\epsilon_{ijk}e_{\alpha m}e_{\beta n}(-1)^{q}(\epsilon^{mnl}S_{l}{}^{i}S^{jk} + \lambda\varepsilon^{mni}S^{jk}) = 4(-1)^{q}\delta^{m}_{[i}\delta^{n}_{j]}e_{\alpha m}e_{\beta n}[s^{j},s^{i}] + 8\lambda e_{\alpha m}e_{\beta m}S^{mn}\\ &= 8(1+\lambda)S_{\alpha\beta} \equiv 8(1+\lambda)\Sigma_{\alpha\beta},\end{aligned}$$

where

$$s_{i} \equiv \tfrac{1}{2}\varepsilon_{ijk}S^{jk}, \quad S_{ij} = (-1)^{q}\varepsilon_{ijk}s^{k},$$
$$[s_{i},s_{j}] = -\varepsilon_{ijk}s^{k} = -(-1)^{q}S_{ij},$$
$$\varepsilon_{\alpha\beta\gamma}e^{\gamma j} = (-1)^{q}\varepsilon^{mnj}e_{\alpha m}e_{\beta n},$$
$$\varepsilon_{ijk}\varepsilon^{mnk} = (-1)^{q}\delta^{m}_{[i}\delta^{n}_{j]}$$

was used, so that

$$\varepsilon_{ABCDEF}\Sigma^{CD}\Sigma^{EF} = 8(1+\lambda)\Sigma_{AB},$$

for $N=3$. Multiplying it by Σ^{AB} we obtain

$$\begin{aligned}C_3'(\lambda,3) &= -8(1+\lambda)\Sigma^{AB}\Sigma_{BA}\\ &\equiv -8(1+\lambda)C_2(\lambda,3)\\ &= -8(1+\lambda)(6\lambda(\lambda+2))\\ &\equiv -48\lambda(\lambda+1)(\lambda+2)\end{aligned}$$

in agreement with (53), for $N=3$. The same procedure is possible for arbitrary N.

APPENDIX C: PROOF OF (46)

We are to prove

$$\varepsilon_{i_1\alpha_1\cdots i_N\alpha_N} = (-1)^{N(N-1)/2}\varepsilon_{i_1\cdots i_N\alpha_1\cdots\alpha_N}.$$

Clearly

$$\varepsilon_{i_1\alpha_1\cdots i_N\alpha_N} = \varkappa(N)\varepsilon_{i_1\cdots i_N\alpha_1\cdots\alpha_N},$$

where $\varkappa(N)$ is 1 or (-1). Then

$$\begin{aligned}\varepsilon_{i_1\alpha_1\cdots i_N\alpha_N i_{N+1}\alpha_{N+1}} &= \varkappa(N)\varepsilon_{i_1\cdots i_N\alpha_1\cdots\alpha_N i_{N+1}\alpha_{N+1}}\\ &= (-1)^{N}\varkappa(N)\varepsilon_{i_1\cdots i_{N+1}\alpha_1\cdots\alpha_{N+1}}\\ &= \varkappa(N+1)\varepsilon_{i_1\cdots i_{N+1}\alpha_1\cdots\alpha_{N+1}},\end{aligned}$$

so that

$$\begin{aligned}\varkappa(N+1) &= (-1)^{N}\varkappa(N) = (-1)^{N}(-1)^{N-1}\varkappa(N-1)\\ &= \cdots = \varkappa(1)\prod_{k=0}^{N-1}(-1)^{N-k}\\ &= \prod_{k=1}^{N}(-1)^{k} = (-1)^{\Sigma_{k=1}^{N}k} = (-1)^{N(N+1)/2},\end{aligned}$$

and

$$\varkappa(N) = (-1)^{N(N-1)/2}.$$

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