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Surfaces Which Behave Like Vortex Lines

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Abstract. In general setting of theory of integral invariants, due to Poincaré and Cartan, one can find a d-dimensional integrable distribution (given by a possibly higher-rank form) whose integral surfaces behave like vortex lines: they move with (abstract) fluid. Moreover, in a special case they reduce to true vortex lines and, in this case, we get the celebrated Helmholtz theorem.

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1. Introduction

In hydrodynamics, *vortex lines* are field lines of the *vorticity* vector field $\boldsymbol{\omega}$, which is curl of velocity field $\mathbf{v}$.

A classical theorem due to Helmholtz says that, in the case of ideal and barotropic fluid that is only subject to conservative forces, vortex lines "move with the fluid" (see Ref. [1] and Refs. [2, 3]; one also says that the lines are "frozen into the fluid" or that "vortex lines are material lines").

Hydrodynamics of ideal fluid may be viewed, albeit it is not quite standard, as an application of the theory of *integral invariants* due to Poincaré and Cartan (see Refs. [4, 5], or, in modern presentation, Refs. [6–8]). Then, the original Poincaré version of the theory refers to the *stationary* (time-independent) flow, described by the stationary Euler equation, whereas Cartan's extension embodies the full, possibly time-dependent, situation.

In this picture, one can base a proof of the Helmholtz theorem upon the concept of a *distribution*. Namely, first, vortex lines are identified with integral surfaces of a 1-dimensional integrable distribution, defined in terms of the appropriate 2-form. Second, the structure of the (Euler) equation of motion immediately reveals that the 2-form is *Lie-invariant* w.r.t. the flow of the fluid. So, third, the corresponding distribution is invariant as well and, consequently, its integral surfaces are invariant w.r.t. the flow of the fluid. And this is exactly what the Helmholtz statement says.

Now, it turns out that the same reasoning may be repeated within the *general integral invariant* setting (so beyond even the "n-dimensional Riemannian hydrodynamics", discussed, e.g., in [9]). What differs is that we have an integrable distribution based upon a possibly *higher-degree* Lie-invariant differential form, there. In particular, the distribution may be *higher-dimensional* and, consequently, its integral surfaces become then higher-dimensional, too. Nevertheless, they still obey the Helmholtz-like rule of "moving with the fluid" (i.e., the *abstract* flow in the general theory translates the integral surfaces into one another).

2. Integral invariants – Poincaré and Cartan

Before considering the main subject of the paper, let us briefly recall key concepts and state main results of Poincaré and Cartan on general theory of integral invariants. See Ref. [8] in this volume or, for a more detailed account, Ref. [7].

2.1. Poincaré integral invariants

Following *Poincaré*, one starts from a manifold M endowed with dynamics (time evolution) given by a *vector field* v (via its flow)

$$(M, \Phi_t \leftrightarrow v) \qquad \textit{phase space} \tag{1}$$

Now, consider integrals of a k-form α over various k-chains (k-dimensional surfaces) c on M. Compare the value of the integral of α over the original c and the integral over $\Phi_t(c)$. If, *for any chain* c, the two integrals are equal, we call it (absolute) integral invariant:

$$\int_{\Phi_t(c)} \alpha = \int_c \alpha \qquad \Leftrightarrow \qquad \int_c \alpha \quad \text{is } \textit{integral invariant.} \tag{2}$$

If we only restrict to k-*cycles* (i.e., chains whose boundaries vanish, $\partial c = 0$), we speak of *relative* integral invariants. It turns out that one can recognize the relative invariant by the *differential* equation

$$i_v d\alpha = d\beta, \tag{3}$$

i.e., the following statement is true

$$i_v d\alpha = d\beta \qquad \Leftrightarrow \qquad \oint_c \alpha = \quad \textit{relative} \text{ invariant.} \tag{4}$$

2.2. Cartan integral invariants

Cartan proposed to study dynamics on $M \times \mathbb{R}$ (*extended* phase space; *time* coordinate is added) rather than on M. Analogs of the forms α and β (from the Poincaré theory) are combined into a *single* k-form

$$\sigma = \hat{\alpha} + dt \wedge \hat{\beta}. \tag{5}$$

Here, $\hat{\alpha}$ and $\hat{\beta}$ are the most general *spatial* forms on $M \times \mathbb{R}$. (In coordinate presentation, they do not contain the dt factor. They may be, however, *time-dependent*, i.e., their *components* may depend on time.) In a similar way, the dynamical vector field v sits in the combination

$$\xi = \partial_t + v. \tag{6}$$

Then, according to Cartan, one has to replace the crucial equation of Poincaré, viz. Eq. (3), with

$$i_\xi d\sigma = 0. \tag{7}$$

And the main statement of Poincaré, viz. Eq. (4), takes the form

$$\boxed{i_\xi d\sigma = 0} \qquad \Leftrightarrow \qquad \oint_c \sigma = \quad \textit{relative} \text{ invariant.} \tag{8}$$

It turns out that the proof of (8) does not use any details of the decomposition. The *structure* of equation (7) is all one needs. One can check that

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta} \tag{9}$$

(the term $\mathcal{L}_{\partial_t}\hat{\alpha}$ is new w.r.t. (3)). Here $\hat{d}$ denotes the *spatial* exterior derivative. (In coordinate presentation – as if the variable t in components was *constant*.) So, the equation

$$\boxed{\mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta}} \tag{10}$$

is *the* equation that (possibly) *time-dependent* forms $\hat{\alpha}$ and $\hat{\beta}$ are to satisfy in order that the integral of σ is to be a relative integral invariant.

3. Surfaces and their motion

3.1. Stationary case

Return back to equation (3). Application of d on both sides results in

$$\mathcal{L}_v(d\alpha) = 0, \qquad \text{i.e.,} \qquad \Phi_t^*(d\alpha) = d\alpha \qquad \Phi_t \leftrightarrow v \tag{11}$$

So, the form $d\alpha$ is *invariant* w.r.t. the flow Φ_t.

Let us define a *distribution* $\mathcal{D}$ in terms of $d\alpha$:

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \ i_w d\alpha = 0 \ \text{ holds}\}. \tag{12}$$

[Motivation for this definition comes from *hydrodynamics*. Namely, see Ref. [8] in this volume, integral submanifolds of this distribution for the particular choice $\alpha = \tilde{v} \equiv g(v, \ \cdot\ )$, where v is the velocity field in hydrodynamics, are one-dimensional and coincide with *vortex lines*.]

Due to the Frobenius criterion the distribution is integrable. Indeed, let $w_1, w_2 \in \mathcal{D}$, i.e., $i_{w_1} d\alpha = 0$ and $i_{w_2} d\alpha = 0$. Then, because of the identity

$$i_{[w_1,w_2]} = [\mathcal{L}_{w_1}, i_{w_2}] \equiv \mathcal{L}_{w_1} i_{w_2} - i_{w_2}\mathcal{L}_{w_1} \tag{13}$$

(see, e.g., Ref. [11]) plus Cartan's formula

$$i_u d + d i_u = \mathcal{L}_u \tag{14}$$

one immediately sees that

$$i_{[w_1,w_2]}d\alpha = 0, \tag{15}$$

i.e., $[w_1, w_2] \in \mathcal{D}$, too. So $\mathcal{D}$ is integrable. Since the distribution $\mathcal{D}$ is invariant w.r.t. $\Phi_t \leftrightarrow v$, its integral surfaces are invariant w.r.t. $\Phi_t \leftrightarrow v$, too. But this means that a "Helmholtz-*like*" theorem is true: *whenever* we encounter the *general* context of the Poincaré integral invariants, the integral surfaces of the distribution $\mathcal{D}$ are *frozen into* "the fluid".

3.2. General, non-stationary case

Now, a question arises whether or not a similar statement is true in a much more complex, *time-dependent*, situation. The answer turns out to be still positive, although the proof is more involved.

Let us start with the application of d on (7). It results in

$$\mathcal{L}_\xi(d\sigma) = 0, \qquad \text{i.e.,} \qquad \Phi^*_\tau(d\sigma) = d\sigma \qquad \Phi_\tau \leftrightarrow \xi. \tag{16}$$

So, $d\sigma$ is *invariant* w.r.t. the flow.

Define the distribution $\mathcal{D}$ (on $M \times \mathbb{R}$, now) in terms of annihilation of as many as *two* exact forms:

$$\mathcal{D} \quad \leftrightarrow \quad i_w d\sigma = 0 = i_w dt. \tag{17}$$

So, we are interested in *spatial* vectors ($i_w dt = 0$) which, in addition, annihilate $d\sigma$.

The new distribution $\mathcal{D}$ is *integrable* as well. The Frobenius criterion shows this easily, again: We assume

$$i_{w_1} d\sigma = 0 = i_{w_1} dt \qquad i_{w_2} d\sigma = 0 = i_{w_2} dt \tag{18}$$

and, using (13) and (14), we see that

$$i_{[w_1,w_2]} d\sigma = 0 = i_{[w_1,w_2]} dt. \tag{19}$$

So, our new distribution $\mathcal{D}$ (on $M \times \mathbb{R}$) defined via annihilation of $d\sigma$ and dt is *integrable* and *invariant* w.r.t. the flow. Consequently, its integral submanifolds (surfaces) are *frozen into* "the fluid".

What is not yet clear, however, is the exact relation of this result to the result of the time-independent case from Section 3.1. (Recall that the distribution considered there was spanned by vectors which annihilate $d\alpha$ rather than $d\sigma$.)

It is here where Eq. (7) comes to rescue again, now in a more subtle way. Indeed, applying d on (5) and then using the decomposed version (10) of (7), we can write

$$d\sigma = \hat{d}\hat{\alpha} + dt \wedge (\mathcal{L}_{\partial_t}\hat{\alpha} + \hat{d}\hat{\beta}) \qquad \text{always} \tag{20}$$

$$= \hat{d}\hat{\alpha} + dt \wedge (-i_v \hat{d}\hat{\alpha}) \qquad \textit{on solutions}. \tag{21}$$

Now, let w be arbitrary *spatial* vector. Denote, for a while, $i_w \hat{d}\hat{\alpha} =: \hat{b}$ (it is a *spatial* 1-form). Then, from (21),

$$i_w d\sigma = \hat{b} - dt \wedge i_v \hat{b} \tag{22}$$

from which immediately

$$i_w(d\sigma) = 0 \qquad \Leftrightarrow \qquad \hat{b} \equiv i_w \hat{d}\hat{\alpha} = 0. \tag{23}$$

This says that we can, alternatively, describe the distribution $\mathcal{D}$ as consisting of those *spatial* vectors which, in addition, annihilate $\hat{d}\hat{\alpha}$ (rather than $d\sigma$, as it is expressed in the definition (17)). But this means that we speak of "the same" distribution as in (12). (The language of σ is more advantageous for proving invariance of the distribution w.r.t. the flow as well as for its integrability, whereas the "decomposed" language of $\hat{\alpha}$ and $\hat{\beta}$ is needed for identification of the distribution as the one from the time-independent case.) So, the Helmholtz-like statement from Section 3.1 is also true in the general, time-dependent, case. (Notice that the system of the surfaces, if regarded as living on M, looks, in general, different in different times. This is because its generating object, the form $\hat{d}\hat{\alpha}$, depends on time.)

[*On solutions* in Eq. (21) means on solutions of equation (7) or, equivalently, of (10). In hydrodynamics, (7) turns out to be (see Ref. [8] in this volume) nothing but the *Euler* equation, i.e., the *equation of motion* of ideal fluid. So, the fact that vortex lines are frozen into the fluid is only true in the case of *real dynamics* of the fluid. It is, unlike the Helmholtz statement on strength of vortex tubes, a *dynamical*, rather than kinematical, statement.]

4. Conclusions

Theory of integral invariants due to Poincaré and Cartan enables one, when applied to hydrodynamics, to get a simple and convincing proof of Helmholtz' classical theorem on motion of vortex lines. Moreover, this approach reveals that, actually, there is *a generalization* of the phenomenon still in the original theory (prior to application to hydrodynamics). In this case, vortex *lines* are to be replaced by appropriate distinguished *surfaces.*

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