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U(1)-GAUGE STRUCTURE ASSOCIATED WITH A MOTION OF A GUITAR STRING

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U(1)-connection equivalent to the kinematics of a classical U(1)-gauge theory (electrodynamics) in 1+1 dimensions is associated with a motion of a string. The connection is non-integrable, leading to standard path-dependence (anholonomy) phenomena.

1. Introduction

In the last decade it became clear that the applications of the ideas and techniques of the mathematical theory of connections in fibre bundles are not restricted to the well established context of gauge fields in elementary particle physics (including general relativity as a peculiar special case) but rather this theory was found to be natural and useful in several quite different physical situations, too [1-6].

In this paper we show how a connection can be associated with such an elementary mechanical system as an ordinary (e.g. guitar) string is. This connection has non-zero curvature giving rise to typical path-dependence (anholonomy) phenomena.

2. A wheel on a road

Let us begin with the situation shown in Fig.1 which represents a wheel on a road. Configuration space of this system is $P = R\times$ U(1), (x,φ) being standard local coordinates. The condition that the wheel *rolls* is expressed by the differential constraint

$$a d\varphi = dx \tag{1}$$

(a - radius of the wheel), or equivalently

$$\theta \equiv i(d\varphi - \frac{1}{a}dx) = 0 \tag{2}$$

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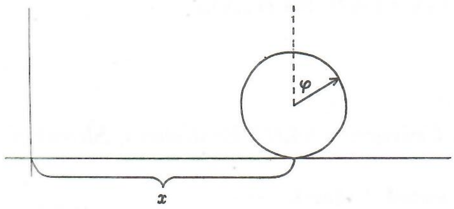

Fig. 1. A wheel on a road

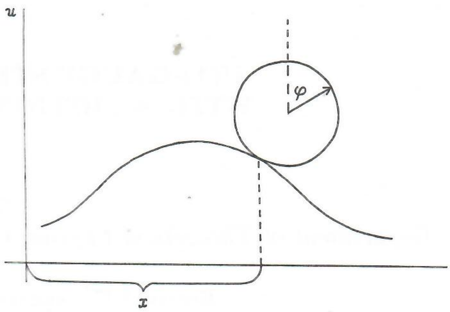

Fig. 2. A wheel on a road given by $u(x)$.

A comparison with the most general connection 1-form on the $U(1)$-principal fibre bundle (PFB)

$$\pi : R \times U(1) \to R \tag{3}$$

namely

$$\theta = i(d\varphi + A(x)dx) \tag{4}$$

shows that the configuration space of a rolling wheel is naturally endowed with the structure of the $U(1)$ - *PFB with a connection*; according to (2),(4)

$$A(x) = -\frac{1}{a}.$$

In other words the change of the angle φ due to the rolling of the wheel can be interpreted in terms of a *parallel transport.*

A slightly more general case of a road which is not necessarily horizontal (see Fig.2) but rather given by a function $u(x)$ results in the constraint

$$d\varphi = \frac{1}{a}\sqrt{1 + u_{,x}{}^2}dx - d\alpha \tag{5}$$

where

$$\alpha(x) \equiv arctg(u_{,x}) \tag{6}$$

i.e.

$$A(x) = -\frac{1}{a}\sqrt{1 + u_{,x}{}^2} + \alpha_{,x}$$

A connection given by (4) has, however, always zero curvature. This is evident from Fig.2 (going there and back arbitrarily the wheel ends up with its original orientation, i.e. the parallel transport along arbitrary loop does not change the orientation of the wheel) as well as formally (one dimensional base of the PFB).
Note: Usually the constraint (1) is solved explicitly

$$\varphi(x) = \varphi_0 + \frac{1}{a}x \tag{7}$$

and one of two variables x,φ is excluded (resulting configuration space being only 1-dimensional). This is possible just because of the zero curvature (horizontal distribution is integrable) and the spirals (7) on the cylinder $P \equiv R \times U(1)$ represent the horizontal submanifold foliation on P.

3. A wheel on a string

1-dimensional base of the PFB (3) makes it impossible to have non-zero curvature (it is 2-form). Thus we are to modify the system under consideration in such a way as to obtain at least 2-dimensional base. This can be achieved by adding one more space dimension, i.e. to study a wheel on a plane [7]. There is, however, another possibility, which is discussed in this paper, viz. the addition of the *time* dimension, which is to say that the shape of the road is allowed to change in time, being given by a function $u(t,x)$, now. A simple realization is a wheel rolling on a transversaly vibrating *string* (elastic properties of the latter are not relevant as well as the usual limitation to small amplitudes).

The configuration space is now a total space of $U(1)$-PFB

$$\pi : R^2 \times U(1) \to R^2 \tag{8}$$

The constraint of rolling is still given formally by the formulas (5), (6), but α as well as $u_{,x}$ depend on time, now. Thus the corresponding connection form is

$$\theta = i(d\varphi - \frac{1}{a}\sqrt{1 + u_{,x}{}^2(t,x)}dx + d\alpha(t,x)) \tag{9}$$

or

$$\theta = i(d\varphi + A_0(x^0,x^1)dx^0 + A_1(x^0,x^1)dx^1) \tag{10}$$

($x^0 \equiv t, x^1 \equiv x$) where

$$A_0(t,x) \equiv \alpha_{,t}$$

$$A_1(t,x) \equiv -\frac{1}{a}\sqrt{1 + u_{,x}{}^2} + \alpha_{,x}$$

A computation of the curvature 2-form leads to

$$\Omega =: D\theta = d\theta = -\frac{1}{a}\frac{u_{,xt}}{\sqrt{1 + u_{,x}{}^2}}dt \wedge dx \tag{11}$$

Vanishing of this expression is equivalent to

$$u(t,x) = f(t) + g(x)$$

This is, however, a motion of a string as a *rigid body*, which is a trivial part of the motion (e.g. it is zero if the edges are kept fixed).

Thus any non-trivial motion of a string results in non-zero curvature Ω.

4. Gauge transformations

The orientation of a wheel which corresponds to φ=0 is quite arbitrary at each point x of the string in any time t. The choice of this orientation, e.g. the particular one ("up") according to Fig.2 (it represents, however, only one possibility out of infinitely many equally legal candidates) defines a section σ of the bundle (8)

$$\sigma : R^2 \to R^2 \times U(1)$$

which in these coordinates reads

$$(t, x) \mapsto (t, x, 0)$$

and a *gauge potential* A

$$\sigma^* \theta =: iA \equiv i(A_0 dx^0 + A_1 dx^1)$$

(u(1)-valued 1-form on the base R^2). If a different choice is made in such a way that the new φ=0 corresponds to old $\varphi=\chi(t,x)$, a new section σ' is defined

$$(t, x) \mapsto (t, x, \chi(t, x))$$

(in original coordinates). Then the new gauge potential A' is given by

$$\sigma'^* \theta =: iA' = i(A + d\chi)$$

i.e.

$$A_\mu \mapsto A_\mu + \partial_\mu \chi$$

- a standard U(1)-gauge theory ("1+1 electrodynamics") formula. The corresponding *field strength* F

$$\sigma^* \Omega =: iF = \frac{i}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = iF_{01} dt \wedge dx$$

is gauge-independent

$$F \mapsto F' = F$$

(F_{01} corresponds to "electric field" component E_1).

5. Anholonomy phenomena

Non-zero curvature of a connection (9) leads to path-dependence of the change of the angle φ. Let γ_1, γ_2 be two different paths in the (t,x)-plane R^2 connecting two points $Q \equiv (t_Q, x_Q)$, $Q' \equiv (t_{Q'}, x_{Q'})$. According to (9), (10) the total change of the angle φ due to rolling of the wheel is

$$(\Delta\varphi)_a = -\int_{\gamma_a} A$$

(a=1,2). Clearly it is gauge-dependent

$$(\Delta\varphi)_a \mapsto (\Delta\varphi)'_a = (\Delta\varphi)_a + \chi(Q) - \chi(Q')$$

but the *difference*

$$(\Delta\varphi)_{12} =: (\Delta\varphi)_2 - (\Delta\varphi)_1 = \int_{\gamma_1} A - \int_{\gamma_2} A$$

is already gauge-independent quantity since

$$(\Delta\varphi)_{12} = \oint_C A = \int_D F$$

($C \equiv \partial D = \gamma_1 - \gamma_2$ is the closed path enclosing the domain D in R^2). In particular (for γ_2 parallel to t-axis) a wheel rolling there and back (γ_1) changes its orientation differently in comparison with the fiducial one remaining (γ_2) all the time in the position x_Q (cf. Sec. 2.).

6. Discussion

Let

$$\ell(t,x) =: \sqrt{1 + u_{,x}{}^2(t,x)}$$

(it is a measure of the length changes of the elements of a string). (9) can be written as

$$\theta = i(d(\varphi + ln\ell(t,x)) - \frac{1}{a}\ell(t,x)dx)$$

and

$$\Omega = -\frac{i}{a}d\ell \wedge dx = -\frac{i}{a}\ell_{,t}dt \wedge dx$$

This alternative form of (11) reveals clearly the physical origin of the occurence of a curvature in this problem. The rolling wheel measures simply the elementary *lengths* (just like the ancient Egyptians did). If the function ℓ depends on time, a change of the angle $\Delta\varphi$ corresponding to some element of a string depends on time, too. In particular, going there and back can result in non-zero net angle change since the return occurs at different time and consequently the corresponding lengths are in general different.

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