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Citation: *J. Math. Phys.* **54**, 102901 (2013); doi: 10.1063/1.4824684

View online: <http://dx.doi.org/10.1063/1.4824684>

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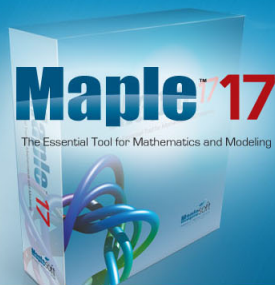
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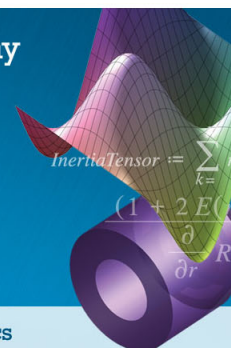
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On symmetries and conserved quantities in Nambu mechanics

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(Received 8 August 2013; accepted 23 September 2013; published online 15 October 2013)

In Hamiltonian mechanics, a (continuous) symmetry leads to conserved quantity, which is a *function* on (extended) phase space. In Nambu mechanics, a straightforward consequence of symmetry is just a *relative integral invariant*, a differential form which only upon integration over a cycle provides a conserved real number. The origin of the difference may be traced back to a shift in degrees of relevant forms present in equations of motion, or, alternatively, to a corresponding shift in degrees of relevant objects in *action integral* for Nambu mechanics. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4824684>]

I. INTRODUCTION

According to seminal paper of Emmy Nöther (Ref. 1, see also Ref. 2), there is a close correspondence between symmetries of action integral and conserved quantities for the dynamics given by the action.

Recall briefly, how it works in *Hamiltonian mechanics*. First, Hamilton equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \quad \dot{p}_a = -\frac{\partial H}{\partial q^a} \quad (1)$$

may be succinctly written as

$$i_{\dot{\gamma}} d\sigma = 0 \quad (2)$$

(see Refs. 3 and 4), where

$$\dot{\gamma} = \dot{q}^a \partial_{q^a} + \dot{p}_a \partial_{p_a} + \partial_t \quad (3)$$

is the velocity vector to curve γ (on *extended* phase space), and

$$\sigma = p_a dq^a - H dt \quad (4)$$

is a distinguished 1-form, Poincaré-Cartan integral invariant (on extended phase space as well). Then, the standard action integral reads (see Refs. 3–5)

$$S[\gamma] = \int_{\gamma} \sigma = \int_{t_1}^{t_2} (p_a \dot{q}^a - H) dt. \quad (5)$$

Conserved quantities extracted from (continuous) symmetries of the action are *functions* on (extended) phase space (e.g., energy from time translations, components of momentum from space translations, components of angular momentum from rotations, etc.). Explicitly (for details, including a derivation, see Appendix A), if the symmetry is given by a vector field ξ (on extended phase space), the corresponding conserved quantity f_{ξ} is given as

$$f_{\xi} = i_{\xi} \sigma - \chi_{\xi}. \quad (6)$$

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In 1973, Nambu proposed a modification of Hamiltonian mechanics (Ref. 6). In its basic version, phase space is three-dimensional (so that extended Nambu phase space is four-dimensional) and equations of motion (Nambu equations) read

$$\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \quad i = 1, 2, 3 \quad (7)$$

or, in vector notation,

$$\dot{\mathbf{r}} = \nabla H_1 \times \nabla H_2. \quad (8)$$

Here, the two ‘‘Hamiltonians’’ are, in general, functions of x_1, x_2, x_3 , and t .

It turns out that construction of action integral for Nambu mechanics is a slightly delicate subject. It was observed (Refs. 7 and 8) that Nambu equations (7) may be succinctly written as

$$i_{\dot{\gamma}} d\hat{\sigma} = 0 \quad (9)$$

too, where

$$\dot{\gamma} = \dot{x}^1 \partial_1 + \dot{x}^2 \partial_2 + \dot{x}^3 \partial_3 + \partial_t \quad (10)$$

is the velocity vector to curve γ on extended Nambu phase space.

Equation (9) formally looks exactly like (2), it has the structure of ‘‘vortex-lines equation’’ (see Ref. 9). However, there is an important difference between the two equations, in that the form $\hat{\sigma}$, the counterpart of the *one*-form (4), is a *two*-form, now. Explicitly, it reads

$$\hat{\sigma} := x^1 dx^2 \wedge dx^3 - H_1 dH_2 \wedge dt. \quad (11)$$

At first sight, the difference might look innocent. Notice, however, that it is no longer possible to write down action integral like (5), since there is no candidate for *one*-form to be integrated along the *curve* γ . Instead, the *two*-form $\hat{\sigma}$ is available. Therefore, the only way to produce a number (the value of action) is to integrate $\hat{\sigma}$ over a two-dimensional *surface*.

In Ref. 7, a possibility to associate a surface with a *single* trajectory γ is investigated. It leads to an action, proposed already before in Ref. 10. This is not very satisfactory, since its extremals are curves on which $\dot{\mathbf{r}}$ is just *proportional*, not necessarily equal, to the rhs of (8).

A more interesting way of how to come to a surface is proposed in Ref. 8. There, the value of action integral is associated with an appropriate one-parameter *family* of trajectories rather than with a single trajectory.

Namely, consider the family constructed as follows: Let, from each point p of a *one-cycle* (loop) c_1 at the time t_1 , emanate the solution $\gamma(t)$ of Nambu equations (9), fulfilling initial condition $\gamma(t_1) = p$. At the time t_2 , the points $\gamma(t_2)$ (for all $p \in c_1$) form a *one-cycle* (loop) c_2 again (the image of c_1 with respect to the Nambu flow for $t_2 - t_1$) and the points $\gamma(t)$, for all $t \in \langle t_1, t_2 \rangle$ and all $p \in c_1$, form a *two-chain* (two-dimensional surface) Σ made of solutions (see Fig. 1; notice that $\partial \Sigma = c_1 - c_2$). The value of the action, assigned to the family, is defined as

$$S[\Sigma] = \int_{\Sigma} \hat{\sigma}. \quad (12)$$

One then easily verifies (see Appendix B) that the surface given by the family of *solutions* of Nambu equations is indeed an *extremal* of the action integral (12).

II. SYMMETRIES OF THE ACTION

Now, let us mimic in Nambu setting, i.e., using Takhtajan’s action integral (12), the standard ‘‘Hamiltonian’’ procedure for obtaining conserved quantities from symmetries (see Appendix A).

So we call, first, vector field ξ a *symmetry* if the action integral (12) evaluated on $\Phi_{\epsilon}(\Sigma)$ (the flow Φ_{ϵ} corresponds to ξ here) gives the same number as on Σ itself,

$$S[\Phi_{\epsilon} \Sigma] = S[\Sigma] \quad (13)$$

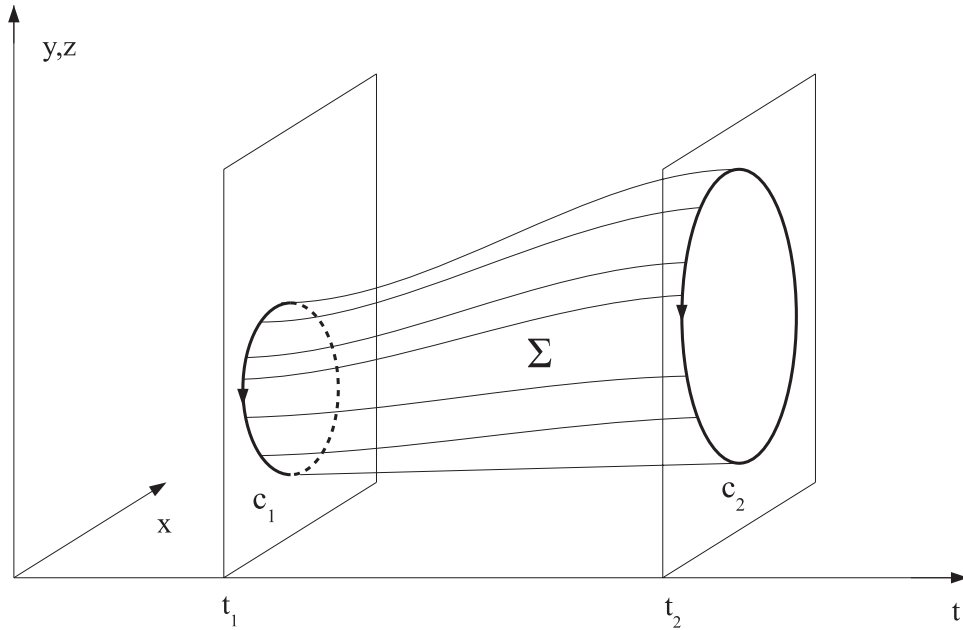


FIG. 1. A two-chain Σ made up from a one-cycle c_1 using solutions of Nambu equations.

(i.e., $\delta S = 0$; there are *no* restrictions on either time component of ξ or the values of ξ at the boundary $\partial \Sigma = c_1 - c_2$ of Σ). By direct computation of δS (see Appendix B), we obtain

$$\delta S = \epsilon \int_{\Sigma} i_{\xi} d\hat{\sigma} + \epsilon \oint_{\partial \Sigma} i_{\xi} \hat{\sigma}. \tag{14}$$

Now, the first integral on the rhs vanishes on the surface Σ given by the family of *solutions* of Nambu equations (by the argument mentioned in Appendix B, $\dot{\gamma}$ is tangent to Σ and, at the same time, it is annihilated by $d\hat{\sigma}$). The second integral is over $\partial \Sigma = c_1 - c_2$ and the sum of both integrals on the rhs of (14) is to vanish. So we get

$$0 = \left(\oint_{c_1} - \oint_{c_2} \right) i_{\xi} \hat{\sigma} \tag{15}$$

or, equivalently,

$$\oint_{c_1} i_{\xi} \hat{\sigma} = \oint_{c_2} i_{\xi} \hat{\sigma}. \tag{16}$$

This is, however, nothing but a *conservation law*: for *solutions* of Nambu equations,

$$f_{\xi}(t_1; c_1) = f_{\xi}(t_2; c_2), \tag{17}$$

where f_{ξ} is given by the *integral*

$$f(t_a; c_a) := \oint_{c_a} i_{\xi} \hat{\sigma} \quad a = 1, 2. \tag{18}$$

In full analogy with the Hamiltonian case (see the text following (A6)), a more general definition of symmetry is possible. Rather than using differential version of (13), vanishing of the Lie derivative

$$\mathcal{L}_{\xi} \hat{\sigma} = 0, \tag{19}$$

we define *symmetry of Nambu system* as a vector field ξ obeying somewhat weaker condition,

$$\mathcal{L}_{\xi} \hat{\sigma} = d\chi_{\xi} \tag{20}$$

(so, *exactness* of the Lie derivative is enough). Or, by Cartan's formula,

$$i_\xi d\hat{\sigma} = -d(i_\xi \hat{\sigma} - \chi_\xi). \quad (21)$$

Upon integration over the surface Σ , we get

$$\int_\Sigma i_\xi d\hat{\sigma} = -\oint_{\partial\Sigma} (i_\xi \hat{\sigma} - \chi_\xi). \quad (22)$$

Since the lhs vanishes (on solutions), it holds

$$\oint_{c_1} (i_\xi \hat{\sigma} - \chi_\xi) = \oint_{c_2} (i_\xi \hat{\sigma} - \chi_\xi). \quad (23)$$

So, we obtain the statement

$$f_\xi(t_1; c_1) = f_\xi(t_2; c_2), \quad (24)$$

where f_ξ is given by the *integral*

$$f_\xi(t_a; c_a) := \oint_{c_a} (i_\xi \hat{\sigma} - \chi_\xi) \quad a = 1, 2. \quad (25)$$

In more wordy formulation: Given a symmetry ξ take, at time t_1 , an arbitrary one-cycle (loop), c_1 . Compute the line integral

$$\oint_{c_1} (i_\xi \hat{\sigma} - \chi_\xi). \quad (26)$$

Then, let each point of c_1 evolve by Nambu flow up to time t_2 . You get another one-cycle (loop), c_2 . Compute, again, the line integral

$$\oint_{c_2} (i_\xi \hat{\sigma} - \chi_\xi). \quad (27)$$

The statement is: You get the same number.

III. INTEGRAL INVARIANTS

What we obtained from symmetry ξ is nothing but a *relative integral invariant* for Nambu dynamics. In general this is, by definition, a differential p -form α such that, when integrated over a p -cycle, it gives an invariant with respect to the dynamical flow. Put another way, if a dynamical vector field Γ generates the flow Φ_t (time evolution) and if c_2 is the Φ_t -image of an *arbitrary* p -cycle c_1 , then

$$\oint_{c_1} \alpha = \oint_{c_2} \alpha \quad (28)$$

(see, e.g., Refs. [3](#), [5](#), [11](#), and [12](#)).

In our case, the result [\(23\)](#) may be regarded as the statement that on Nambu extended phase space endowed with the dynamical vector field Γ defined by

$$i_\Gamma d\hat{\sigma} = 0 \quad (29)$$

(see [\(9\)](#)), we get, as a consequence of existence of a symmetry ξ , a relative integral invariant. So [\(28\)](#) holds for the *one-form*

$$\alpha = i_\xi \hat{\sigma} - \chi_\xi. \quad (30)$$

Of course, as is always the case, our relative integral invariant then automatically yields an *absolute* integral invariant, integral of the *exterior derivative* $d\alpha$ of α over *any* two-chain (two-dimensional surface) s . So, taking into account [\(21\)](#),

$$\int_{s_1} i_\xi d\hat{\sigma} = \int_{s_2} i_\xi d\hat{\sigma}. \quad (31)$$

IV. AN APPROACH VIA MOMENTUM MAP

Momentum map is a powerful tool for studying connection between symmetries and conserved quantities in Hamiltonian framework.

Standardly, it is a map from a symplectic manifold (phase space of a Hamiltonian system) to the dual of the Lie algebra of the symmetry group acting on the phase space (see, e.g., Refs. 3, 4, and 13).

A slightly modified version enables one to treat momentum map as a map from *extended* phase space of the Hamiltonian system to the dual of the Lie algebra (see Appendix C).

Here we mimic the derivation of momentum map within the framework of the *extended Nambu* phase space.

So, consider an action of a Lie group G on extended Nambu phase space. The action satisfies $R_g^* d\hat{\sigma} = d\hat{\sigma}$ or, infinitesimally, $\mathcal{L}_{\xi_X} d\hat{\sigma} = 0$. This can be rewritten as *closedness* of *two-form* β_X :

$$d\beta_X = 0 \quad \beta_X := i_{\xi_X} d\hat{\sigma}. \quad (32)$$

(Notice that (32) is a statement concerning *complete Nambu system*, rather than just the extended Nambu phase space alone, since $\hat{\sigma}$ contains both ‘‘Hamiltonians’’ H_1 and H_2 .) Often β_X happens to be *exact*; then we get

$$i_{\xi_X} d\hat{\sigma} = -dP_X. \quad (33)$$

Here, P_X is a *one-form* (rather than a zero-form, i.e., a function, as in Hamiltonian case). We can achieve linearity of P_X in X in a standard way (see Ref. 4), write $P_X = X^i P_i$ and introduce

$$P := P_i E^i. \quad (34)$$

What we obtained is a \mathcal{G}^* -valued *one-form* on extended Nambu phase space (rather than the corresponding \mathcal{G}^* -valued *zero-form* = *function* known from Hamiltonian framework as *the momentum map*). Its j -th component one-form, P_j , is defined, according to (33), by the equation

$$i_{\xi_{E_j}} d\hat{\sigma} = -dP_j. \quad (35)$$

Comparison with (21) and (23) shows that in this way we get, as a reward for finding a symmetry, *the same* integral invariants as we found in Sec. II,

$$\oint_c P_j = \text{relative integral invariant} \quad (36)$$

(P_j equals $i_{\xi_{E_j}} \hat{\sigma} - \chi_{\xi_{E_j}}$ modulo additive *closed* one-form, vanishing after integration over the cycle) and in Sec. III,

$$\int_s i_{\xi_{E_j}} d\hat{\sigma} = \text{absolute integral invariant} \quad (37)$$

(see (31) and (35)).

So, to conclude, *both* approaches (the one discussed in Sec. II as well as the one discussed here) lead to *the same picture* regarding the relation between symmetries and conserved quantities due to them: each symmetry provides us with a *relative integral invariant* of the form (23) (and, consequently, with an *absolute integral invariant* of the form (31)).

Let us remark that the question of momentum map in the context of Nambu mechanics was already addressed before in Ref. 14. There, it was introduced as a $\mathcal{G}^* \times \mathcal{G}^*$ valued mapping (i.e., function) defined by the formula

$$i_{\xi_X} d\hat{\sigma} = dP_{1X} \wedge dP_{2X}, \quad (38)$$

where both components of the pair (P_1, P_2) are defined by (C3), in which M is *Nambu* phase space now. Notice, however, that the formula (38) is inconsistent, since the lhs is linear in X , whereas the rhs is not.

V. THE CASE OF MORE NAMBU HAMILTONIANS

What we treated in detail was the “basic” version of Nambu mechanics, the situation when the Nambu phase space is *three*-dimensional and there are *two* Nambu “Hamiltonians,” H_1 and H_2 .

Already in the original paper (Ref. 6) Nambu pointed out that the idea may be straightforwardly generalized to more dimensions, n -dimensional (Nambu) phase space and $n - 1$ Nambu “Hamiltonians,” H_1, \dots, H_{n-1} . (There are also other generalizations, see Refs. 6 and 8.)

And it is easily seen that all constructions discussed in this paper work equally well in the n -dimensional version. In particular, $\hat{\sigma}$ becomes $(n - 1)$ -form, c_1 becomes $(n - 2)$ -cycle, Σ is $(n - 1)$ -dimensional surface, and so on. (See Refs. 8 and 9.) Conserved quantities are still integral invariants (formally equally looking formulas (23) and (31) hold, where c_a are $(n - 2)$ -cycles and s_a are $(n - 1)$ -chains).

VI. CONCLUSIONS

Both Hamiltonian and Nambu mechanics study motion of (formally speaking) *points* in phase space (or, by technical reasons, in extended phase space). Therefore, it is natural to expect conserved quantities to be *functions* on (perhaps extended) phase space. Once we study a particular motion, we evaluate the function at the time t_1 at the point where the motion begins, and then we profit from the fact that, at the future points of the trajectory, the same value of the function is guaranteed by the conservation law.

In Hamiltonian mechanics the story really goes like this. *Functions* (like energy or various components of momentum) are often conserved and this fact then makes life much more easy.

In Nambu mechanics, there are conserved *functions* as well. Already in the first paper on the subject (Ref. 6), Nambu discusses, as a key example, dynamical Euler equations for the motion of a free rigid rotator. Here, the Nambu phase space is three-dimensional (actually, it is just a subsystem of a complete *six*-dimensional *Hamiltonian* system of equations; one should add *kinematical* Euler equations to get the standard picture) and both energy and square of the angular momentum are conserved. Nambu shows that it is possible to choose these two (conserved) functions as the two “Hamiltonians” H_1 and H_2 in his approach. In many papers, thereafter, authors write various systems of ordinary differential equations in Nambu mechanics form, exactly to “make explicit” conserved quantities (functions, namely, H_1 and H_2).

However, the message of this paper is that *these* conserved functions *do not* directly follow from symmetries, as is usual in Hamiltonian case. In the case of symmetries, application of more or less standard machinery results, because of a peculiar situation with the action integral (presence of a two-form rather than one-form, necessity of taking a *family* of trajectories rather than a single trajectory), in conserved quantities, which have the character of *integral invariants* rather than usual conserved functions. Namely, the machinery leads to *higher-degree* forms rather than usual zero-forms, that is, functions (one-form for a relative invariant, and its exterior derivative, two-form, for the corresponding absolute invariant). As a reward for finding a symmetry, the conserved *number* is only obtained as *integral* of the form over a one-cycle (or two-chain for the absolute invariant).

In order to make the picture complete, let us note that also *Liouville theorem* holds in Nambu mechanics (phase volume is conserved) irrespective of concrete Nambu “Hamiltonians” (see Refs. 6 and 9). This means that there is an integral invariant available, not related to symmetries, too. (There is also a whole series of well-known *Poincaré-Cartan* integral invariants in Hamiltonian mechanics, with no relation to symmetries as well.)

So, there are altogether as many as *three* kinds of conserved quantities in Nambu mechanics. First, more common, evidently useful quantities (functions), which are, however, not related (at least in a clear way) to symmetries. Second, more exotic quantities (integral invariants), which, on the contrary, result from application of standard machinery on symmetries. And third, the phase volume (integral invariant) which is not related to symmetries again.

APPENDIX A: HAMILTONIAN MECHANICS – EXTREMALS, SYMMETRIES, AND CONSERVED QUANTITIES

Here, in order to make comparison with Nambu mechanics easier, we recall briefly how standard reasoning goes in *Hamiltonian* mechanics. So, our action integral is given by (5).

On extended phase space, consider a vector field with vanishing time component (otherwise yet arbitrary), $W = W^a \partial_{q^a} + W_a \partial_{p_a} + 0 \cdot \partial_t$ (“variational” field). Its infinitesimal flow Φ_ϵ performs (“equal time”) variations of curves $\gamma \mapsto \gamma_\epsilon = \Phi_\epsilon(\gamma)$. Then $S[\gamma] \mapsto S[\gamma_\epsilon]$, where

$$\begin{aligned} S[\gamma_\epsilon] &= \int_{\gamma_\epsilon} \sigma = \int_\gamma \Phi_\epsilon^* \sigma = \int_\gamma (\hat{1} + \epsilon \mathcal{L}_W) \sigma \\ &= S[\gamma] + \epsilon \int_\gamma i_W d\sigma + \epsilon \int_\gamma di_W \sigma \\ &= S[\gamma] + \epsilon \int_\gamma i_W d\sigma + \epsilon \int_{\partial\gamma} i_W \sigma. \end{aligned}$$

So, we get for variation of action, $\delta S \equiv S[\gamma_\epsilon] - S[\gamma]$,

$$\delta S = \epsilon \int_{t_1}^{t_2} \langle -i_\gamma d\sigma, W \rangle dt + \epsilon (p_a W^a)|_{t_1}^{t_2}. \quad (\text{A1})$$

This means that, within the class of curves with fixed q^a at t_1 and t_2 (this corresponds to $W^a(\gamma(t_1)) = 0 = W^a(\gamma(t_2))$), extremals of the action ($\delta S = 0$) coincide with *solutions* of equations of motion (2).

Now, vector field ξ is a *symmetry* if the action integral (5) evaluated on $\Phi_\epsilon \circ \gamma$ (the flow Φ_ϵ already corresponds to ξ here) gives the same number as on γ itself,

$$S[\Phi_\epsilon \circ \gamma] = S[\gamma] \quad (\text{A2})$$

(i.e., $\delta S = 0$; there are *no* restrictions on either time component of ξ or the values of ξ at the ends of γ). By the same direct computation as above, however, we obtain

$$\delta S = \epsilon \int_{t_1}^{t_2} \langle -i_\gamma d\sigma, \xi \rangle dt + \epsilon \int_{\partial\gamma} i_\xi \sigma. \quad (\text{A3})$$

Therefore, combining (A2), (A3), and (2), we see that on *solutions* of Hamilton equations, one has

$$0 = \int_{\partial\gamma} i_\xi \sigma. \quad (\text{A4})$$

This is, however, nothing but a *conservation law*: on solutions of Hamilton equations,

$$f_\xi(\gamma(t_2)) = f_\xi(\gamma(t_1)), \quad (\text{A5})$$

for the function

$$f_\xi := i_\xi \sigma. \quad (\text{A6})$$

Actually, requiring (A2) is too restrictive. To see this notice that it is equivalent, according to the first line of the computation in the beginning of this section, to differential condition

$$\mathcal{L}_\xi \sigma = 0. \quad (\text{A7})$$

It turns out, however, that a more general definition of symmetry may be useful, namely, as a vector field ξ fulfilling just

$$\mathcal{L}_\xi \sigma = d\chi_\xi, \quad \text{i.e.,} \quad i_\xi d\sigma = -d(i_\xi \sigma - \chi_\xi) \quad (\text{A8})$$

(i.e., *exactness* of the Lie derivative is enough for gaining a conserved quantity, its vanishing being too strong requirement). Indeed, integrating (A8) over γ gives

$$\int_{t_1}^{t_2} \langle i_\gamma d\sigma, \xi \rangle dt = \int_{\partial\gamma} (i_\xi \sigma - \chi_\xi). \quad (\text{A9})$$

Since the lhs vanishes (on solutions), we get conservation law (A5) for more general function, namely,

$$f_\xi := i_\xi \sigma - \chi_\xi. \quad (\text{A10})$$

APPENDIX B: TAKHTAJAN'S ACTION AND ITS EXTREMALS

Let us proceed to the *Nambu* mechanics now. Consider Takhtajan's action integral (12). Infinitesimal flow Φ_ϵ of variational field $W = W^x \partial_x + W^y \partial_y + W^z \partial_z + 0 \cdot \partial_t$ performs ("equal time") variations of surfaces $\Sigma \mapsto \Sigma_\epsilon = \Phi_\epsilon(\Sigma)$. Then $S[\Sigma] \mapsto S[\Sigma_\epsilon]$, where

$$\begin{aligned} S[\Sigma_\epsilon] &= \int_{\Sigma_\epsilon} \hat{\sigma} = \int_{\Sigma} \Phi_\epsilon^* \hat{\sigma} = \int_{\Sigma} (\hat{1} + \epsilon \mathcal{L}_W) \hat{\sigma} \\ &= S[\Sigma] + \epsilon \int_{\Sigma} i_W d\hat{\sigma} + \epsilon \int_{\Sigma} di_W \hat{\sigma} \\ &= S[\Sigma] + \epsilon \int_{\Sigma} i_W d\hat{\sigma} + \epsilon \int_{\partial\Sigma} i_W \hat{\sigma}. \end{aligned}$$

Now, on Σ made of *solutions* of the equations of motion (9), integral $\int_{\Sigma} i_W d\hat{\sigma}$ vanishes, since we just sum terms proportional to $(d\hat{\sigma})(W, \dot{\gamma}, u) = -(i_{\dot{\gamma}} d\hat{\sigma})(W, u) = 0$ (u is tangent to the surface Σ , linearly independent of $\dot{\gamma}$; this is the counterpart of summing terms proportional to $(d\sigma)(W, \dot{\gamma}) = -(i_{\dot{\gamma}} d\sigma)(W) = 0$ in the Hamiltonian case). So we get for variation of action, $\delta S \equiv S[\Sigma_\epsilon] - S[\Sigma]$, when computed on surface Σ composed of *solutions*,

$$\delta S = \epsilon \left(\oint_{c_1} - \oint_{c_2} \right) x(W^y dz - W^z dy) \quad (\text{B1})$$

(since $i_W \hat{\sigma} = x(W^y dz - W^z dy) + (\dots)dt$ and dt vanishes on c_1 and c_2). This means that surfaces made of solutions provide extremals ($\delta S = 0$) in the class of surfaces whose boundaries, 1-chains c_1 at t_1 and c_2 at t_2 , respectively, have fixed y and z values (put another way, fixed projections onto the yz -plane). This is because W^y and W^z should vanish at t_1 and t_2 . (There is no need to fix x at the ends. This is a counterpart of the Hamiltonian freedom to move p_a at the ends.)

APPENDIX C: MOMENTUM MAP AND EXTENDED PHASE SPACE

Most frequently, momentum map is associated with (certain) action of a Lie group G on a phase space (symplectic manifold (M, ω) ; see Refs. 3, 4, and 13). The action should preserve the symplectic form, $R_g^* \omega = \omega$, so infinitesimally $\mathcal{L}_{\xi_X} \omega = 0$ (where ξ_X is the generator of the action, $X \in \mathcal{G}$). This can be rewritten as *closedness* of α_X :

$$d\alpha_X = 0 \quad \alpha_X := i_{\xi_X} \omega. \quad (\text{C1})$$

Often α_X happens to be *exact*; then we get

$$i_{\xi_X} \omega = -dP_X \quad P_X : M \rightarrow \mathbb{R}. \quad (\text{C2})$$

Finally, since linearity of P_X with respect to X may always be achieved, we can introduce momentum map as follows:

$$P : M \rightarrow \mathcal{G}^* \quad \langle P(m), X \rangle := P_X(m). \quad (\text{C3})$$

Now, replace ω (on phase space) by $d\sigma$ (on extended phase space, σ is given by (4)):

$$d\sigma = dp_a \wedge dq^a - dH \wedge dt. \quad (\text{C4})$$

So, consider an action of G on *extended* phase space $M \times \mathbb{R}$, such that $R_g^* d\sigma = d\sigma$, so infinitesimally $\mathcal{L}_{\xi_X} d\sigma = 0$. This can be rewritten as *closedness* of β_X :

$$d\beta_X = 0 \quad \beta_X := i_{\xi_X} d\sigma. \quad (\text{C5})$$

(Notice that, unlike (C1), which says something about the phase space alone, with no reference to particular Hamiltonian governing the dynamics, (C5) is a statement concerning *complete Hamiltonian* system, since σ contains H .) Often β_X happens to be *exact*; then we get

$$i_{\xi_X} d\sigma = -dP_X \quad P_X : M \times \mathbb{R} \rightarrow \mathbb{R}. \quad (\text{C6})$$

Finally, we can again introduce "momentum map" as follows:

$$P : M \times \mathbb{R} \rightarrow \mathcal{G}^* \quad \langle P(m, t), X \rangle := P_X(m, t). \quad (\text{C7})$$

(Unlike (C2), we cannot read (C6) as that the field ξ_X is Hamiltonian; it lives on odd-dimensional manifold.) This function (i.e., all component “ordinary” functions P_i , given by $P = P_i E^i$, $P_X = X^i P_i$) is *conserved*. Indeed, because of (C6) and (2) we can write

$$\begin{aligned}\dot{P}_X &\equiv \dot{\gamma} P_X = \langle dP_X, \dot{\gamma} \rangle = -(d\sigma)(\xi_X, \dot{\gamma}) \\ &= \langle i_{\dot{\gamma}} d\sigma, \xi_X \rangle = 0,\end{aligned}$$

so that

$$\dot{P}_i = 0 \quad i = 1, \dots, \dim \mathcal{G}. \quad (\text{C8})$$

Comparison of (A8) and (C6) reveals that the two ways of obtaining conserved quantities from symmetry, discussed in Appendices A and C, respectively, yield *the same result*. (Modulo, of course, an additive constant function. One has to *fix* X and call $\xi_X \equiv \xi$. Then P_X from (C6) coincides with f_{ξ} from (A10).)

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