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On 3+1 decompositions with respect to an observer field via differential forms

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3+1 decompositions of differential forms on a Lorentzian manifold $(M, g; +---)$ with respect to arbitrary observer field and the decomposition of the standard operations acting on them are studied, making use of the ideas of the theory of connections on principal bundles. Simple explicit general formulas are given as well as their application to the Maxwell equations. © *1997 American Institute of Physics.* [S0022-2488(97)01809-4]

I. INTRODUCTION

There is rather extensive literature devoted to 3+1 split of the laws of physics in curved space–time (cf. Ref. 1 for a review and also the references therein). According to Sec. II of Ref. 1 there are two rather different methods available to cope with the problem, viz. the congruence method and the hypersurface one.

Here we present a systematic method of 3+1 split within the *congruence* method using the language of *differential forms* on both (4 and 3+1) levels.

The use of forms within the 3+1 decomposition program can be traced back to the classical paper on geometrodynamics[2] (p. 581), where it was applied, however, in the framework of the *hypersurface* method (cf. also Ref. 3, pp. 93–94); the time serves there as a parameter labeling the spacelike hypersurfaces and the time derivative of a form is interpreted as a differentiation with respect to a parameter.

The observer field approach similar to ours can be found in Ref. 4 (pp. 193–197). What we add here is the introduction of the (simply realized) operator hor (cf. Sec. III) and a spatial exterior derivative within the general congruence approach (Sec. IV B). These objects turn out to be very convenient to manipulate and enable one to derive very simple and at the same time general decomposition formulas and rules.

The following data are assumed in the article: a four-dimensional Lorentzian (g of the signature $+---$) manifold M with orientation ($\equiv$ space–time), and an *observer* (velocity) *field*, i.e., a future oriented vector field on M obeying

$$g(V, V) \equiv \|V\|^2 = 1 \tag{1.1}$$

(the integral curves of V provide then the congruence of proper-time parametrized world lines of observers).

All the constructions are in fact local, i.e., it is enough that the objects mentioned above are available only in a domain $\mathscr{U} \subset M$ rather than globally and consequently no global properties of M are assumed.

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II. THE DECOMPOSITION OF FORMS

For any vector field W let us define the following standard operations on forms:

$$i_W : \Omega^p(M) \to \Omega^{p-1}(M),$$

$$j_W : \Omega^p(M) \to \Omega^{p+1}(M),$$

$$i_W \alpha(U,\dots) := \alpha(W,U,\dots), \tag{2.1}$$

$$j_W \alpha := \widetilde{W} \wedge \alpha, \quad \widetilde{W} \equiv g(W,.) \equiv \flat_g W. \tag{2.2}$$

Then the following identity holds:

$$j_W i_U + i_U j_W = g(U,W), \tag{2.3}$$

in particular,

$$j_V i_V + i_V j_V = 1 \equiv \text{identity on } \Omega(M). \tag{2.4}$$

Further, introducing

$$\mathscr{P} := i_V j_V, \quad \mathscr{Q} := j_V i_V, \tag{2.5}$$

one checks easily that they represent the set of *projection operators* on $\Omega^p(M)$, i.e.,

$$\mathscr{P}^2 = \mathscr{P}, \quad \mathscr{Q}^2 = \mathscr{Q}, \quad \mathscr{P}\mathscr{Q} = 0 = \mathscr{Q}\mathscr{P}, \quad \mathscr{P} + \mathscr{Q} = 1. \tag{2.6}$$

Then for any $\alpha \in \Omega^p(M)$ one has

$$\alpha = (\mathscr{Q} + \mathscr{P})\alpha = \widetilde{V} \wedge i_V \alpha + i_V j_V \alpha, \tag{2.7}$$

i.e., one obtains the *decomposition*

$$\alpha = \widetilde{V} \wedge \hat{s} + \hat{r}, \tag{2.8}$$

where

$$\hat{s} \equiv i_V \alpha, \quad \hat{r} \equiv i_V j_V \alpha. \tag{2.9}$$

III. OPERATOR HOR AND SPATIAL FORMS

At any point $m \in M$ we define *vertical* (instantaneous time) direction—parallel to V and *horizontal* (instantaneous 3-space) directions—perpendicular to V. Then for any vector there is the unique decomposition

$$U = U_\parallel + U_\perp \equiv \text{ver } U + \text{hor } U$$

and one can define (in the spirit of the theory of connections on principal bundles, cf. Ref. 5)

$$(\text{hor } \alpha)(U,W,\dots) := \alpha(\text{hor } U, \text{hor } W, \dots). \tag{3.1}$$

It turns out (cf. Appendix A) that this operation is realized explicitly as

$$\text{hor } \alpha = i_V j_V \alpha \equiv \mathscr{P}\alpha \equiv \hat{r}, \tag{3.2}$$

so that the decomposition in Eq. (2.8) can be rewritten also as

$$\alpha = \widetilde{V} \wedge i_V \alpha + \text{hor } \alpha. \tag{3.3}$$

We also introduce the space of purely *spatial* (horizontal) *p-forms* by

$$\hat{\Omega}^p(M) := \{\alpha \in \Omega^p(M) | \alpha = \text{hor } \alpha\} \tag{3.4}$$

[i.e., $\hat{s} = 0$ in the decomposition (2.8)] and the Cartan algebra of spatial forms

$$\hat{\Omega}(M) := \oplus_p \hat{\Omega}^p(M) \tag{3.5}$$

(it is closed with respect to $\wedge$). One readily verifies [inserting arguments and using the definition (3.1)] that the projection operator $\mathscr{P} \equiv \text{hor}$ is compatible with the algebra structure in $\Omega(M)$,

$$\text{hor}(\alpha + \lambda \beta) = \text{hor } \alpha + \lambda \text{ hor } \beta, \quad \alpha, \beta \in \Omega(M), \quad \lambda \in \mathbb{R}, \tag{3.6}$$

$$\text{hor}(\alpha \wedge \beta) = \text{hor } \alpha \wedge \text{hor } \beta, \quad \alpha, \beta \in \Omega(M), \tag{3.7}$$

i.e.,

$$\text{hor}: \Omega(M) \rightarrow \text{Im hor} \equiv \hat{\Omega}(M) \leqslant \Omega(M) \tag{3.8}$$

is an *endomorphism* of the Cartan algebra $\Omega(M)$. From Eq. (3.3) we obtain useful criterion:

$$\alpha = \text{spatial form} \Leftrightarrow i_V \alpha = 0. \tag{3.9}$$

Then we see that $\hat{r}, \hat{s}$ in the decomposition (2.8) are spatial [Eq. (2.9) plus $i_V i_V = 0$].

Note: If a local orthonormal frame field $e_a \equiv (e_0 \equiv V, e_i)$ and its dual $e^a \equiv (e^0 \equiv \widetilde{V}, e^i)$ are used and if

$$\alpha = \frac{1}{p!} \alpha_{a\cdots b} e^a \wedge \cdots \wedge e^b, \tag{3.10}$$

then the decomposition (2.8) is just split into two parts which do and do not contain, respectively, the basis 1-form $e^0 \equiv \widetilde{V}$, i.e.,

$$\alpha = e^0 \wedge \hat{s} + \hat{r}, \tag{3.11}$$

where $\hat{s}, \hat{r}$, being spatial, do not already contain the local ''time'' basis 1-form e^0, but rather only the ''spatial'' basis 1-forms e^i; explicitly

$$\hat{s} = \frac{1}{(p-1)!} \alpha_{0i\cdots j} \underbrace{e^i \wedge \cdots \wedge e^j}_{p-1}, \quad \hat{r} = \frac{1}{p!} \alpha_{k\cdots j} \underbrace{e^k \wedge \cdots \wedge e^j}_{p}. \tag{3.12}$$

IV. THE DECOMPOSITION OF THE OPERATIONS ON FORMS

According to Eq. (2.8) any form on (M, g, V) can be 3+1 decomposed as

$$\alpha = \widetilde{V} \wedge \hat{s} + \hat{r},$$

so that the full information about $\alpha \in \Omega^p(M)$ is encoded (in an observer-dependent way) into a *pair of spatial* forms $\hat{s} \in \hat{\Omega}^{p-1}(M)$ and $\hat{r} \in \hat{\Omega}^p(M)$. In this section we perform the decomposition of the standard *operations* on forms, viz. the *Hodge star* $*$ and the *exterior derivative d* (some other important operators are then easily obtained by their combinations). By this, we mean to introduce some ''spatial'' operations (acting directly on $\hat{s}, \hat{r}$ and dependent on the observer) producing the same effect as does the given operator acting on α.

A. The Hodge star

The horizontal subspace of a tangent space at each point inherits natural metric tensor $\hat{h}$ (with signature $+ + +$ by definition, i.e., $g = \widetilde{V} \otimes \widetilde{V} - \hat{h}$) and orientation [a spatial frame (e_1, e_2, e_3) is declared to be right-handed if $(V \equiv e_0, e_1, e_2, e_3)$ is right-handed]. These data are just enough for the unique *spatial Hodge* operator

$$\hat{*} := *_{\hat{h}} : \hat{\Omega}^p(M) \rightarrow \hat{\Omega}^{3-p}(M) \tag{4.1}$$

(it is to be applied only on spatial forms). Using the operator

$$\hat{\eta}\alpha := (-1)^{\deg\alpha}\alpha$$

one readily computes (cf. Appendix B) that the decomposition of the ''full'' Hodge star reads

$$*(\widetilde{V}\wedge\hat{s}+\hat{r})=\widetilde{V}\wedge\hat{*}\hat{r}+\hat{*}\hat{\eta}\hat{s}. \tag{4.2}$$

As an example, applying this to $1 \in \Omega^0(M)$ ($\hat{s}=0$, $\hat{r}=1$) results in the decomposition of the 4-volume form

$$*1\equiv\omega=\widetilde{V}\wedge\hat{*}1=:\widetilde{V}\wedge\hat{\omega}, \tag{4.3}$$

where

$$\hat{\omega}:=\hat{*}1 \tag{4.4}$$

is the *spatial volume form*. In the local orthonormal right-handed coframe field e^a it is just

$$\omega\equiv e^0\wedge e^1\wedge e^2\wedge e^3=e^0\wedge(e^1\wedge e^2\wedge e^3)\equiv\widetilde{V}\wedge\hat{\omega}. \tag{4.5}$$

B. The exterior derivative

Let $\hat{b}$ be a spatial form, $\mathscr{D}$ a spatial ($\equiv$horizontal) domain (i.e., the domain of any possible dimension with the property that any vector tangent to it is horizontal). Then

$$\int_{\mathscr{D}} d\hat{b} \overset{1.}{=} \int_{\partial\mathscr{D}} \hat{b} \quad \text{due to Stokes's theorem}$$

$$\overset{2.}{=} \int_{\mathscr{D}} \text{hor } d\hat{b} \equiv \int_{\mathscr{D}} \hat{d}\hat{b} \quad \text{since } \mathscr{D} \text{ is horizontal}$$

$\Rightarrow$

$$\int_{\mathscr{D}} \hat{d}\hat{b} = \int_{\partial\mathscr{D}} \hat{b}, \tag{4.6}$$

where we introduced the *spatial exterior derivative*

$$\hat{d} : \hat{\Omega}^p(M) \to \hat{\Omega}^{p+1}(M),$$

$$\hat{d} := \text{hor } d \equiv i_V j_V d \tag{4.7}$$

(exactly like the *covariant* exterior derivative of forms on principal bundle with connection). Thus for spatial forms and domains the ''full'' operator d in the Stokes formula can be replaced by $\hat{d}$. This means that $\hat{d}$ and $\hat{*}$ provide the basic building blocks for the ''three-dimensional vector analysis'' operations, being the natural generalizations of div, curl etc. used in Minkowski space (div~$\hat{*}\hat{d}\hat{*}$, curl~$\hat{*}\hat{d}$,...). We emphasize that the validity of the *spatial Stokes formula* (4.6) for $\hat{d}$ is essential for the usefulness and naturality of the latter, e.g., as a means to relate the usual differential 3+1 laws to the corresponding integral ones (like div $\mathbf{B}=0 \leftrightarrow \oint \mathbf{B} \cdot d\mathbf{S}=0$).

So our task now is to express the action of the full d operator in terms of $\hat{d}$ (and possibly some other ones) directly on $\hat{s}, \hat{r}$ present in the decomposition (2.8) of α. We have

$$d\alpha = d\widetilde{V} \wedge \hat{s} - \widetilde{V} \wedge d\hat{s} + d\hat{r}$$

so that we are to focus our attention on two particular issues, viz. d of $\widetilde{V}$ and d of a spatial form.

The decomposition of the 2-form $d\widetilde{V}$ according to Eq. (2.8) results in

$$d\widetilde{V} = \widetilde{V} \wedge \hat{a} + \hat{y} \tag{4.8}$$

with $\hat{a} \in \hat{\Omega}^1(M)$, $\hat{y} \in \hat{\Omega}^2(M)$. The forms $\hat{a}, \hat{y}$ are the *kinematical characteristics* of the observer field V, which can be easily extracted from any given V using Eq. (2.9). Their physical meaning is discussed in Appendix C. It turns out (see also Refs. 6–8) that $\hat{a}$ equals the *acceleration 1-form*

$$\hat{a} = g(\nabla_V V, \cdot) \equiv g(a, \cdot) \equiv \widetilde{a} \tag{4.9}$$

($a \equiv \nabla_V V$ is the *acceleration field* corresponding to V) and the 2-form $\hat{y}$, the *vorticity form* (tensor) is the measure of the (non)integrability of the spatial (horizontal) distribution, i.e., it encodes whether or not the instantaneous 3-spaces mesh together to form a (local) spatial 3-domain $\mathscr{D}$ (or, equivalently, whether or not the *time synchronization* is possible). These properties of $\hat{a}$ and $\hat{y}$ are reflected in the terminology: V is said to be *geodesic* if $\hat{a}=0$, *irrotational* or *time synchronizable* if $\hat{y}=0$ and *proper-time synchronizable* if both $\hat{a}$ and $\hat{y}$ vanish (then $V=\partial_t$, $\widetilde{V}=dt$ in adapted coordinates).

The computation of the action of d on a spatial form, as well as on a general form α then, is performed in Appendix D and the result reads

$$d(\widetilde{V} \wedge \hat{s} + \hat{r}) = \widetilde{V} \wedge (-\hat{d}\hat{s} + \mathscr{L}_V \hat{r} + \hat{a} \wedge \hat{s}) + (\hat{d}\hat{r} + \hat{y} \wedge \hat{s}). \tag{4.10}$$

The formula (4.10) gives the desired 3+1 decomposition of the full d operator. Notice the explicit occurrence of both kinematical characteristics $\hat{a}$ and $\hat{y}$.

The spatial exterior derivative $\hat{d}$ shares some important properties with the full d. In particular, it *is* the *graded derivation* of the spatial Cartan algebra $\hat{\Omega}(M)$. Indeed, according to Eq. (3.7) we have

$$\hat{d}(\hat{r} \wedge \hat{R}) = \text{hor}(d\hat{r} \wedge \hat{R} + \hat{\eta}\hat{r} \wedge d\hat{R}) = \hat{d}\hat{r} \wedge \underbrace{\text{hor } \hat{R}}_{\hat{R}} + \hat{\eta} \underbrace{\text{hor } \hat{r}}_{\hat{r}} \wedge \hat{d}\hat{R} = \hat{d}\hat{r} \wedge \hat{R} + \hat{\eta}\hat{r} \wedge \hat{d}\hat{R}.$$

$$\tag{4.11}$$

On the other hand, it is *not nilpotent* in general, but rather (see Appendices D and G)

$$\hat{d}\hat{d}\hat{b} = -\hat{y} \wedge \mathscr{L}_V \hat{b}, \quad \hat{b} \in \hat{\Omega}(M) \tag{4.12}$$

holds. This may seem to contradict Eq. (4.6), since the (full) exterior derivative can be uniquely defined by the (full) Stokes formula[9] (and it *is* then nilpotent due to the nilpotence of the boundary operator). The situation can be clarified as follows: For *any* domain $\mathscr{D}$ and $\alpha \in \Omega(M)$ one has

$$\int_{\mathscr{D}} dd\alpha = \int_{\partial\partial\mathscr{D}} \alpha = 0 \tag{4.13}$$

(since $\partial\partial = 0$) which leads to $dd\alpha = 0$ identically, i.e., d *is* nilpotent. For a *spatial* domain $\mathscr{D}$ and *spatial* form $\hat{b}$ one has similarly

$$\int_{\mathscr{D}} \hat{d}\hat{d}\hat{b} = \int_{\partial\partial\mathscr{D}} \hat{b} = 0. \tag{4.14}$$

This does not mean, however, that $\hat{d}\hat{d}\hat{b} = 0$ identically, now, but rather $\hat{d}\hat{d}\hat{b}$ should vanish upon restriction to any *spatial* $\mathscr{D}$. The only nontrivial cases are for the dimension of $\mathscr{D}$ being 3 or 2. For dim $\mathscr{D}=3$, $\hat{y} \neq 0$ [and thus $\hat{d}\hat{d}\hat{b} \neq 0$ due to Eq. (4.12)] means (via Frobenius theorem) that spatial $\mathscr{D}$ [to be used in Eq. (4.14)] does not exist at all. For dim $\mathscr{D}=2$ we have $\hat{b}$=function$\equiv f$ and the question is whether $(Vf)\hat{y}$ vanishes (for any f) upon restriction on any spatial two-dimensional domain $\mathscr{D}$. This *is*, however, the case as a result of $\hat{y}$ being the measure of nonintegrability (the bracket of any two vectors tangent to $\mathscr{D}$ is trivially again tangent to $\mathscr{D}$). Thus there is *no conflict* between Eqs. (4.6) and (4.12).

V. MATRIX NOTATION

For the computation of more complex expressions [e.g., the *codifferential* in Eq. (5.3)] it is quite useful to introduce matrix realization of the operators. If the decomposition (2.8) of α is represented by a column

$$\alpha \equiv \widetilde{V} \wedge \hat{s} + \hat{r} \leftrightarrow \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}$$

then, e.g.,

$$*(\widetilde{V} \wedge \hat{s} + \hat{r}) = \widetilde{V} \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \leftrightarrow \begin{pmatrix} \hat{*}\hat{r} \\ \hat{*}\hat{\eta}\hat{s} \end{pmatrix} \equiv \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}$$

so that

$$* \leftrightarrow \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix}.$$

For the exterior derivative we obtain similarly

$$d(\widetilde{V} \wedge \hat{s} + \hat{r}) \leftrightarrow \begin{pmatrix} -\hat{d}\hat{s} + \mathscr{L}_V \hat{r} + \hat{a} \wedge \hat{s} \\ \hat{d}\hat{r} + \hat{y} \wedge \hat{s} \end{pmatrix} \equiv \begin{pmatrix} -\hat{d} + \hat{a} & \mathscr{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}$$

or

$$d \leftrightarrow \begin{pmatrix} -\hat{d}+\hat{a} & \mathscr{L}_V \\ \hat{y} & \hat{d} \end{pmatrix}.$$

In the same sense we can then also express other useful operations in terms of such matrices; for the sake of convenience we collect them here together:

$$* \leftrightarrow \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix}, \quad *^{-1} \leftrightarrow \begin{pmatrix} 0 & -\hat{*}\hat{\eta} \\ \hat{*} & 0 \end{pmatrix}, \tag{5.1}$$

$$\hat{\eta} \leftrightarrow \begin{pmatrix} -\hat{\eta} & 0 \\ 0 & \hat{\eta} \end{pmatrix}, \quad d \leftrightarrow \begin{pmatrix} -\hat{d}+\hat{a} & \mathscr{L}_V \\ \hat{y} & \hat{d} \end{pmatrix}, \tag{5.2}$$

$$\delta := *^{-1}d*\hat{\eta} \leftrightarrow \begin{pmatrix} 0 & -\hat{*}\hat{\eta} \\ \hat{*} & 0 \end{pmatrix} \begin{pmatrix} -\hat{d}+\hat{a} & \mathscr{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix} \begin{pmatrix} -\hat{\eta} & 0 \\ 0 & \hat{\eta} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\delta} & \hat{*}(\hat{y}\wedge\hat{*}) \\ -\hat{*}\mathscr{L}_V\hat{*} & -\hat{\delta}+\hat{*}(\hat{a}\wedge\hat{*}\hat{\eta}) \end{pmatrix}, \tag{5.3}$$

where

$$\hat{\delta} := \hat{*}^{-1}\hat{d}\hat{*}\hat{\eta} \tag{5.4}$$

is the *spatial codifferential*,

$$i_V \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j_V \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.5}$$

$$\text{hor} = i_V j_V \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.6}$$

$$\mathscr{L}_V \equiv i_V d + d i_V \leftrightarrow \begin{pmatrix} \mathscr{L}_V & 0 \\ \hat{a} & \mathscr{L}_V \end{pmatrix}. \tag{5.7}$$

VI. THE MAXWELL EQUATIONS

According to the standard conventions on the relationship between the components of the electromagnetic field 2-form $F \equiv \frac{1}{2}F_{ab}e^a \wedge e^b$ (e_a is g-orthonormal frame) and the 3-space vectors of the electric and magnetic fields, respectively,

$$F_{0\alpha} = E_\alpha = E^\alpha, \quad F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B^\gamma \equiv -\epsilon_{\alpha\beta\gamma}B_\gamma \tag{6.1}$$

(e_α is $\hat{h}$-orthonormal frame; $\alpha, \beta,...$ run from 1 to 3, being raised and lowered by the *spatial* metric tensor $\hat{h}_{\alpha\beta} \equiv +\delta_{\alpha\beta} \equiv -\eta_{\alpha\beta}$), one can associate with the electric and magnetic fields the *spatial forms*

$$\hat{E} = E_\alpha e^\alpha =: \mathbf{E} \cdot d\mathbf{r}, \quad \hat{B} = B^\alpha dS_\alpha =: \mathbf{B} \cdot d\mathbf{S},$$

$$dS_\alpha := \frac{1}{2}\epsilon_{\alpha\beta\gamma}e^\beta \wedge e^\gamma. \tag{6.2}$$

Then

$$F = \widetilde{V} \wedge \hat{E} - \hat{B} \leftrightarrow \begin{pmatrix} \hat{E} \\ -\hat{B} \end{pmatrix} \tag{6.3}$$

(so that $\hat{s} = \hat{E}$, $\hat{r} = -\hat{B}$ here). Similarly the electric *4-current 1-form* decomposes to

$$j = j_a e^a = j_0 e^0 + j_i e^i \equiv \rho \widetilde{V} - \hat{j} \leftrightarrow \begin{pmatrix} \rho \\ -\hat{j} \end{pmatrix}, \quad \hat{j} := j_\alpha e^\alpha = j^\alpha e^\alpha. \tag{6.4}$$

Then

$$*F = \widetilde{V} \wedge (-\hat{*}\hat{B}) - \hat{*}\hat{E} \leftrightarrow \begin{pmatrix} -\hat{*}\hat{B} \\ -\hat{*}\hat{E} \end{pmatrix}, \tag{6.5}$$

$$*j = \widetilde{V} \wedge (-\hat{*}\hat{j}) + \rho \hat{\omega} \leftrightarrow \begin{pmatrix} -\hat{*}\hat{j} \\ \rho \hat{\omega} \end{pmatrix} \tag{6.6}$$

and so the 3+1 decomposition of the Maxwell equations

$$d*F = -4\pi *j, \tag{6.7}$$

$$dF = 0 \tag{6.8}$$

and the continuity equation

$$d*j = 0, \tag{6.9}$$

respectively, result in

$$\hat{d}\hat{*}\hat{E} + \hat{y} \wedge \hat{*}\hat{B} = 4\pi\rho\hat{\omega}, \tag{6.7a}$$

$$\hat{d}\hat{*}\hat{B} - \mathscr{L}_V \hat{*}\hat{E} - \hat{a} \wedge \hat{*}\hat{B} = 4\pi\hat{*}\hat{j}, \tag{6.7b}$$

$$\hat{d}\hat{E} + \mathscr{L}_V \hat{B} - \hat{a} \wedge \hat{E} = 0, \tag{6.8a}$$

$$\hat{d}\hat{B} - \hat{y} \wedge \hat{E} = 0, \tag{6.8b}$$

and

$$\mathscr{L}_V(\rho\hat{\omega}) + \hat{d}\hat{*}\hat{j} - \hat{a} \wedge \hat{*}\hat{j} = 0. \tag{6.10}$$

In particular, in the simplest situation, viz. for the irrotational ($\hat{y} = 0$), geodesic ($\hat{a} = 0$) observer field V (then $V = \partial_t$, $\widetilde{V} = dt$) we get

$$\hat{d}\hat{*}\hat{E} = 4\pi\rho\hat{\omega}, \tag{6.7a$'$}$$

$$\hat{d}\hat{*}\hat{B} - \mathscr{L}_{\partial_t}\hat{*}\hat{E} = 4\pi\hat{*}\hat{j}, \tag{6.7b$'$}$$

$$\hat{d}\hat{E} + \mathscr{L}_{\partial_t}\hat{B} = 0, \tag{6.8a$'$}$$

$$\hat{d}\hat{B}=0, \tag{6.8b$'$}$$

and

$$\mathscr{L}_{\partial_t}(\rho\hat{\omega})+\hat{d}\hat{\ast}\hat{j}=0 \tag{6.10$'$}$$

(there is a simple rule to modify these equations to the case $\hat{a}\neq0$ but still $\hat{y}=0$; cf. Appendix I).

Since Eqs. (6.7a)–(6.10) are written in terms of differential forms and standard well-behaved operations with respect to integrals, one can readily write down their corresponding *integral versions*: let spatial domains of necessary dimensions exist (two-dimensional surface $\mathscr{S}$, three-dimensional volume $\mathscr{D}$—the latter case needs $\hat{y}=0$, therefore we put $\hat{y}=0$ in the equations where the integration over the three-dimensional domain is performed); then

$$\oint_{\partial\mathscr{D}}\hat{\ast}\hat{E}=4\pi\int_{\mathscr{D}}\rho\hat{\omega}\equiv4\pi Q, \tag{6.11a}$$

$$\oint_{\partial\mathscr{S}}\hat{\ast}\hat{B}-\frac{d}{d\tau}\bigg|_0\int_{\Phi_\tau(\mathscr{S})}\hat{\ast}\hat{E}-\int_{\mathscr{S}}\hat{a}\wedge\hat{\ast}\hat{B}=4\pi\int_{\mathscr{S}}\hat{\ast}\hat{j}, \tag{6.11b}$$

$$\oint_{\partial\mathscr{S}}\hat{E}+\frac{d}{d\tau}\bigg|_0\int_{\Phi_\tau(\mathscr{S})}\hat{B}-\int_{\mathscr{S}}\hat{a}\wedge\hat{E}=0, \tag{6.12a}$$

$$\oint_{\partial\mathscr{D}}\hat{B}=0, \tag{6.12b}$$

and

$$\frac{d}{d\tau}\bigg|_0\int_{\Phi_\tau(\mathscr{D})}\rho\hat{\omega}+\oint_{\partial\mathscr{D}}\hat{d}\hat{\ast}\hat{j}-\int_{\mathscr{D}}\hat{a}\wedge\hat{\ast}\hat{j}=0, \tag{6.13}$$

where Φ_τ is the (local) flow generated by V.

Equations (6.7a)–(6.10) can be also expressed in more familiar form, making use of three-dimensional vector analysis operators div, curl, etc.; this is done in Appendix H [cf. (H8)–(H12)].

Equivalently, if instead of Eq. (6.7)

$$\delta F=4\pi j \tag{6.14}$$

is used, Eqs. (6.7a) and (6.7b) are to be replaced by

$$\hat{\delta}\hat{E}-\hat{\ast}(\hat{y}\wedge\hat{\ast}\hat{B})=4\pi\rho, \tag{6.14a}$$

$$\hat{\delta}\hat{B}-\hat{\ast}\mathscr{L}_V\hat{\ast}\hat{E}-\hat{\ast}(\hat{a}\wedge\hat{\ast}\hat{B})=4\pi\hat{j} \tag{6.14b}$$

[they can be obtained directly by also applying $\hat{\ast}$ on Eqs. (6.7a) and (6.7b)].

The decomposition of the *4-potential* 1-form

$$A\leftrightarrow\begin{pmatrix}\phi\\-\hat{A}\end{pmatrix} \tag{6.15}$$

gives

$$
\begin{pmatrix} \hat{E} \\ -\hat{B} \end{pmatrix} \leftrightarrow F \equiv dA \leftrightarrow \begin{pmatrix} -\hat{d}+\hat{a} & \mathscr{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} \phi \\ -\hat{A} \end{pmatrix} = \begin{pmatrix} -\hat{d}\phi+\phi\hat{a}-\mathscr{L}_V\hat{A} \\ \phi\hat{y}-\hat{d}\hat{A} \end{pmatrix}, \tag{6.16}
$$

so that

$$
\hat{E} = -\hat{d}\phi + \phi\hat{a} - \mathscr{L}_V\hat{A}, \tag{6.17}
$$

$$
\hat{B} = \hat{d}\hat{A} - \phi\hat{y}. \tag{6.18}
$$

Finally, the gauge transformation:

$$
A \mapsto A' \equiv A + d\chi \leftrightarrow \begin{pmatrix} \phi \\ -\hat{A} \end{pmatrix} + \begin{pmatrix} -\hat{d}+\hat{a} & \mathscr{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{6.19}
$$

is

$$
\phi \mapsto \phi' \equiv \phi + V\chi, \tag{6.20}
$$

$$
\hat{A} \mapsto \hat{A}' \equiv \hat{A} - \hat{d}\chi. \tag{6.21}
$$

VII. CONCLUSIONS AND SUMMARY

In this article we presented a simple method of 3+1 decomposition of the physical equations written in terms of differential forms on space–time (M,g) with respect to a general observer field V.

The method consists of the decomposition of both forms and operations on them. The decomposition of forms is based technically on a simple identity (2.4), which can be interpreted in terms of projection operators on $\Omega^p(M)$. The decomposition of the operations on forms consists first in the decomposition (4.2) of the Hodge star operator and then the decomposition of the exterior derivative d. Here the formalism mimics the approach used standardly in the theory of connections on principal bundle, viz. we first introduce the operator hor [projecting on the ''spatial part'' of the form; its simple realization is given by Eq. (3.2)] and then define the *spatial* exterior derivative as $\hat{d} := \text{hor } \hat{d}$ (the counterpart of the *covariant* exterior derivative on principal bundle with connection). The decomposition of d is then given by Eq. (4.10). The essential property of $\hat{d}$, which makes it a useful object, is the validity of the *spatial Stokes formula* (4.6). It provides the usual link between the differential and integral formulations of the physical laws, respectively. The language of differential forms on both 4 and 3+1 levels turns out to be the most convenient tool for realization of this link, since forms are the objects directly present under the integral signs.

Let us also mention that the quantities of physical interest which ''are not'' forms (energy–momentum tensor, Ricci and Einstein tensors, etc.) admit description in terms of forms;[4] it is then possible to apply the decomposition presented here also to them.

ACKNOWLEDGMENT

APPENDIX A: PROOF OF EQ. (3.2)

The formula to be proved

$$
(\text{hor } \alpha)(U,\ldots,W) \equiv \alpha(\text{hor } U,\ldots,\text{hor } W) = (i_V j_V \alpha)(U,\ldots,W) \tag{A1}
$$

is $\mathscr{F}$ linear $\Rightarrow$ it is enough to take either all vector fields $(U,\ldots,W)$ horizontal or one vertical (then V is enough) and the rest horizontal. The former case means to check

$$\alpha(U,\ldots,W)=\alpha(U,\ldots,W)-(\widetilde{V}\wedge i_V\alpha)(U,\ldots,W), \tag{A2}$$

the latter case

$$0=(i_V i_V j_V\alpha)(U,\ldots,W). \tag{A3}$$

Both are easily seen to hold.

APPENDIX B: PROOF OF EQ. (4.2)

In general, one has in any orthonormal right-handed frame by definition

$$\underbrace{*e^a\wedge\cdots\wedge e^b}_{p}=\frac{1}{(n-p)!}\,\eta^{ac}\cdots\eta^{bd}\epsilon_{c\cdots de\cdots f}e^e\wedge\cdots\wedge e^f. \tag{B1}$$

Let

$$e^{0i\cdots j}:=e^0\wedge\underbrace{e^i\wedge\cdots\wedge e^j}_{p-1},\quad e^{k\cdots l}:=\underbrace{e^k\wedge\cdots\wedge e^l}_{p} \tag{B2}$$

be mixed and spatial basis p forms in $(M^n;g;\underbrace{+\ -\cdots\ -}_{n})$, respectively, and let us treat the orthogonal complement to $V\equiv e_0$ as the Euclidean space with the signature $(+\cdots+)$ (its dimension is $n-1$). Then

$$*e^{0i\cdots j}=\frac{1}{(n-p)!}\,\underbrace{\eta^{00}\eta^{ii}\cdots\eta^{jj}}_{(-)^{p-1}}\,\underbrace{\epsilon_{0i\cdots jk\cdots l}e^{k\cdots l}}_{((n-1)-(p-1))!\,\hat{*}e^{i\cdots j}}\ =(-1)^{p-1}\hat{*}e^{i\cdots j}=\hat{*}\hat{\eta}e^{i\cdots j} \tag{B3}$$

and

$$*e^{k\cdots l}=\frac{1}{(n-p)!}\,\underbrace{\eta^{kk}\cdots\eta^{ll}}_{(-)^p}\epsilon_{k\cdots la\cdots b}e^{a\cdots b}=\frac{(-1)^p}{(n-p)!}\,(n-p)\epsilon_{k\cdots l0r\cdots s}e^{0r\cdots s}$$

$$=\frac{1}{(n-p-1)!}\,\epsilon_{0k\cdots lr\cdots s}e^{0r\cdots s}=e^0\wedge\hat{*}e^{k\cdots l}. \tag{B4}$$

Then if

$$\hat{s} \equiv \frac{1}{(p-1)!}\, \hat{s}_{i\cdots j} e^{i\cdots j}, \quad (p-1)-\text{form},$$

$$\hat{r} \equiv \frac{1}{p!}\, \hat{r}_{k\cdots l} e^{k\cdots l}, \quad p-\text{form},$$

and

$$\alpha = e^0 \wedge \hat{s} + \hat{r}, \tag{B5}$$

we have

$$*(e^0 \wedge \hat{s} + \hat{r}) = \frac{1}{(p-1)!}\, \hat{s}_{i\cdots j} *e^{0i\cdots j} + \frac{1}{p!}\, \hat{r}_{k\cdots l} *e^{k\cdots l} = e^0 \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s}. \tag{B6}$$

APPENDIX C: INTERPRETATION OF $\hat{a}$ AND $\hat{y}$ IN EQ. (4.8)

Let

$$d\widetilde{V} = \widetilde{V} \wedge \hat{a} + \hat{y}$$

be the decomposition (4.8) of the 2-form $d\widetilde{V}$. Then according to Eq. (2.9) one has

$$\hat{a} = i_V d\widetilde{V} = \underbrace{(i_V d + d i_V)}_{\mathscr{L}_V}\widetilde{V} - d\underbrace{i_V\widetilde{V}}_{\|V\|^2=1} = \mathscr{L}_V \widetilde{V} = \mathscr{L}_V(g(V,\cdot)) = (\mathscr{L}_V g)(V,\cdot) + g(\underbrace{\mathscr{L}_V V}_{[V,V]=0},\cdot)$$

$$= (\mathscr{L}_V g)(V,\cdot). \tag{C1}$$

However, for the Levi-Civita connection one has in arbitrary coordinates

$$(\mathscr{L}_V g)_{ij} = V_{i;j} + V_{j;i} \tag{C2}$$

$\Rightarrow$

$$((\mathscr{L}_V g)(V,\cdot))_i = V^j V_{i;j} + V^j V_{j;i} = (\nabla_V \widetilde{V})_i + (d\underbrace{\tfrac{1}{2}\|V\|^2}_{1})_i$$

so that

$$\hat{a} = (\mathscr{L}_V g)(V,\cdot) = \nabla_V \widetilde{V} = \underbrace{(\nabla_V g)}_{0}(V,\cdot) + g(\nabla_V V,\cdot) \equiv g(a,\cdot), \tag{C3}$$

where

$$a := \nabla_V V \tag{C4}$$

is the *acceleration field* corresponding to the observer field V; thus

$$\hat{a} = g(a,\cdot) \equiv \widetilde{a}. \tag{C5}$$

The 2-form $\hat{y}$: According to the Frobenius theorem, $\hat{y} \neq 0$ means the nonintegrability of the *horizontal* (3-space) *distribution*. This can also be rephrased as the impossibility of the synchronization of the clocks within the 3-space for the observers moving along V. Indeed, let

$$t: \mathcal{U} \to \mathbb{R}$$

be a time function (coordinate) in a 4-region $\mathcal{U}$, synchronized for any two nearby space-related points, i.e.,

(1) $Wt = 0$ for any horizontal W (t is constant along the instantaneous 3-space),
(2) $Vt \equiv \chi > 0$ (time increases along any observer's world line).

Condition (1) can also be rewritten as

$$0 = (\text{hor } W)t = \langle dt, \text{hor } W \rangle = \langle \text{hor } dt, W \rangle \equiv \langle \hat{d}t, W \rangle$$

for *any* W, i.e.,

$$\hat{d}t = 0 \tag{C6}$$

as a 1-form. According to Eq. (D 1)

$$0 = \hat{d}t = dt - (Vt)\widetilde{V} \equiv dt - \chi\widetilde{V}$$

or

$$\widetilde{V} = \psi dt, \quad \psi \equiv \chi^{-1} > 0. \tag{C7}$$

Then

$$d\widetilde{V} = d\psi \wedge dt = \frac{d\psi}{\psi} \wedge \psi dt = \widetilde{V} \wedge \left(-\frac{d\psi}{\psi} \right) = \widetilde{V} \wedge \left(-\frac{\hat{d}\psi + (V\psi)\widetilde{V}}{\psi} \right) = \widetilde{V} \wedge \left(-\frac{\hat{d}\psi}{\psi} \right).$$

Comparison with Eq. (4.8) then gives

$$\hat{y} = 0, \quad \hat{a} = -\frac{\hat{d}\psi}{\psi} \equiv -\hat{d}\Phi, \quad \psi \equiv e^{\Phi}. \tag{C8}$$

Thus $\hat{y} \neq 0$ is the obstacle for existence of a time function t synchronized in 3-space and (if $\hat{y} = 0$) $\Phi := \ln \psi$ *is the "gravitational potential"* (*Ref.* 6, *p.* 33).

APPENDIX D : THE PROOFS OF EQS.(4.10) AND (4.12)

Let $\hat{b}$ be any spatial form, i.e., $i_V\hat{b} = 0$; then

$$d\hat{b} = (j_V i_V + i_V j_V)d\hat{b} = \widetilde{V} \wedge \underbrace{i_V d}_{\mathscr{L}_V - d i_V} \hat{b} + \underbrace{i_V j_V d\hat{b}}_{\hat{d}} = \widetilde{V} \wedge \mathscr{L}_V \hat{b} + \hat{d}\hat{b} - \widetilde{V} \wedge d \underbrace{i_V \hat{b}}_{0},$$

so that on *spatial* forms

$$d\hat{b} = \widetilde{V} \wedge \mathscr{L}_V\hat{b} + \hat{d}\hat{b}. \tag{D1}$$

Then on a *general* forms α,

$$d\alpha = d(\widetilde{V}\wedge\hat{s}+\hat{r}) = d\widetilde{V}\wedge\hat{s} - \widetilde{V}\wedge d\hat{s} + d\hat{r}$$

$$= (\widetilde{V}\wedge\hat{a}+\hat{y})\wedge\hat{s} - \widetilde{V}\wedge(\widetilde{V}\wedge\mathscr{L}_V\hat{s}+\hat{d}\hat{s}) + \widetilde{V}\wedge\mathscr{L}_V\hat{r} + \hat{d}\hat{r}$$

$$= \widetilde{V}\wedge(\mathscr{L}_V\hat{r} - \hat{d}\hat{s} + \hat{a}\wedge\hat{s}) + (\hat{d}\hat{r} + \hat{y}\wedge\hat{s}), \qquad (D2)$$

where all forms in the brackets are already spatial.

The computation of $\hat{d}\hat{d}$: for arbitrary horizontal form $\hat{b}$

$$\hat{d}\hat{d}\hat{b} \overset{(D1)}{=} \text{hor } d(d\hat{b} - \widetilde{V}\wedge\mathscr{L}_V\hat{b}) = -\text{hor}(d\widetilde{V}\wedge\mathscr{L}_V\hat{b} - \widetilde{V}\wedge d\mathscr{L}_V\hat{b}) \overset{(3.7)}{=}$$

$$= -\underbrace{\text{hor } d\widetilde{V}}_{\hat{y}} \wedge \underbrace{\text{hor } \mathscr{L}_V\hat{b}}_{\mathscr{L}_V\hat{b}} + \underbrace{\text{hor } \widetilde{V}}_{0} \wedge\text{hor } d\mathscr{L}_V\hat{b} = -\hat{y}\wedge\mathscr{L}_V\hat{b}, \qquad (D3)$$

since

$$\text{hor } \mathscr{L}_V\hat{b} = i_V j_V(i_V d + d i_V)\hat{b} = i_V \underbrace{j_V i_V}_{1-i_V j_V} d\hat{b} = i_V d\hat{b} - \underbrace{i_V i_V}_{0} j_V d\hat{b} = (i_V d + d\ i_V)\hat{b} = \mathscr{L}_V\hat{b}. \quad (D4)$$

APPENDIX E: THE VOLUME EXPANSION COEFFICIENT θ

According to Eq. (4.2)

$$\omega \equiv *1 = \widetilde{V}\wedge\hat{*}1 \equiv \widetilde{V}\wedge\hat{\omega},$$

where $\hat{\omega} \equiv \hat{*}1$ is the *spatial* volume form. The standard definition of the *volume expansion* coefficient θ is (Ref. 6, p. 9)

$$\theta := V^\mu_{;\mu} \equiv \nabla\cdot V \equiv \text{div } V. \qquad (E1)$$

Since

$$\mathscr{L}_V\omega = (\text{div } V)\omega \equiv \theta\omega$$

one can write

$$\theta\omega = \widetilde{V}\wedge(\theta\hat{\omega}) = \mathscr{L}_V\omega = (\mathscr{L}_V\widetilde{V})\wedge\hat{\omega} + \widetilde{V}\wedge\mathscr{L}_V\hat{\omega} = (\text{cf. Appendix C}) = \hat{a}\wedge\hat{\omega} + \widetilde{V}\wedge\mathscr{L}_V\hat{\omega}.$$

The first term vanishes (spatial 4-form), $\mathscr{L}_V\hat{\omega}$ is spatial (the end of Appendix D) so that

$$\mathscr{L}_V\hat{\omega} = \theta\hat{\omega}, \qquad (E2)$$

which means that θ is the rate of change of *3-volumes* along the observer's word line γ (with $\dot{\gamma} = V$). As an example $\mathscr{L}_V(\rho\hat{\omega}) = (V\rho + \theta)\hat{\omega}$ [see Eq. (6.10)].

APPENDIX F: THE IDENTITIES RESULTING FROM $dd = 0$

Applying d on the decomposition (4.8) and using (4.10) one obtains

$$0 = dd\widetilde{V} = d\widetilde{V}\wedge\hat{a} - \widetilde{V}\wedge d\hat{a} + d\hat{y} = \widetilde{V}\wedge(\mathscr{L}_V\hat{y} - \hat{d}\hat{a}) + (\hat{d}\hat{y} + \hat{y}\wedge\hat{a})$$

so that $\hat{a}$ and $\hat{y}$ are always related by

$$\hat{d}\hat{a} = \mathscr{L}_V \hat{y}, \tag{F1}$$

$$\hat{d}\hat{y} = -\hat{y} \wedge \hat{a}. \tag{F2}$$

In particular, for $\hat{y} = 0$ we get

$$\hat{d}\hat{a} = 0. \tag{F3}$$

From Eq. (C8) we even know that

$$\hat{a} = -\hat{d}\Phi \tag{F4}$$

in this case.

Applying $dd = 0$ on a spatial form $\hat{r}$ and taking into account Eqs. (F1) and (F2) we obtain another useful identity:

$$[\mathscr{L}_V, \hat{d}] = -\hat{a} \wedge \mathscr{L}_V \quad \text{on} \ \ \hat{\Omega}(M) \tag{F5}$$

[this shows that ''time'' derivative $\mathscr{L}_V$ and ''space'' derivatives hidden in $\hat{d}$ do not commute in general; they do commute, however, for the geodesic observer field ($\hat{a} = 0$)].

APPENDIX G: FORMAL LINKS WITH THE THEORY OF CONNECTIONS ON PRINCIPAL BUNDLES

The formulation used in this article resembles in many respects the theory of connections on principal bundles.[5] There is a (right) action R_g of a Lie group G on the total space P, if we set $G \equiv (\mathbb{R}, +)$, $P \equiv M$ (globally one needs V to be *complete* for this) and the action is identified with the flow generated by V. The difference is, however, that the horizontal distribution here is *not G invariant* in general: since $\widetilde{V}$ is the counterpart of the *connection form* ω—both define the horizontal distribution via annihilation—and the group is one dimensional, the $(\mathbb{R}, +)$ invariance means $\mathscr{L}_V \widetilde{V} = 0$. But

$$\mathscr{L}_V \widetilde{V} \stackrel{(C3)}{=} \hat{a} \neq 0 \quad \text{in general} \tag{G1}$$

[thus for *geodesic* ($\equiv$*non*accelerating) observer field there *is* in fact $\mathbb{R}$ connection available].

Many formulas here are very similar to those in the connection theory, e.g.,

$$\hat{d}\hat{b} = d\hat{b} - \widetilde{V} \wedge \mathscr{L}_V \hat{b} \tag{G2}$$

[cf. Eq. (D1)] is the counterpart of the standard formula valid for the computation of the *covariant exterior derivative* of the horizontal form α of type ρ, viz.

$$D\alpha = d\alpha + \rho'(\omega) \dot{\wedge} \alpha, \tag{G3}$$

where ω is the connection form, ρ a representation of G. To see this more explicitly, one has to notice that the forms of type ρ satisfy

$$\mathscr{L}_{\xi_i} \alpha = -\rho'(E_i)\alpha \tag{G4}$$

($\xi_i \equiv \xi_{E_i}$ being the fundamental field corresponding to the basis element E_i of the Lie algebra $\mathscr{G}$ of G) and consequently

$$\rho'(\omega)\dot{\wedge}\alpha = \omega^i\wedge\rho'(E_i)\alpha = -\omega^i\wedge\mathscr{L}_{\xi_i}\alpha \tag{G5}$$

or

$$D\alpha = d\alpha - \omega^i\wedge\mathscr{L}_{\xi_i}\alpha. \tag{G6}$$

This formula is valid for *all horizontal* forms on the principal bundle (not only of type ρ). For the one-dimensional group (as is the case here) we have exactly the form (G2).

In the same way one can see the similarity of

$$\hat{d}\hat{d}\hat{b} = -\hat{y}\wedge\mathscr{L}_V\hat{b} \tag{G7}$$

[Eq. (4.12)] with the standard formula

$$DD\alpha = \rho'(\Omega)\dot{\wedge}\alpha \equiv \Omega^i\wedge\rho'(E_i)\alpha, \tag{G8}$$

where Ω, the curvature 2-form, is the counterpart of our $\hat{y}$: both encode the (non)integrability of the horizontal distribution and consequently both are computed by the same rule, viz.

$$\hat{y}:=\text{hor } d\widetilde{V}\equiv\hat{d}\widetilde{V} \quad \text{vs} \quad \Omega:=\text{hor } d\omega\equiv D\omega.$$

The counterpart of the *Bianchi identity* $DD\omega=0$ is

$$\hat{d}\hat{d}\widetilde{V} = \hat{d}\hat{y} \overset{(F2)}{=} -\hat{y}\wedge\hat{a}.$$

This is *not* zero in general, but it *is* zero for $\hat{a}=0$, when $\widetilde{V}$ *does* define a connection.

APPENDIX H: THE MAXWELL EQUATIONS IN THE STANDARD VECTOR ANALYSIS NOTATIONS

We use the standard three-dimensional Euclidean space relations [cf. the definitions in Eq. (6.2)]

$$\hat{E}=\mathbf{E}\cdot d\mathbf{r}, \quad \hat{B}=\mathbf{B}\cdot d\mathbf{S}, \quad \hat{j}=\mathbf{j}\cdot d\mathbf{r}, \quad \hat{y}=\mathbf{y}\cdot d\mathbf{S}, \quad \hat{a}=\mathbf{a}\cdot d\mathbf{r}, \tag{H1}$$

$$\hat{*}\hat{E}=\mathbf{E}\cdot d\mathbf{S}, \quad \hat{*}\hat{B}=\mathbf{B}\cdot d\mathbf{r}, \quad \hat{*}\hat{j}=\mathbf{j}\cdot d\mathbf{S}, \quad \hat{*}\hat{y}=\mathbf{y}\cdot d\mathbf{r}, \quad \hat{*}\hat{a}=\mathbf{a}\cdot d\mathbf{S},$$

$$\hat{y}\wedge\hat{*}\hat{B}=(\mathbf{y}\cdot\mathbf{B})\hat{\omega}, \quad \hat{a}\wedge\hat{*}\hat{B}=(\mathbf{a}\times\mathbf{B})\cdot d\mathbf{S}, \quad \hat{a}\wedge\hat{*}\hat{j}=(\mathbf{a}\cdot\mathbf{j})\hat{\omega},$$

$$\hat{y}\wedge\hat{E}=(\mathbf{y}\cdot\mathbf{E})\hat{\omega}, \quad \hat{a}\wedge\hat{E}=(\mathbf{a}\times\mathbf{E})\cdot d\mathbf{S}.$$

One can then introduce curl and div operations according to

$$\hat{d}\hat{E}=:(\text{curl }\mathbf{E})\cdot d\mathbf{S}, \tag{H2}$$

$$\hat{d}\hat{*}\hat{E}=:(\text{div }\mathbf{E})\hat{\omega}, \tag{H3}$$

and consequently,

$$\hat{d}\hat{B}=:(\text{div }\mathbf{B})\hat{\omega}, \tag{H4}$$

$$\hat{d}\hat{*}\hat{B}=:(\text{curl }\mathbf{B})\cdot d\mathbf{S}. \tag{H5}$$

These definitions guarantee the validity of ''standard'' integral formulas like

$$\oint_{\partial \mathscr{D}} \mathbf{E} \cdot d\mathbf{S} = \int_{\mathscr{D}} (\operatorname{div} \mathbf{E}) \hat{\omega}, \tag{H6}$$

$$\oint_{\partial \mathscr{S}} \mathbf{B} \cdot d\mathbf{r} = \int_{\mathscr{S}} (\operatorname{curl} \mathbf{B}) \cdot d\mathbf{S} \tag{H7}$$

as a consequence of the ''spatial'' Stokes formula (4.6). Then from Eqs. (6.7a)–(6.10) we obtain

$$\operatorname{div} \mathbf{E} + \mathbf{y} \cdot \mathbf{B} = 4\pi\rho, \tag{H8}$$

$$(\operatorname{curl} \mathbf{B}) \cdot d\mathbf{S} - \mathscr{L}_V(\mathbf{E} \cdot d\mathbf{S}) - (\mathbf{a} \times \mathbf{B}) \cdot d\mathbf{S} = 4\pi \mathbf{j} \cdot d\mathbf{S}, \tag{H9}$$

$$(\operatorname{curl} \mathbf{E}) \cdot d\mathbf{S} + \mathscr{L}_V(\mathbf{B} \cdot d\mathbf{S}) - (\mathbf{a} \times \mathbf{E}) \cdot d\mathbf{S} = 0, \tag{H10}$$

$$\operatorname{div} \mathbf{B} - \mathbf{y} \cdot \mathbf{E} = 0, \tag{H11}$$

and

$$\mathscr{L}_V(\rho\hat{\omega}) + (\operatorname{div} \mathbf{j}) \hat{\omega} - (\mathbf{a} \cdot \mathbf{j}) \hat{\omega} = 0 \tag{H12}$$

or (cf. Appendix E)

$$V\rho + \theta\rho + \operatorname{div} \mathbf{j} - \mathbf{a} \cdot \mathbf{j} = 0 \tag{H13}$$

as well as the corresponding integral versions

$$\oint_{\partial \mathscr{D}} \mathbf{E} \cdot d\mathbf{S} = 4\pi \int_{\mathscr{D}} \rho\hat{\omega} \equiv 4\pi \mathscr{Q}, \tag{H14}$$

$$\oint_{\partial \mathscr{S}} \mathbf{B} \cdot d\mathbf{r} - \frac{d}{d\tau}\Big|_0 \int_{\Phi_\tau(\mathscr{S})} \mathbf{E} \cdot d\mathbf{S} - \int_{\mathscr{S}} (\mathbf{a} \times \mathbf{B}) \cdot d\mathbf{S} = 4\pi \int_{\mathscr{S}} \mathbf{j} \cdot d\mathbf{S}, \tag{H15}$$

$$\oint_{\partial \mathscr{S}} \mathbf{E} \cdot d\mathbf{r} + \frac{d}{d\tau}\Big|_0 \int_{\Phi_\tau(\mathscr{S})} \mathbf{B} \cdot d\mathbf{S} - \int_{\mathscr{S}} (\mathbf{a} \times \mathbf{E}) = 0, \tag{H16}$$

$$\oint_{\partial \mathscr{D}} \mathbf{B} \cdot d\mathbf{S} = 0, \tag{H17}$$

and

$$\frac{d}{d\tau}\Big|_0 \int_{\Phi_\tau(\mathscr{D})} \rho\omega + \oint_{\partial \mathscr{D}} \mathbf{j} \cdot d\mathbf{S} - \int_{\mathscr{D}} (\mathbf{a} \cdot \mathbf{j}) \hat{\omega} = 0. \tag{H18}$$

In the simplest situation, i.e., for irrotational ($\mathbf{y} = 0$), geodesic ($\mathbf{a} = 0$) observer field V (then $V = \partial_t$, $\widetilde{V} = dt$) we get

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \tag{H19}$$

$$(\operatorname{curl} \mathbf{B}) \cdot d\mathbf{S} - \mathscr{L}_{\partial_t}(\mathbf{E} \cdot d\mathbf{S}) = 4\pi \mathbf{j} \cdot d\mathbf{S}, \tag{H20}$$

$$(\text{curl } \mathbf{E}) \cdot d\mathbf{S} + \mathscr{L}_{\partial_t}(\mathbf{B} \cdot d\mathbf{S}) = 0, \tag{H21}$$

$$\text{div } \mathbf{B} = 0, \tag{H22}$$

and

$$\partial_t \rho + \theta \rho + \text{div } \mathbf{j} = 0. \tag{H23}$$

APPENDIX I: THE EQUATIONS $d\alpha = \beta$ AND $\delta\alpha = \gamma$ FOR IRROTATIONAL OBSERVER FIELD

Let V be an *irrotational* ($\hat{y} = 0$) observer field. Then [cf. Eq. (C 8)]

$$\hat{a} = -\hat{d}\Phi, \quad \Phi \equiv \ln \psi, \quad \widetilde{V} = \psi dt, \quad V = \psi^{-1}\partial_t \tag{I1}$$

($e^{\Phi} \equiv \psi$—*lapse function*, cf. Ref. 10). Let us study the equation of the structure

$$d\alpha = \beta. \tag{I2}$$

If

$$\alpha \leftrightarrow \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}, \quad \beta \leftrightarrow \begin{pmatrix} \hat{S} \\ \hat{R} \end{pmatrix}, \tag{I3}$$

we have

$$\begin{pmatrix} -\hat{d} + \hat{a} & \mathscr{L}_V \\ 0 & \hat{d} \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} \hat{S} \\ \hat{R} \end{pmatrix} \tag{I4}$$

or

$$(-\hat{d} + \hat{a})\hat{s} + \mathscr{L}_V \hat{r} = \hat{S}, \quad \hat{d}\hat{r} = \hat{R}. \tag{I5}$$

Now

$$(-\hat{d} + \hat{a})\hat{s} = -\hat{d}\hat{s} - \hat{d}\Phi \wedge \hat{s} = -e^{-\Phi}\hat{d}(e^{\Phi}\hat{s}),$$

$$\mathscr{L}_V \hat{r} = e^{-\Phi} \mathscr{L}_{\partial_t} \hat{r}$$

so that we obtain

$$-\hat{d}(e^{\Phi}\hat{s}) + \mathscr{L}_{\partial_t}\hat{r} = e^{\Phi}\hat{S}, \quad \hat{d}\hat{r} = \hat{R}. \tag{I6}$$

Thus we have the simple rule: The acceleration term $\hat{a} = -\hat{d}\Phi$ manifests itself only through the replacement

$$\hat{s} \mapsto e^{\Phi}\hat{s} \equiv \psi\hat{s}, \quad \hat{S} \mapsto e^{\Phi}\hat{S} \equiv \psi\hat{S} \tag{I7}$$

of the *upper* components of Eq. (I3) (the lower ones being unchanged) in the corresponding equations with $\hat{a} = 0$, i.e., in

$$-\hat{d}\hat{s} + \mathscr{L}_{\partial_t}\hat{r} = \hat{S}, \quad \hat{d}\hat{r} = \hat{R}. \tag{I8}$$

The similar analysis repeated for the equation

$$\delta\alpha = \gamma$$

where

$$\gamma \mapsto \begin{pmatrix} \mathfrak{G} \\ \mathfrak{R} \end{pmatrix} \tag{I9}$$

shows that the replacement to be performed in the corresponding equations with $\hat{a}=0$ is

$$\hat{r} \mapsto e^{\Phi}\hat{r} \equiv \psi\hat{r}, \quad \mathfrak{R} \mapsto e^{\Phi}\mathfrak{R} \equiv \psi\mathfrak{R}, \tag{I10}$$

i.e., only the *lower* components do change now.

For the case of the Maxwell equations (6.14), (6.8), and the continuity equation (6.9) it results in the replacements

$$\hat{E} \mapsto e^{\Phi}\hat{E} \equiv \psi\hat{E}, \quad \rho \mapsto e^{\Phi}\rho \equiv \psi\rho \tag{I11}$$

in the *homogeneous* pair (ρ is, however, trivial since it is not present there),

$$\hat{B} \mapsto e^{\Phi}\hat{B} \equiv \psi\hat{B}, \quad \hat{j} \mapsto e^{\Phi}\hat{j} \equiv \psi\hat{j} \tag{I12}$$

in the *inhomogeneous* pair and

$$\hat{j} \mapsto e^{\Phi}\hat{j} \equiv \psi\hat{j} \tag{I13}$$

in the continuity equation, i.e., the equations for $\hat{a} = -\hat{d}\Phi$ read (cf. Ref. 10, pp. 18–19 and Appendix H here)

$$\hat{d}\hat{*}\hat{E} = 4\pi\rho\hat{\omega}, \quad \hat{d}\hat{*}(e^{\Phi}\hat{B}) - \mathscr{L}_{\partial_t}\hat{*}\hat{E} = 4\pi\hat{*}(e^{\Phi}\hat{j}), \tag{I14}$$

$$\hat{d}(e^{\Phi}\hat{E}) + \mathscr{L}_{\partial_t}\hat{B} = 0, \quad \hat{d}\hat{B} = 0, \tag{I15}$$

and

$$\mathscr{L}_{\partial_t}(\rho\hat{\omega}) + \hat{d}\hat{*}(e^{\Phi}\hat{j}) = 0. \tag{I16}$$

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