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Integral Invariants (Poincaré–Cartan) and Hydrodynamics

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1. Introduction

There are several ways how hydrodynamics of ideal fluid may be treated geometrically. In particular, it may be viewed as an application of the theory of *integral invariants* due to Poincaré and Cartan (see Refs. [1, 2], or, in modern presentation, Refs. [3, 4]). Then, the original Poincaré version of the theory refers to the *stationary* (time-independent) flow, described by the stationary *Euler equation*, whereas Cartan's extension embodies the full, possibly time-dependent, situation.

Although the approach via integral invariants is far from being the best known, it has some nice features which, hopefully, make it worth spending some time. Namely, the form in which the Euler equation is expressed in this approach, turns out to be ideally suited for extracting important (and useful) classical consequences of the equations remarkably easily (see more details in Ref. [4]). This refers, in particular, to the behavior of *vortex lines*, discovered long ago by Helmholtz.

2. Poincaré integral invariants

Consider a manifold M endowed with dynamics given by a *vector field* v

$$\dot{\gamma} = v \qquad\qquad \dot{x}^i = v^i(x). \tag{1}$$

The field v generates the dynamics (time evolution) via its flow $\Phi_t \leftrightarrow v$. We will call the structure *phase space*

$$(M, \Phi_t \leftrightarrow v) \qquad\qquad \textit{phase space}. \tag{2}$$

In this situation, let us have a k-form α and consider its integrals over various k-chains (k-dimensional surfaces) c on M. Due to the flow Φ_t corresponding to v, the k-chains flow away, $c \mapsto \Phi_t(c)$. Compare the value of the integral of α over the original c and integral over $\Phi_t(c)$. If, *for any chain* c, the two integrals are equal, it reflects a remarkable property of the form α with respect to the field v. We call it (absolute) integral invariant:

$$\int_{\Phi_t(c)} \alpha = \int_c \alpha \qquad \Leftrightarrow \qquad \int_c \alpha \quad \text{is } \textit{integral invariant}. \tag{3}$$

For *infinitesimal* $t \equiv \epsilon$ we have

$$\int_{\Phi_\epsilon(c)} \alpha = \int_c \alpha + \epsilon \int_c \mathcal{L}_v \alpha \tag{4}$$

(here $\mathcal{L}_v$ is the *Lie derivative* along v). If (3) is to be true *for each* c, we get from (4)

$$\mathcal{L}_v \alpha = 0. \tag{5}$$

Sometimes, however, it may be enough that the integral only behaves invariantly when restricted to k-*cycles* (i.e., chains whose boundary vanish, $\partial c = 0$). We speak of *relative* integral invariants. Then the condition (5) can be weakened to

$$\mathcal{L}_v \alpha = d\tilde{\beta} \tag{6}$$

for some $\tilde{\beta}$. (So, α is to be *Lie-invariant modulo exact* form.) Using Cartan's formula $i_v d + d i_v = \mathcal{L}_v$, the condition (6) may also be rewritten as

$$i_v d\alpha = d\beta. \tag{7}$$

Therefore, the *main statement* on relative (Poincaré) invariants reads:

$$\boxed{i_v d\alpha = d\beta} \qquad \Leftrightarrow \qquad \oint_c \alpha = \quad \textit{relative} \text{ invariant w.r.t. } \Phi_t \leftrightarrow v. \tag{8}$$

2.1. Stationary Euler equation

The *Stationary Euler equation* for the ideal (inviscid) fluid reads (see, e.g., Ref. [5])

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p/\rho - \boldsymbol{\nabla}\Phi. \tag{9}$$

Here the mass density ρ, the velocity field $\mathbf{v}$, the pressure p and the potential Φ of the volume force field (gz for the usual gravitational field) are functions of $\mathbf{r}$.

It turns out (see Ref. [4]) that for *barotropic* fluid (when $\boldsymbol{\nabla} p/\rho = \boldsymbol{\nabla} P$, where P is the *enthalpy* (heat function) *per unit mass*) it may be rewritten in the form of Eq. (7) with a particular choice of α and β:

$$\boxed{i_v d\tilde{v} = -d\mathcal{B}} \qquad \textit{Euler equation} \tag{10}$$

where

$$\tilde{v} := \mathbf{v} \cdot d\mathbf{r} \qquad (\equiv g(v,\ \cdot\ )\ \equiv \flat_g v) \tag{11}$$

is the *velocity* 1-*form* standardly associated with the velocity vector field $v = v^i \partial_i$ in terms of "lowering of index" ($\equiv \flat_g$ procedure) and

$$\mathcal{B} := v^2/2 + P + \Phi \qquad \textit{Bernoulli function}. \tag{12}$$

2.2. Vortex lines equation

Vortex lines, $\gamma(\lambda) \leftrightarrow \mathbf{r}(\lambda)$, are field lines of the *vorticity vector field* $\boldsymbol{\omega}$, which is the curl of the *velocity* field $\mathbf{v}$. So, they satisfy $\boldsymbol{\omega} \times \mathbf{r}' = 0$ (the prime symbolizes *tangent* vector).

Now we have (see the machinery explained in § 8.5 of Ref. [6])

$$\tilde{v} = \mathbf{v} \cdot d\mathbf{r} \tag{13}$$

$$d\tilde{v} = (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S} \;\equiv\; \boldsymbol{\omega} \cdot d\mathbf{S} \tag{14}$$

$$i_{\gamma'} d\tilde{v} = (\boldsymbol{\omega} \times \mathbf{r}') \cdot d\mathbf{r} \tag{15}$$

The *vorticity* 2-*form* $d\tilde{v}$, present in Eq. (10), is of crucial importance. It encodes complete information about the vorticity vector field $\boldsymbol{\omega}$ and, as we see from (15),

$$\boxed{i_{\gamma'} d\tilde{v} = 0} \qquad \textit{vortex line equation} \tag{16}$$

expresses the fact that $\gamma(\lambda)$ is a vortex line.

2.3. Why the form of Eq. (10) is so convenient

For several reasons:

1. Application of i_v on both sides gives

$$v\mathcal{B} = 0 \qquad \textit{Bernoulli equation} \tag{17}$$

(saying that $\mathcal{B}$ is constant along *stream*lines).

2. Application of $i_{\gamma'}$ on both sides (where γ' is from (16)) gives

$$\gamma' \mathcal{B} = 0 \tag{18}$$

(saying that $\mathcal{B}$ is constant along *vortex*-lines).

3. Setting $d\tilde{v} = 0$ (when the flow is *irrotational*) leads to

$$\mathcal{B} = \text{const.} \tag{19}$$

(a version of Bernoulli equation; $\mathcal{B}$ is, then, constant *in bulk* of the fluid).

4. Just looking at (8), (10) and (11) we get

$$\oint_c \mathbf{v} \cdot d\mathbf{r} = \text{const.} \qquad \textit{Kelvin's theorem} \tag{20}$$

(velocity circulation is conserved *w.r.t. the flow*).

5. Just looking at (8), (16) and using the Stokes theorem gives

$$\int_S \boldsymbol{\omega} \cdot d\mathbf{S} = \text{const.} \qquad \textit{Helmholtz theorem} \tag{21}$$

(the *strength* of the vortex tube is constant *along the tube*).

6. Application of d on both sides gives very quickly… see Section 2.4.

2.4. Helmholtz theorem on frozen vortex lines – stationary case

Application of d on both sides of (10) results in

$$\mathcal{L}_v(d\tilde{v}) = 0, \qquad \text{i.e.,} \qquad \Phi_t^*(d\tilde{v}) = d\tilde{v} \qquad \Phi_t \leftrightarrow v. \tag{22}$$

So, the vorticity 2-form $d\tilde{v}$ is *invariant* w.r.t. the flow of the fluid.

Let us define a *distribution* $\mathcal{D}$ in terms of $d\tilde{v}$:

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \; i_w d\tilde{v} = 0 \; \text{ holds}\}. \tag{23}$$

Due to the Frobenius criterion the distribution is integrable (see Refs. [4], [6]). From (15) and (16) we see that the distribution is one-dimensional (at those points where $\boldsymbol{\omega} \neq 0$) and that its integral surfaces coincide with vortex lines. Since the distribution $\mathcal{D}$ is invariant w.r.t. $\Phi_t \leftrightarrow v$, its integral surfaces (i.e., vortex lines) are invariant w.r.t. $\Phi_t \leftrightarrow v$, too. But this means that (another) *Helmholtz* theorem is true: vortex lines *move with the fluid* (are *frozen into* the fluid; see Refs.[7–9]).

3. Cartan integral invariants

Cartan proposed, as a first step, to study the dynamics given in (1) and (2) on $M \times \mathbb{R}$ (the *extended* phase space; the *time* coordinate is added) rather than on M. Using the natural projection

$$\pi : M \times \mathbb{R} \to M \quad (m, t) \mapsto m \quad (x^i, t) \mapsto x^i \tag{24}$$

the forms α and β (from the Poincaré theory) may be pulled-back from M onto $M \times \mathbb{R}$ and then combined into a single k-form

$$\sigma = \hat{\alpha} + dt \wedge \hat{\beta}. \tag{25}$$

(Here, we denote $\hat{\alpha} = \pi^*\alpha$ and $\hat{\beta} = \pi^*\beta$.) In a similar way, define a vector field

$$\xi = \partial_t + v. \tag{26}$$

Its flow clearly consists of the flow $\Phi_t \leftrightarrow v$ on the M factor combined with the trivial lapsing of time in the $\mathbb{R}$ factor (so, it is "the same flow"). A simple check (see Ref. [4]) reveals that the equation

$$i_\xi d\sigma = 0 \tag{27}$$

is equivalent to (7). And the main statement (8) takes the form

$$\boxed{i_\xi d\sigma = 0} \qquad \Leftrightarrow \qquad \oint_c \sigma = \quad \textit{relative} \text{ invariant.} \tag{28}$$

The first *new* result by Cartan (w.r.t. Poincaré) is the following observation: Take *any* two cycles in $M \times \mathbb{R}$ which encircle the common *tube of solutions* (here "solutions" mean integral curves of ξ, i.e., solutions of the dynamics as seen from $M \times \mathbb{R}$). Then, *still*, integrals of σ over c_1 and c_2 give the same number (a simple proof see in Ref. [4]).

The further Cartan generalization is stronger and much more interesting for us. Namely, (25) might also be regarded as a decomposition of the *most general*

k-form σ on $M\times\mathbb{R}$. In this case, $\hat{\alpha}$ and $\hat{\beta}$ need not be obtained by the pull-back from M. Rather, they are the most general *spatial* forms on $M\times\mathbb{R}$. In comparison with just pull-backs, they may be *time-dependent*, i.e., it *may happen* that $\mathcal{L}_{\partial_t}\hat{\alpha}\neq 0$ and/or $\mathcal{L}_{\partial_t}\hat{\beta}\neq 0$. (In coordinate presentation, their *components* may depend on time.)

It turns out that the proof of (28) does not use any details of the decomposition. The *structure* of the equation (27) is all one needs. Notice, however, that the equivalence of (27) and (7) is no longer true, now. Instead, one can check that

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{\alpha} + i_v\hat{d}\hat{\alpha} = \hat{d}\hat{\beta} \tag{29}$$

(the term $\mathcal{L}_{\partial_t}\hat{\alpha}$ is new). Here $\hat{d}$ denotes the *spatial* exterior derivative. (In coordinate presentation – as if the variable t in components was *constant*.) So, the equation

$$\boxed{\mathcal{L}_{\partial_t}\hat{\alpha} + i_v\hat{d}\hat{\alpha} = \hat{d}\hat{\beta}} \tag{30}$$

is *the* equation that *time-dependent* forms $\hat{\alpha}$ and $\hat{\beta}$ are to satisfy in order that the integral of σ is to be relative integral invariant (in the new, more general, sense of encircling the common tube of solutions).

3.1. Non-stationary Euler equation

Retell Cartan's results in the context of hydrodynamics, i.e., for

$$\sigma = \hat{v} - \mathcal{B}dt \tag{31}$$

where, in usual coordinates $(\mathbf{r},t)$ on $E^3\times\mathbb{R}$,

$$\hat{v} := \mathbf{v}\cdot d\mathbf{r} \equiv \mathbf{v}(\mathbf{r},t)\cdot d\mathbf{r} \tag{32}$$

From (29) we get

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{v} + i_v\hat{d}\hat{v} = -\hat{d}\mathcal{B}. \tag{33}$$

One easily checks that the r.h.s. of (33) is nothing but the *complete, time-dependent*, Euler equation. Therefore the time-dependent Euler equation may also be written in remarkably succinct form

$$\boxed{i_\xi d\sigma = 0} \qquad \textit{Euler equation.} \tag{34}$$

Just looking at (28), (34), (31) and (32) shows that *Kelvin's theorem* is still true (the two loops c_1 and c_2 are usually in constant-time hyper-planes $t=t_1$ and $t=t_2$, so that the $\mathcal{B}dt$ term does not contribute).

3.2. Helmholtz theorem on frozen vortex lines – non-stationary case

Application of d on (34) results in

$$\mathcal{L}_\xi(d\sigma) = 0, \qquad \text{i.e.,} \qquad \Phi^*_\tau(d\sigma) = d\sigma \qquad \Phi_\tau \leftrightarrow \xi \tag{35}$$

So, $d\sigma$ is *invariant* w.r.t. the flow of the fluid.

Define the distribution $\mathcal{D}$ in terms of annihilation of as many as *two* exact forms:

$$\mathcal{D} \quad \leftrightarrow \quad i_w d\sigma = 0 = i_w dt. \tag{36}$$

The new distribution $\mathcal{D}$ is *integrable* as well. It is, however, also *invariant* w.r.t. the flow of the fluid. (Because of (35) and the trivial fact that $\mathcal{L}_\xi(dt) = 0$.) So, integral submanifolds (surfaces) *move with the fluid*.

What do they look like? Although it is not visible at first sight, they are nothing but vortex lines (see Ref. [10] or, in more detail, Ref. [4]). So, the *Helmholtz* theorem is also true in the non-stationary case: vortex lines *move with the fluid* (are *frozen into* the fluid).

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