AIP | Journal of Mathematical Physics

Transformation of the solutions of the Maxwell equations on a Lorentzian manifold by means of a certain class of diffeomorphisms

Marián Fecko

Citation: J. Math. Phys. 36, 1198 (1995); doi: 10.1063/1.531114
View online: http://dx.doi.org/10.1063/1.531114
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v36/i3
Published by the American Institute of Physics.

Transformation of the solutions of the Maxwell equations on a Lorentzian manifold by means of a certain class of diffeomorphisms

Marián Fecko[a)]
Department of Theoretical Physics, Comenius University, 84215 Bratislava, Slovakia

(Received 26 May 1994; accepted for publication 19 October 1994)

Poincaré invariance of the Maxwell equations in Minkowski space enables one to generate in a usual way new solutions (corresponding to new sources) from a given one by means of the pullback operation. A method is proposed generalizing this to a wider class of diffeomorphisms $f:M \to M$ of a Lorentzian manifold (M,g). It results in the simple explicit formulas for obtaining the new field F^f and its source j^f from a given pair (F,j). Some applications, e.g., a uniformly rotating or radially pulsating source, are presented. © *1995 American Institute of Physics.*

I. INTRODUCTION

Let $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})\equiv\mathbf{0}$ represent the electric and magnetic field vectors, respectively, generated by a static charge density $\rho(\mathbf{r})$ in Minkowski space (M,η). Then it is clear that the fields generated by the shifted source $\rho(\mathbf{r}-\mathbf{r}_0)$ are $\mathbf{E}(\mathbf{r}-\mathbf{r}_0)$, $\mathbf{B}(\mathbf{r}-\mathbf{r}_0)\equiv\mathbf{0}$. This is a consequence of the Poincaré invariance of the Maxwell equations, expressed technically by the transformation rule

$$F_{\mu\nu}(x) \mapsto \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}F_{\alpha\beta}(\Lambda x+a), \tag{1}$$

$$j_{\mu}(x) \mapsto \Lambda^{\alpha}_{\mu}j_{\alpha}(\Lambda x+a) \tag{2}$$

or in the language of differential forms

$$F \mapsto f^*F, \tag{1'}$$

$$j \mapsto f^*j, \tag{2'}$$

when a diffeomorphism of the Minkowski space given in the standard Cartesian coordinates by

$$f:x \mapsto \Lambda x+a$$

(general Poincaré transformation) is performed [in the above mentioned case $\Lambda\equiv 1$, $a^{\mu}\equiv(0,-\mathbf{r}_0)$]. Thus there is a simple algorithm (pullback operation) for computing the source and the corresponding field for the new situation from those for the old one (i.e., transformation of solution—not solving the Maxwell equations once more).

Imagine, now, that the originally static source $\rho(\mathbf{r})$ rotates uniformly with the (constant) angular velocity ω. The new source term in Maxwell equations contains a nonzero three-current, too ($j^{\varphi} \neq 0$ in cylindrical or spherical polar coordinates). Is it still possible to transform the original electrostatic solution to the solution of the new problem? It may seem one can proceed as follows: take the point of view of a comoving (rotating) observer. Since he sees just our original source, he can use our original solution ($\mathbf{E} \neq \mathbf{0}$, $\mathbf{B}=\mathbf{0}$) and it suffices to transform it back according to the (once more pullback $F \mapsto f^*F$) rule

a)Present address: Department of Theoretical Physics, Comenius University, Mlynská dolina F2, 842 15 Bratislava, Slovakia; E-mail: fecko@fmph.uniba.sk

0022-2488/95/36(3)/1198/10/$6.00

$$F_{\mu\nu}(x) \mapsto \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} F_{\alpha\beta}(x') \tag{3}$$

valid for a general coordinate transformation. Note, however, that the geometry of the space–time is not Minkowskian for the rotating observer (cf. Sec. IV C for $v=0$) and that is why he *cannot* use *our* electrostatic solution. Thus the strategy used in the first example does not work, here.

In what follows we propose a method for obtaining a new pair (F^f, j^f) from the original one (F,j) using a diffeomorphism $f:M \to M$ of a Lorentzian manifold (M,g). The pullback f^* is replaced by a more general operator on differential forms $\hat{R}^f$ [defined in Eq. (10)]. It reduces to f^* for isometries of (M,g) and reproduces standard results for the conformal transformations as well. The method can be used sometimes, however, for more general transformations f than for conformal ones (cf. Sec. IV C, where a special rotating source is discussed, or Sec. IV D—radially pulsating source).

II. (F^f, j^f) FROM (F,j)

Let

$$f:M \to M$$

be a diffeomorphism of a Lorentzian manifold (M,g) with orientation. Applying the pullback f^* to the Maxwell equations[1,2]

$$d*_g F = -4\pi *_g j, \tag{4}$$

$$dF = 0 \tag{5}$$

and making use of the formulas

$$f^* d = d f^*, \tag{6}$$

$$f^* *_g = \pm *_{f^*g} f^* \tag{7}$$

(see Appendix A) one obtains

$$d*_{f^*g}(f^*F) = -4\pi *_{f^*g} f^* j,$$

$$d(f^*F) = 0.$$

This shows that f^*F is the field generated by f^*j, but (unfortunately) on the manifold (M, f^*g) rather than (M,g). Only in the case, when

$$f^* g = g, \tag{8}$$

i.e., when f is an *isometry* of (M,g), does the field f^*F correspond to f^*j *on* (M,g). [This is the reason why the method discussed in the Sec. I failed: the map f corresponding to the passing to the rotating observer frame of reference is *not* the isometry of the Minkowski space (M,η), $f^*\eta \neq \eta$].

Let us restrict now to Eq. (4) for a moment. Applying f^* and inserting the identity operator $(-1)^{(p+1)} *_g *_g$ (when acting on p-forms) one comes to

$$d*_g(-*_g f^* *_g F) = -4\pi *_g (*_g f^* *_g j). \tag{9}$$

Define the operator on differential forms on M

$$\hat{R}^f:\Omega^p(M) \to \Omega^p(M)$$

by

$$\hat{R}^f =: (-1)^{p+1} *_g f^* *_g . \tag{10}$$

Then Eq. (9) reads

$$d *_g F^f = -4\pi *_g j^f \tag{11}$$

and shows that if F is the solution of Eq. (4) corresponding to the source j then for *any* diffeomorphism $f:M \to M$, $F^f \equiv \hat{R}^f F$ is the solution of Eq. (4), too, corresponding to the source $j^f \equiv \hat{R}^f j$ [the continuity equation for j^f is guaranteed due to the structure of Eq. (11)].

Equation (5) was ignored, however. In fact it is *not* fulfilled by F^f in general and one has to restrict the choice of f to the class of diffeomorphisms for which

$$dF^f = 0 \tag{12}$$

does hold (it is an equation relating F, f, and g). Fortunately, examples show that this class is wider than well-known conformal transformations (in particular isometries) of (M,g).

III. OTHER EXPRESSIONS FOR THE OPERATOR $\hat{R}^f$

It may be useful for some computations to know other expressions for the crucial operator $\hat{R}^f$ given by Eq. (10), too.

According to Eq. (7) one has

$$\hat{R}^f =: (-1)^{p+1} *_g f^* *_g = \pm (-1)^{p+1} *_g *_{f^*g} f^*$$

and the formula (Appendix B)

$$*_g *_h = \operatorname{sgn} g (-1)^{(n+1)p} \frac{\sqrt{|h|}}{\sqrt{|g|}} \flat_g \sharp_h \tag{13}$$

(where $\sharp_g$, $\flat_g$ are the raising and lowering of *all* indices operations respectively) leads to (sgn $g = -1$, $n=4$ for Lorentzian manifold)

$$\hat{R}^f = \pm \frac{\sqrt{|f^*g|}}{\sqrt{|g|}} \flat_g \sharp_{f^*g} f^*$$

or (cf. Appendix B) for the components with respect to a basis e^μ

$$(\hat{R}^f F)_{\mu\nu} = \pm \frac{1}{\sqrt{|\det \mathscr{R}|}} \mathscr{R}^\alpha_\mu \mathscr{R}^\beta_\nu (f^*F)_{\alpha\beta},$$

$$(\hat{R}^f j)_\mu = \pm \frac{1}{\sqrt{|\det \mathscr{R}|}} \mathscr{R}^\alpha_\mu (f^* j)_\alpha,$$

where

$$\mathscr{R}^\alpha_\mu =: (f^*g)^{\alpha\nu} g_{\nu\mu}, \tag{14}$$

i.e.,

$$F^f \equiv \hat{R}^f F = \pm \frac{1}{\sqrt{|\det \mathscr{R}|}} \frac{1}{2} \mathscr{R}^\alpha_\mu \mathscr{R}^\beta_\nu (f^* F)_{\alpha\beta} e^\mu \wedge e^\nu = \pm \frac{1}{\sqrt{|\det \mathscr{R}|}} \frac{1}{2} (f^* F)_{\alpha\beta} \mathscr{E}^\alpha \wedge \mathscr{E}^\beta, \tag{15}$$

$$j^f \equiv \hat{R}^f j = \pm \frac{1}{\sqrt{|\det \mathscr{R}|}} \mathscr{R}^\alpha_\mu (f^* j)_\alpha e^\mu = \pm \frac{1}{\sqrt{|\det \mathscr{R}|}} (f^* j)_\alpha \mathscr{E}^\alpha, \tag{16}$$

where

$$\mathscr{E}^\mu =: \mathscr{R}^\mu_\nu e^\nu. \tag{17}$$

Thus one way of performing the $\hat{R}^f$ operation consists of the following steps:

(i) compute $f^* F, f^* j$,
(ii) compute $\mathscr{R}^\mu_\nu$ and $\mathscr{E}^\mu$ according to Eqs. (14), (17),
(iii) replace $e^\mu \mapsto \mathscr{E}^\mu$ in e^μ basis expansion of $f^* F, f^* j$,
(iv) multiply the result by the factor $\pm 1/\sqrt{|\det \mathscr{R}|}$.

This makes sense especially in the case when $\det \mathscr{R} = 1$ and $f^* F = F, f^* j = j$ [often, however, the direct use of Eq. (10) may prove equally efficient].

IV. EXAMPLES

In this section we apply the method of Sec. II first to isometries and conformal transformations, reproducing standard results.[3] Then two simple applications follow in which the method works [i.e., Eq. (12) holds] though the diffeomorphisms used are *not* conformal transformations of (M,g).

A. Isometries

If f is an isometry of (M,g), i.e., Eq. (8) holds, then Eqs. (7) and (10) result in

$$\hat{R}^f = \pm f^*.$$

Thus $F^f \equiv f^* F$ corresponds to $j^f \equiv f^* j$ [in particular the right-hand sides of Eqs. (1) and (2) for $g = \eta$]. Due to Eq. (6), Eq. (12) *does* hold.

B. Conformal transformations

Let

$$f^* g = \sigma g$$

for some function $\sigma: M \to \mathrm{R}$. Then

$$*_{f^* g} = *_{\sigma g} = \sigma^{(n/2) - p} *_g = \sigma^{2-p} *_g$$

and

$$\hat{R}^f = \sigma^{2-p} f^*.$$

Thus $F^f = f^* F$, $j^f = \sigma f^* j$ are the new solutions (cf. Ref. 3) generated by the conformal transformation f [Eq. (12) *does* hold since F^f is again only a pullback of the *closed* form F].

C. Helix motion of an infinite charged cylinder in Minkowski space

Let us have a purely electrostatic source

$$j=\rho(r)dt$$

depending only on the coordinate r in cylindrical coordinates (t,r,φ,z) in Minkowski space (e.g., an infinite charged cylinder). The field generated by j is a radial electrostatic field

$$F=E(r)dt\wedge dr\equiv E(r)e^0\wedge e^r,$$

where the usual orthonormal basis

$$e^0=dt,\quad e^r=dr,\quad e^\varphi=r\,d\varphi,\quad e^z=dz$$

was used [Eq. (4) gives then $r^{-1}(rE)'\equiv\text{div}\,\mathbf{E}=4\pi\rho$; Eq. (5) is satisfied]. Take a diffeomorphism given in the cylindrical coordinates by

$$f:(t,r,\varphi,z)\mapsto(t,r,\varphi-\omega t,z-vt)$$

[note that it is *not* a conformal transformation of (M,η)]. Then

$$f^*e^\varphi=e^\varphi-\omega re^0,\quad f^*e^z=e^z-ve^0$$

(the rest being unchanged) and we obtain

$$F^f\equiv\hat{R}^fF=-*_gf^**_gF=-*_gf^*(E(r)*_g(e^0\wedge e^r))=E(r)*_gf^*e^\varphi\wedge f^*e^z$$

$$=E(r)*_g(e^\varphi\wedge e^z-\omega re^0\wedge e^z-ve^\varphi\wedge e^0)=F+\omega rE(r)e^r\wedge e^\varphi+vE(r)e^r\wedge e^z,$$

$$j^f\equiv\hat{R}^fj=*_gf^**_g(\rho(r)e^0)=\rho(r)*_g(f^*e^r\wedge f^*e^\varphi\wedge f^*e^z)=j-\omega r\rho(r)e^\varphi-v\rho(r)e^z.$$

So

$$F^f=F+\omega rE(r)e^r\wedge e^\varphi+vE(r)e^r\wedge e^z,\quad j^f=j-\omega r\rho(r)e^\varphi-v\rho(r)e^z.$$

One verifies easily that $dF^f=0$ holds. Thus a new solution was generated. The new source has the components

$$(j^f)^\mu\equiv(\rho(r),0,\omega r\rho(r),v\rho(r))$$

(with respect to the dual orthonormal basis e_t, e_r, e_φ, e_z) and the new electromagnetic field has the nonzero components

$$E^r=E(r),\quad B^z=-\omega rE(r),\quad B^\varphi=vE(r),$$

[magnetostatic field $\mathbf{B}=(-\omega rE(r))e_z+(vE(r))e_\varphi$ is present in addition to the original electrostatic one].

D. Radial motion of a spherically symmetric source

Let us have a spherically symmetric electrostatic source, i.e.,

$$j=\rho(r)dt,$$

expressed using the spherical polar coordinates (t,r,ϑ,φ) in Minkowski space. The field generated by this j is

$$F=E(r)dt\wedge dr\equiv E(r)e^0\wedge e^r.$$

Here the orthonormal basis

$$e^0=dt,\quad e^r=dr,\quad e^\vartheta=r\,d\vartheta,\quad e^\varphi=r\sin\vartheta\,d\varphi$$

is used (note the different meaning of "r" here and in Sec. IV C).

Consider now a diffeomorphism

$$f:(t,r,\vartheta,\varphi)\mapsto(t,e^{\Psi(t,r)}r,\vartheta,\varphi),$$

where $\Psi(t,r)$ is an arbitrary function (e^Ψ is not to be confused with any of the basis covectors). Since

$$f^*e^0=e^0,\quad f^*e^r=e^\Psi[(1+r\Psi')e^r+r\dot{\Psi}e^0],$$

$$f^*e^\vartheta=e^\Psi e^\vartheta,\quad f^*e^\varphi=e^\Psi e^\varphi$$

($\dot{\Psi}\equiv\partial_t\Psi$, $\Psi'\equiv\partial_r\Psi$), we have

$$F^f=-*_gf^**_g(E(r)e^0\wedge e^r)=*_gf^*(E(r)e^\vartheta\wedge e^\varphi)=e^{2\Psi}E(e^\Psi r)*_g(e^\vartheta\wedge e^\varphi)\equiv E^f(r,t)e^0\wedge e^r,$$

where

$$E^f(r,t)\equiv e^{2\Psi(t,r)}E(e^{\Psi(t,r)}r). \tag{18}$$

Thus a new solution was generated (note that $dF^f=0$), represented by a time-dependent radial electric field.

[Since $E(r)=Qr^{-2}$ *outside* the source, Eq. (18) reveals that the same holds for the new solution: in the region outside the new source, which depends, however, on time, the new electric field is still *static*, viz., $E^f(r,t)=Qr^{-2}$.]

The source j^f is

$$j^f=*_gf^**_g(\rho(r)e^0)=*_gf^*(\rho(r)e^r\wedge e^\vartheta\wedge e^\varphi)=e^{3\Psi}\rho(e^\Psi r)[(1+r\Psi')e^0+r\dot{\Psi}e^r]\equiv\rho^fe^0+j^f_re^r,$$

where

$$\rho^f(t,r)\equiv e^{3\Psi(t,r)}\rho(e^{\Psi(t,r)}r)(1+r\Psi'(t,r)),$$

$$j^f_r(t,r)\equiv e^{3\Psi(t,r)}\rho(e^{\Psi(t,r)}r)r\dot{\Psi}(t,r).$$

Thus the new source contains the r-component of the three-current in addition to the (now time-dependent in general) charge density.

V. THE CHARGE Q^f GIVEN BY j^f

The total charge Q corresponding to the four-current j is given by

$$Q=\int_\Sigma *_gj,$$

with Σ being a three-dimensional spacelike hypersurface in M (e.g., t=const in Minkowski space). Then

$$Q^f =: \int_\Sigma *_g j^f = \int_\Sigma *_g *_g f^* *_g j = \int_\Sigma f^* *_g j = \int_{f(\Sigma)} *_g j.$$

If $f(\Sigma)$ happens to be homological to Σ, i.e.,

$$f(\Sigma) = \Sigma + \partial\Omega$$

for some four-dimensional domain Ω [in particular if $f(\Sigma)=\Sigma$—this is the case, e.g., in Secs. IV C, IV D], then $Q^f = Q$, since

$$Q^f = \int_\Sigma *_g j + \int_{\partial\Omega} *_g j = Q + \int_\Omega d *_g j = Q$$

due to the continuity equation $d *_g j = 0$ for j.

VI. INFINITESIMAL TRANSFORMATIONS

Let f_λ be a one-parameter group of diffeomorphisms generated by a vector field ξ. We want to determine those ξ for which the corresponding *infinitesimal* transformation $f_\epsilon (0 < \epsilon \ll 1)$ generates $F_\epsilon \equiv F^{f_\epsilon} \equiv \hat{R}^{f_\epsilon} F$ which satisfies Eq. (5) [and thus *both* Maxwell equations (4) and (5)]. The expressions of Sec. III simplify in the following way: since $f_\epsilon^* g = g + \epsilon \mathscr{L}_\xi g + \cdots\cdots$, one has

$$\mathscr{R}^\alpha_\mu =: (f^*_\epsilon g)^{\alpha\nu} g_{\nu\mu} = \delta^\alpha_\mu - \epsilon \tau^\alpha_\mu + \cdots ,$$

where the *deformation tensor* $\tau_{\mu\nu}$ is defined as

$$\tau_{\mu\nu} =: (\mathscr{L}_\xi g)_{\mu\nu}.$$

Then $\sqrt{|\det \mathscr{R}|} = 1 - \epsilon \frac{1}{2} \tau^\mu_\mu + \cdots\cdots$ and inserting this in Eq. (15) results in

$$F_\epsilon = F + \epsilon\left(\tfrac{1}{2}\tau^\alpha_\alpha F - \tau^{(1)} F + \mathscr{L}_\xi F\right) + \cdots ,$$

where

$$(\tau^{(1)} F)_{\mu\nu} =: \tau^\alpha_\mu F_{\alpha\nu} + \tau^\alpha_\nu F_{\mu\alpha}.$$

Then the requirement $dF_\epsilon = 0$ is equivalent to

$$d\left(\tfrac{1}{2}\tau^\alpha_\alpha F\right) = d(\tau^{(1)} F) \tag{19}$$

or in (coordinate basis) components

$$\epsilon^{\kappa\mu\nu\rho}(F_{\mu\nu}\tau^\alpha_{\alpha,\rho} + 4F_{\alpha\mu}\tau^\alpha_{\nu,\rho} + 4F_{\alpha\mu,\rho}\tau^\alpha_\nu) = 0.$$

This is an equation relating ξ, g (through τ) and F. For example, in the case of conformal transformations $\tau^\mu_\nu = \psi(x)\delta^\mu_\nu$ for some ψ and one readily verifies that *for any* F Eq. (19) *is* satisfied (in particular $\tau=0$ for isometries, for which it is clearly the case).

In example IV C $\xi = c_1 t \partial_\phi + c_2 t \partial_z$, $g_{\mu\nu} = \eta_{\mu\nu}$, $F_{\mu\nu}$ has a single nonzero component $F_{01}(r)$ and one verifies that Eq. (19) *is* satisfied, too.

VII. CONCLUSION

The message of this article is a simple observation, that if F is a solution of the Maxwell equations (4) and (5) on a Lorentzian manifold (M,g) generated by the source j, then for *any* diffeomorphism $f:M \to M$.

$$F^f \equiv \hat{R}^f F \equiv -*_g f^* *_g F$$

is a solution of Eq. (4) with the source term

$$j^f \equiv \hat{R}^f j \equiv *_g f^* *_g j.$$

In order to generate new solutions of the *whole* system (4), (5), one has to restrict the class of the diffeomorphisms to those for which $dF^f = 0$ holds.

For the conformal transformations (in particular isometries) of (M,g), F^f reduces to f^*F and consequently $dF^f = 0$ *does* hold independently of the original solution F (j^f is then $\sigma f^* j$, σ being given by $f^*g = \sigma g$; in particular $\sigma \equiv 1$ for isometries).

In general the equation $dF^f = 0$ is an equation relating g, f, and F (or, implicitly, j). It restricts one member of the triplet when the other two are fixed, e.g., restricts f for given g, F (which class of diffeomorphisms can be used for a given field), or restricts F for given f, g [which fields can be used as the original solution for a given class of diffeomorphisms—e.g., which fields of uniformly rotating sources can be obtained by this method from the fields of static sources; there are no restrictions if f is a conformal transformation of (M,g)].

APPENDIX A: A PROOF OF EQUATION (7)

Let M be a pseudo-Riemanian manifold (M,g) with orientation, e_a orthonormal right-handed local field of frames, e^a its dual, i.e.,

$$g(e_a, e_b) = \eta_{ab} \equiv \mathrm{diag}(\pm 1), \tag{A1}$$

$$\omega(e_a, \ldots, e_b) = +\epsilon_{a\cdots b}, \tag{A2}$$

with ω being the metric and orientation-compatible volume form and ϵ the Levi–Civita symbol. By definition

$$*_g(e^a \wedge \cdots \wedge e^b) = \frac{1}{(n-p)!}\, \eta^{a\alpha} \cdots \eta^{b\beta} \epsilon_{\alpha\cdots\beta c\cdots d} e^c \wedge \cdots \wedge e^d. \tag{A3}$$

Then if

$$f:M \to M$$

is a diffeomorphism, one obtains

$$f^* *_g(e^a \wedge \cdots \wedge e^b) = \frac{1}{(n-p)!}\, \eta^{a\alpha} \cdots \eta^{b\beta} \epsilon_{\alpha\cdots\beta c\cdots d} E^c \wedge \cdots \wedge E^d, \tag{A4}$$

where

$$E^a =: f^* e^a. \tag{A5}$$

The dual frame E_a is orthonormal *with respect to* f^*g

$$(f^*g)(E_a, E_b) = g(f_* f^* e_a, f_* f^* e_b) = g(e_a, e_b) = \eta_{ab} \tag{A6}$$

but it may be not right handed (depending on whether f does or does not preserve the orientation on M)

$$\omega(E_a,\ldots,E_b)=\pm\epsilon_{a\cdots b}. \tag{A7}$$

Thus

$$*_{f^*g}f^*(e^a\wedge\cdots\wedge e^b)=*_{f^*g}(E^a\wedge\cdots\wedge E^b)=\frac{1}{(n-p)!}\,\eta^{a\alpha}\cdots\eta^{b\beta}(\pm\epsilon_{\alpha\cdots\beta c\cdots d})E^c\wedge\cdots\wedge E^d$$
$$=\pm f^**_g(e^a\wedge\cdots\wedge e^b)$$

so that

$$f^**_g=\pm *_{f^*g}f^*, \tag{A8}$$

which is Eq. (7).

APPENDIX B: A PROOF OF EQUATIONS (13), (15), AND (16)

Equation (13) is a generalization of the well-known formula

$$*_g*_g=\operatorname{sgn} g(-1)^{(n+1)p} \tag{B1}$$

to the case of the two possibly different metric tensors g and h on M, where $\sharp_g$, $\flat_g$ are the raising and lowering of *all* indices operations, respectively, and $|g|$, $|h|$ are (the absolute values of) the determinants of the matrices g_{ij}, h_{ij}. The proof of Eq. (13) consists of a straightforward modification of the standard one for Eq. (B1). Let α be a p-form. Then

$$(\sharp_g*_g*_h\alpha)^{i\cdots j}=g^{ia}\cdots g^{jb}(*_g*_h\alpha)_{a\cdots b}=\frac{1}{(n-p)!}\,g^{ia}\cdots g^{jb}g^{r\alpha}\cdots g^{s\beta}(*_h\alpha)_{\alpha\cdots\beta}(\omega_g)_{r\cdots sa\cdots b}$$
$$=\frac{1}{(n-p)!}\,(\sharp_g\omega_g)^{\alpha\cdots\beta i\cdots j}(*_h\alpha)_{\alpha\cdots\beta}$$
$$=\frac{(\operatorname{sgn} g)(-1)^{p(n-p)}}{(n-p)!p!}\,\epsilon_{\alpha\cdots\beta r\cdots s}\epsilon^{\alpha\cdots\beta i\cdots j}\,\frac{\sqrt{|h|}}{\sqrt{|g|}}\,(\sharp_h\alpha)^{r\cdots s}$$
$$=\operatorname{sgn} g(-1)^{p(n-p)}\,\frac{\sqrt{|h|}}{\sqrt{|g|}}\,(\sharp_h\alpha)^{i\cdots j}.$$

Thus

$$*_g*_h=\operatorname{sgn} g(-1)^{p(n+1)}\,\frac{\sqrt{|h|}}{\sqrt{|g|}}\,\flat_g\sharp_h. \tag{B2}$$

Now

$$(\flat_g\sharp_h\alpha)_{i\cdots j}=g_{ia}\cdots g_{jb}h^{ak}\cdots h^{bl}\alpha_{k\cdots l}=\mathscr{R}^k_i\cdots\mathscr{R}^l_j\alpha_{k\cdots l},$$

where

$$\mathscr{R}^k_i=:h^{ka}g_{ai} \tag{B3}$$

and

$$\frac{\sqrt{|h|}}{\sqrt{|g|}}=\frac{1}{\sqrt{|\det \mathscr{R}|}}.$$

Thus

$$(*_g *_h \alpha)_{i\cdots j}=\text{sgn } g(-1)^{p(n+1)}\frac{1}{\sqrt{|\det \mathscr{R}|}}\mathscr{R}^a_i\cdots\mathscr{R}^b_j\alpha_{a\cdots b}$$

or

$$*_g *_h \alpha=\frac{1}{p!}\text{ sgn } g\frac{(-1)^{p(n+1)}}{\sqrt{|\det \mathscr{R}|}}\alpha_{a\cdots b}\mathscr{E}^a\wedge\cdots\wedge\mathscr{E}^b,$$

where

$$\mathscr{E}^a=:\mathscr{R}^a_i e^i. \tag{B4}$$

Finally

$$\hat{R}^f\alpha=\pm(-1)^{np+1}\text{ sgn } g\frac{1}{p!}\frac{1}{\sqrt{|\det \mathscr{R}|}}(f^*\alpha)_{i\cdots j}\mathscr{E}^i\wedge\cdots\wedge\mathscr{E}^j$$

or, in particular, for space–time (sgn $g=-1, n=4$)

$$\hat{R}^f\alpha=\pm\frac{1}{p!}\frac{1}{\sqrt{|\det \mathscr{R}|}}(f^*\alpha)_{i\cdots j}\mathscr{E}^i\wedge\cdots\wedge\mathscr{E}^j,$$

which is used in Eqs. (15) and (16).

[1] Ch. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[2] N. Straumann, *General Relativity and Relativistic Astrophysics* (Springer-Verlag, Berlin, 1991).
[3] F. Fulton, F. Rohrlich, and L. Witten, "Conformal invariance in physics," Rev. Mod. Phys. **34**, 442 (1962).