Geometric Methods in Physics. XXXV Workshop 2016
Trends in Mathematics, 27–33

Nambu Mechanics: Symmetries and Conserved Quantities

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Abstract. In Nambu mechanics, continuous symmetry leads to a *relative integral invariant*, a differential form which only upon integration over a cycle provides a conserved real number. This differs sharply from what is the case in Hamiltonian mechanics, where conserved quantities are *functions* on (extended) phase space, which are constant on trajectories. The origin of the difference may be traced back to a shift in degrees of relevant form present in *action integral* for Nambu mechanics.

Mathematics Subject Classification (2010). Primary 53Z05; Secondary 49S05.

Keywords. Hamiltonian mechanics, Nambu mechanics, symmetry, conserved quantity, integral invariant.

1. Introduction

From times when seminal paper of Emmy Noether [1] was published (see also nice account in [2]), we know that there is close correspondence between symmetries of action integral and conserved quantities for the dynamics given by the action.

In Hamiltonian mechanics, as an example, the conserved quantity is represented by a *function* on the phase space of the system, which is constant on trajectories (see, e.g., [3, 4] or [5]). In practical applications of Hamiltonian mechanics, valuable information may then be obtained by evaluating the function (say, energy, a component of linear or angular momentum, etc.) in two points of particular trajectory and using the fact that the two numbers are guaranteed to be the same.

In 1973, Nambu [6] proposed a different dynamics, which later became known as *Nambu mechanics.* It is governed, in its basic version, by *two* "Nambu Hamiltonians" H_1 and H_2, each of them being a function on "Nambu phase space". Now, one easily proves that *both* H_1 and H_2 are *conserved* in the sense described above. So, one could conjecture that there are two symmetries of the corresponding action integral which lead to these particular conserved quantities.

However, construction of action integral for Nambu mechanics turns out to be a delicate matter (see [7] and [8]). Namely, the action is given by a *surface* (rather than line) integral in spite of the fact that equations of motion describe motion of *points* along *trajectories* in phase space (along “world-lines” in extended phase space; exactly like it is the case for Hamiltonian mechanics). This peculiarity then leads to the fact, that standard machinery for obtaining conserved quantity from symmetry leads, in Nambu mechanics, to a strange result: conserved quantity that one obtains for a continuous symmetry turns out to be a *relative integral invariant* rather than a function on the phase space.

2. Nambu mechanics – equations and action integral

In its basic version, Nambu equations read

$$\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \qquad i = 1, 2, 3. \tag{1}$$

Here, H_1 and H_2 are, in general, functions of x_1, x_2, x_3 and t.

As was observed in [7] and [8], equations (1) may be rewritten as “vortex lines equations”

$$i_{\dot{\gamma}} d\hat{\sigma} = 0, \tag{2}$$

where

$$\dot{\gamma} = \dot{x}^1 \partial_1 + \dot{x}^2 \partial_2 + \dot{x}^3 \partial_3 + \partial_t \tag{3}$$

is the velocity vector to curve γ on extended Nambu phase space and

$$\hat{\sigma} := x^1 dx^2 \wedge dx^3 - H_1 dH_2 \wedge dt \tag{4}$$

(see also [9]). Formally, Eq. (2) looks exactly like geometrical version of *Hamilton* equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a} \tag{5}$$

except for the fact, that for Hamilton equations the role of $\hat{\sigma}$ is played by

$$\sigma = p_a dq^a - H dt. \tag{6}$$

The similarity suggests that one could construct action integral for Nambu mechanics simply repeating the way it is done in Hamilton mechanics. Namely, it is well known (see again [3, 4] or [5]) that the action integral for the Hamiltonian case reads

$$S[\gamma] = \int_\gamma \sigma = \int_{t_1}^{t_2} (p_a \dot{q}^a - H) dt. \tag{7}$$

Then, replacing σ by $\hat{\sigma}$ might probably lead to the action for the Nambu case. The idea, however, does *not* work since one can not integrate *two*-form over *one*-dimensional object (curve). Instead, one is forced to integrate $\hat{\sigma}$ over a *surface*. A problem then arises how a surface may be naturally associated with Nambu trajectories.

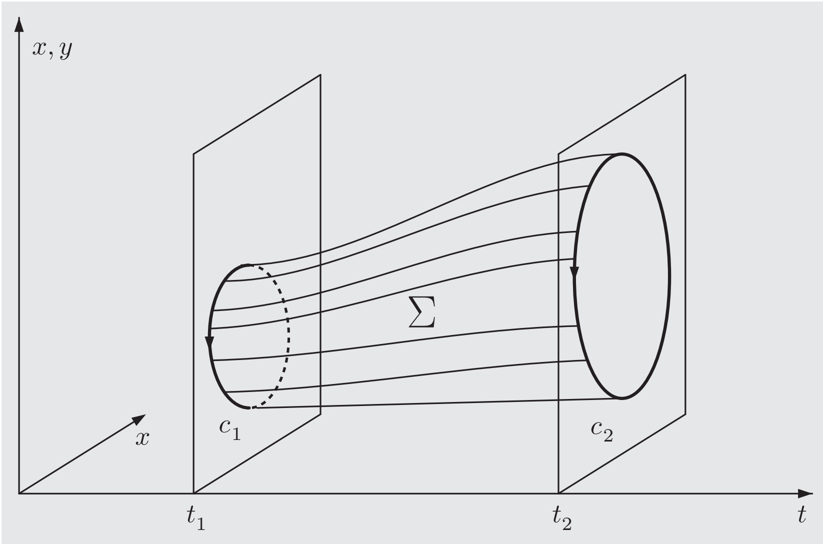

FIGURE 1. A two-chain Σ made up from a one-cycle c_1 using solutions of Nambu equations.

In Takhtajan's paper [8] it is done by the following trick: The value of action integral is associated with an appropriate one-parameter *family* of trajectories rather than with a single trajectory.

Namely, consider the family constructed as follows: Let, from each point p of a *one-cycle* (loop) c_1 at the time t_1, emanate the solution $\gamma(t)$ of Nambu equations (2), fulfilling initial condition $\gamma(t_1) = p$. At the time t_2, the points $\gamma(t_2)$ (for all $p \in c_1$) form a *one-cycle* (loop) c_2 again (image of c_1 w.r.t. the Nambu flow for $t_2 - t_1$) and the points $\gamma(t)$, for *all* $t \in \langle t_1, t_2 \rangle$ and *all* $p \in c_1$, form a *two-chain* (2-dimensional surface) Σ made of solutions (see Fig. 1; notice that $\partial\Sigma = c_1 - c_2$). The value of the action, assigned to the family, is defined to be

$$S[\Sigma] = \int_\Sigma \hat{\sigma}. \tag{8}$$

One then verifies [8, 10] that the surface given by the family of *solutions* of Nambu equations is indeed an *extremal* of the action integral (8).

3. Conserved quantity from a symmetry

Having introduced action integral for Nambu mechanics, we can mimic steps which lead from a symmetry of *Hamiltonian* action (7) to corresponding conserved quantity (function, there). And see what we get in this way in Nambu mechanics. (See more details in [10].)

First, call vector field ξ a *symmetry* if the action integral (8) evaluated on $\Phi_\epsilon(\Sigma)$ (the flow Φ_ϵ corresponds to ξ, here) gives the same number as on Σ itself

$$S[\Phi_\epsilon \Sigma] = S[\Sigma] \tag{9}$$

(i.e., $\delta S = 0$). By direct computation of δS, we obtain

$$\delta S = \epsilon \int_\Sigma i_\xi d\hat{\sigma} + \epsilon \oint_{\partial\Sigma} i_\xi \hat{\sigma}. \tag{10}$$

Now, the first integral on the r.h.s. vanishes on the surface Σ given by the family of *solutions* of Nambu equations ($\dot{\gamma}$ is tangent to Σ and, at the same time, it is annihilated by $d\hat{\sigma}$). The second integral is over $\partial\Sigma = c_1 - c_2$ and so the sum of both integrals on the r.h.s. of (10) is to vanish. We get

$$0 = \left(\oint_{c_1} - \oint_{c_2} \right) i_\xi \hat{\sigma} \tag{11}$$

or, equivalently,

$$\oint_{c_1} i_\xi \hat{\sigma} = \oint_{c_2} i_\xi \hat{\sigma}. \tag{12}$$

This is, however, nothing but a *conservation law*: for *solutions* of Nambu equations,

$$f_\xi(t_1; c_1) = f_\xi(t_2; c_2), \tag{13}$$

where f_ξ is given by the *integral*

$$f_\xi(t_a; c_a) := \oint_{c_a} i_\xi \hat{\sigma} \qquad a = 1, 2. \tag{14}$$

In full analogy with the Hamiltonian case, a more general definition of symmetry is possible. Rather than using differential version of (9), vanishing of the Lie derivative

$$\mathcal{L}_\xi \hat{\sigma} = 0 \ , \tag{15}$$

we define *symmetry of Nambu system* as a vector field ξ obeying somewhat weaker condition,

$$\mathcal{L}_\xi \hat{\sigma} = d\chi_\xi \tag{16}$$

(*exactness* of the Lie derivative is enough). Or, by Cartan's formula,

$$i_\xi d\hat{\sigma} = -d(i_\xi \hat{\sigma} - \chi_\xi). \tag{17}$$

Upon integration over the surface Σ we get

$$\int_\Sigma i_\xi d\hat{\sigma} = -\oint_{\partial\Sigma} (i_\xi \hat{\sigma} - \chi_\xi). \tag{18}$$

Since the l.h.s. vanishes (on solutions), it holds

$$\oint_{c_1} (i_\xi \hat{\sigma} - \chi_\xi) = \oint_{c_2} (i_\xi \hat{\sigma} - \chi_\xi). \tag{19}$$

So, we obtain the statement

$$f_\xi(t_1; c_1) = f_\xi(t_2; c_2), \tag{20}$$

where (more general, cf. (14)) f_ξ is given by the *integral*

$$f_\xi(t_a; c_a) := \oint_{c_a} (i_\xi \hat{\sigma} - \chi_\xi) \qquad a = 1, 2. \tag{21}$$

In words: Given a symmetry ξ take, at time t_1, an arbitrary one-cycle (loop) c_1. Compute the line integral

$$\int_{c_1} (i_\xi \hat{\sigma} - \chi_\xi). \tag{22}$$

Then, let each point of c_1 evolve by Nambu flow up to time t_2. You get another one-cycle (loop), c_2. Compute, again, the line integral

$$\int_{c_2} (i_\xi \hat{\sigma} - \chi_\xi). \tag{23}$$

The conservation law says: You get the same number.

4. Conserved quantities as relative integral invariants

In Nambu mechanics, conserved quantity associated with symmetry ξ turns out to be a *relative integral invariant*. This is, by definition, a differential p-form α such that, when integrated over a *p-cycle*, it gives an invariant w.r.t. the dynamical flow. Put in another way, if a dynamical vector field V generates the flow Φ_t (time evolution) and if c_2 is the Φ_t-image of an *arbitrary p-cycle* c_1, then,

$$\oint_{c_1} \alpha = \oint_{c_2} \alpha \tag{24}$$

(see, e.g., [4, 11] and [12]).

In our case, the result (19) may be regarded as the statement that on Nambu extended phase space endowed with the dynamical vector field V defined by

$$i_V d\hat{\sigma} = 0 \tag{25}$$

(see (2)) we get, as a consequence of existence of a symmetry ξ, a relative integral invariant. Namely, (24) holds for the *one-form*

$$\alpha = i_\xi \hat{\sigma} - \chi_\xi. \tag{26}$$

Of course, as is always the case, our relative integral invariant then automatically yields an *absolute* integral invariant, integral of the *exterior derivative* $d\alpha$ of α over *any* two-chain (two-dimensional surface) s. So, taking into account (17),

$$\int_{s_1} i_\xi d\hat{\sigma} = \int_{s_2} i_\xi d\hat{\sigma}. \tag{27}$$

5. More Nambu Hamiltonians

Already in the original paper [6] Nambu pointed out that the idea of *three*-dimensional phase space and *two* Nambu "Hamiltonians", H_1 and H_2, may be straightforwardly generalized to more dimensions, n-dimensional (Nambu) phase space and $n-1$ Nambu "Hamiltonians", $H_1, \dots, H_{n-1}$. (There are also other generalizations, see Refs. [6, 8].)

And it is easily seen that all constructions discussed in this paper work equally well in the n-dimensional version. In particular, $\hat{\sigma}$ becomes $(n-1)$-form, c_1 becomes $(n-2)$-cycle, Σ is $(n-1)$-dimensional surface and so on (see [8, 9]). Conserved quantities are still integral invariants (formally equally looking formulas (19) and (27) hold, where c_a are $(n-2)$-cycles and s_a are $(n-1)$-chains).

6. Conclusions

Both Hamiltonian and Nambu mechanics study motion of *points* in phase space along their *trajectories*. Therefore it is natural to expect conserved quantities to be *functions* on phase space. Once we study particular motion, we evaluate the function at the time t_1 at the point where the motion begins, and then we profit from the fact that, at the future points of the trajectory, the same value of the function is guaranteed by the conservation law.

In Hamiltonian mechanics it is really so. In Nambu mechanics, there are conserved *functions* as well. Namely, the two "Hamiltonians" H_1 and H_2 are conserved.

However, as we have seen, *these* conserved functions *do not* directly follow from symmetries, as we might expect from the Hamiltonian case. Instead, in the case of symmetries, application of more or less standard machinery results, because of a peculiar situation with the action integral, in conserved quantities which have the character of *integral invariants* rather then usual conserved functions. (The machinery leads to *higher-degree* forms rather than usual zero-forms, that is, functions.) As a reward for finding a symmetry, the conserved *number* is only obtained as *integral* of the form over a one-cycle.

We stress again that the reason lies in the peculiar structure of the *action* integral: Since we only can associate the action with a *family* of trajectories, conserved quantities also reflect properties *of the family* and they are, therefore, constructed using *integration* "over the family".

Let us note that there is the whole series of well-known *Poincaré–Cartan* integral invariants in Hamiltonian mechanics, where numbers only come out from integration "over (an appropriate) family" of trajectories. These integral invariants, however, have nothing to do with symmetries of *particular* Hamiltonian system (they hold in general, irrespective of the concrete form of the Hamiltonian).

Acknowledgment

This research was supported by VEGA project 1/0985/16.

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