GALILEAN AND CARROLLIAN HODGE STAR OPERATORS

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The standard Hodge star operator is naturally associated with metric tensor (and orientation). It is routinely used to concisely write down physics equations on, say, Lorentzian spacetimes. On Galilean (Carrollian) spacetimes, there is no canonical (nonsingular) metric tensor available. So, the usual construction of the Hodge star does not work. Here we propose analogs of the Hodge star operator on Galilean (Carrollian) spacetimes. They may be used to write down important physics equations, e.g. equations of Galilean (Carrollian) electrodynamics.

Keywords: Galilean spacetime, Carrollian spacetime, Galilean electrodynamics, Carrollian electrodynamics, Hodge star operator.

1. Introduction — standard Hodge star operator

The *Hodge star* operator $* \equiv *_{g,o}$ is a well-known *canonical* linear mapping acting on differential forms on (pseudo)Riemannian oriented manifold (M, g, o),

$$* : \Omega^p \to \Omega^{d-p}, \qquad \Omega^p \equiv \Omega^p(M, g, o), \tag{1}$$

(d beeing the dimension of M; see e.g. [1–7]). In components (w.r.t. a local frame field e_a), it is defined as

$$(*\alpha)_{a\ldots b} := \frac{1}{p!}\alpha_{c\ldots d}\ \omega^{c\ldots d}{}_{a\ldots b}, \qquad \omega^{c\ldots d}{}_{a\ldots b} := g^{ce}\ldots g^{df}\omega_{e\ldots fa\ldots b}, \tag{2}$$

where $\omega \equiv \omega_{g,o}$ stands for the *metric volume form* on (M, g, o).

The Hodge star $*$ is a remarkably effective tool for concise and coordinate-free expression of important physics equations. For example, the Maxwell equations may be written, on any (Lorentzian) spacetime, as

$$dF = 0, \qquad d * F = -J, \tag{3}$$

(here J stands for $(d-1)$-form of current). Notice that no linear connection is needed for that (no semicolons in local coordinates).

Recently, much research has been devoted to the study of physics in *Galilean* and *Carrollian* spacetimes, see e.g. [8–17] or a long lists of references in recent PhD theses [18] and [19]. Now if we wanted to use Hodge star operator on these spacetimes, a serious problem would immediately arise: There is *no canonical* metric tensor on either Galilean or Carrollian spacetime. And since metric tensor g turns out to be key element in construction of the $*_g$, the routine procedure fails, there.

2. Galilean and Carrollian Hodge stars

From (2) we see that the role of the metric tensor g in construction of the "mixed" tensor $\omega^{c\dots d}{}_{a\dots b}$, i.e. in construction of the (standard) Hodge star, is twofold: It enters the scene via

– the *metric* volume form ω_g and
– the co*metric* g^{-1} (for raising indices).

Consequently, *if* the two objects, *canonical* volume form and $\binom{2}{0}$-type tensor field (for raising indices) were also available on Galilean/Carrollian spacetimes, one could also construct $\omega^{c\dots d}{}_{a\dots b}$ there, i.e. potentially useful *analogs* of the Hodge star were possible on *Galilean* (*Carrollian*) spacetime as well.

More generally, analogs of the standard Hodge star could be defined on Galilean and Carrollian spacetimes, if the necessary mixed tensor field $\omega^{c\dots d}{}_{a\dots b}$ could be constructed using *canonical* tensor fields *available* there.

It turns out that such a construction is indeed possible.

Let us start with the fact (see Appendix A.1) that, on *both* Galilean and Carrollian spacetimes, *both* (canonical) top-degree form and top-degree polyvector

$$\omega_{a\dots b}, \quad \tilde{\omega}^{a\dots b}, \tag{4}$$

are available, see (65) and (70).

So we have at least two different possibilities for constructing the needed *mixed* tensor $\omega^{a\dots b}{}_{c\dots d}$: we

– *either* start from $\omega_{a\dots bc\dots d}$ and *raise* (somehow) the group $(a\dots b)$,
– *or* start from $\tilde{\omega}^{a\dots bc\dots d}$ and *lower* (somehow) the group $(c\dots d)$.

Since we start from *canonical* tensors $\omega_{a\dots bc\dots d}$ and $\tilde{\omega}^{a\dots bc\dots d}$, the resulting tensor $\omega^{a\dots b}{}_{c\dots d}$ will be canonical as well provided that the *procedures* of raising and lowering indices will be canonical, too.

Now recall (see Sections (A.1.1) and (A.1.2)) what canonical tensors (with the potential of raising and lowering indices) are available on the spacetimes of our interest:

– Galilean: $(M,\xi,h), \quad h\in\binom{2}{0}, \quad \xi\in\binom{0}{1} \quad \Rightarrow \quad k:=\xi\otimes\xi\in\binom{0}{2}$,
– Carrollian: $(M,\tilde{\xi},\tilde{h}), \quad \tilde{h}\in\binom{0}{2}, \quad \tilde{\xi}\in\binom{1}{0} \quad \Rightarrow \quad \tilde{k}:=\tilde{\xi}\otimes\tilde{\xi}\in\binom{2}{0}$.

So indeed, on both spacetimes of interest, there is a possibility to both raise and lower indices in a *canonical* way:

– Galilean: raising with $h \in \binom{2}{0}$, lowering with $k \in \binom{0}{2}$,
– Carrollian: raising with $\tilde{k} \in \binom{2}{0}$, lowering with $\tilde{h} \in \binom{0}{2}$.

This means that there are, on *both* spacetimes of interest, as many as two *analogs of the Hodge star* operator (in the sense of the canonical mapping of p-forms to $(d-p)$-forms on d-dimensional M), namely those based on tensors $h^{ab}/\tilde{h}_{ab}$ and those based on tensors $k_{ab}/\tilde{k}^{ab}$.

In order to foresee some important features of the resulting operators, just recall (see again Sections (A.1.1) and (A.1.2)) how (matrices of) components of the necessary canonical tensors look like (in *any adapted* frame/coframe):

$$\text{Galilean: } h^{ab} \leftrightarrow \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix}, \tag{5}$$

$$k_{ab} \leftrightarrow \begin{pmatrix} k_{00} & k_{0i} \\ k_{i0} & k_{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

$$\text{Carrollian: } \tilde{h}_{ab} \leftrightarrow \begin{pmatrix} \tilde{h}_{00} & \tilde{h}_{0i} \\ \tilde{h}_{i0} & \tilde{h}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix}, \tag{7}$$

$$\tilde{k}^{ab} \leftrightarrow \begin{pmatrix} \tilde{k}^{00} & \tilde{k}^{0i} \\ \tilde{k}^{i0} & \tilde{k}^{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

Since the matrices h^{ab} and $\tilde{h}_{ab}$ are typically much *less* degenerate (their rank being n) than the matrices k_{ab} and $\tilde{k}^{ab}$ (rank 1), the procedure of raising/lowering indices using tensors $h^{ab}/\tilde{h}_{ab}$ is much less "lossy" than that using tensors $k_{ab}/\tilde{k}^{ab}$. (Recall that raising and lowering indices is perfectly "lossless" in the standard Riemannian case.) Put it differently, analogs of Hodge star based on $h^{ab}/\tilde{h}_{ab}$ are *in general* expected to be *more useful* than those based on tensors $k_{ab}/\tilde{k}^{ab}$.

All the operators are computed in detail in Appendix A.2 and the results are summarized and discussed in Section 3.

3. Explicit formulae and some properties

3.1. General explicit formulae

In both Galilean and Carrollian spacetimes, exactly as is the case in Lorentzian spacetime (see e.g. [25, 26] and Section16.1 in [7]), any p-form α may be uniquely written, w.r.t. (local) adapted coframe field $e^a = (e^0, e^i)$, as

$$\alpha = e^0 \wedge \hat{s} + \hat{r}, \tag{9}$$

where the two hatted forms, $(p-1)$-form $\hat{s}$ and p-form $\hat{r}$, respectively, are *spatial*, meaning that they do not contain the ("temporal") 1-form e^0.

When applying both versions of Hodge star operator $*$ onto $(1+n)$-decomposed presentation (9) of α, we get (in any dimension $d = 1+n$) the following results (see Appendix A.2 for more details).

First, Hodge stars based on (rank n) tensors h^{ab} and $\tilde{h}_{ab}$, respectively, act as follows:[1]

$$*(e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \qquad \textit{Lorentzian} \text{ Hodge star,} \tag{10}$$

$$*(e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge \hat{*}\hat{r} \qquad \textit{Galilean} \text{ Hodge star,} \tag{11}$$

$$*(e^0 \wedge \hat{s} + \hat{r}) = \hat{*}\hat{\eta}\hat{s} \qquad \textit{Carrollian} \text{ Hodge star.} \tag{12}$$

(see e.g. [25, 26] for (10), (80) and (89) for (11) and (12)). Here $\hat{*}$ stands for standard "Euclidean" (n-dimensional) Hodge star (it only operates on spatial forms) and $\hat{\eta}$ is the "main automorhism" on forms (just a multiple by $(-1)^p$; see e.g. Section 5.3 in [7]).

Notice that this type of Galilean $*$ vanishes for $p = d = 1 + n$ (top-degree forms) and the Carrollian $*$ vanishes for $p = 0$ (functions).

Second, there are Hodge stars based on (rank 1) tensors k_{ab} (for Galilean case) and $\tilde{k}^{ab}$ (for Carrollian case). Their action is displayed in (100)–(102) for Galilean case and (110)–(112) for Carrollian case. The comparison shows the following:

Galilean: The "Hodge star" $*$ based on k_{ab}

- vanishes for all $p < n$ (unlike those given by (11)),
- coincides for $p = n$ with (11),
- does not vanish for $p = n + 1$ (unlike the one given by (11)).

Carrollian: The "Hodge star" $*$ based on $\tilde{k}^{ab}$

- vanishes for all $p > 1$ (unlike those given by (12)),
- coincides for $p = 1$ with (12),
- does not vanish for $p = 0$ (unlike the one given by (12)).

Now, it is clear that *zero* operators from p-forms to q-forms are, although being formally canonical, not very interesting, notably if there are also *nonzero* canonical operators available in the same situation.

So, from this point of view, we can conclude that we should

- *in general* choose (11) and (12),

i.e. star operators based on (less degenerate) tensors h^{ab} and $\tilde{h}_{ab}$, the *only exceptions* being

- *Galilean*: $p = d = 1 + n$, when (102) is more useful (= nonzero) than (11),
- *Carrollian*: $p = 0$, when (110) is more useful (= nonzero) than (12).

So the two *exceptional* cases (which differ from (11) and (12)) read

$$*(e^0 \wedge \hat{\omega}) = 1 \qquad \textit{Galilean} \text{ Hodge star} \qquad \text{for} \quad p = 1 + n, \tag{13}$$

$$*1 = e^0 \wedge \hat{\omega} \qquad \textit{Carrollian} \text{ Hodge star} \qquad \text{for} \quad p = 0. \tag{14}$$

[1]We display the results, just for comparison, together with the Lorentzian case, based on standard metric g_{ab}. Notice that all this is compatible with an observation that, naively, Galilean limit ($c \to \infty$) of the Lorentzian case may be written as $e^0 \gg e^i$ and for Carrollian limit ($c \to 0$) we have $e^0 \ll e^i$.

For even more explicit expressions of (10)–(12) and (13)–(14) for the usual $1+3$ (rather than $1+n$) case, see Appendix A.3.

For a completely different approach (based on the concept of *intertwining operator*) to finding formulae (125)–(139) from Appendix A.3, which also provides an interesting additional insight, see paper [27].

3.2. Basic properties

Exactly as is the case for the standard Hodge star, both new Hodge stars map linearly p-forms on $(d-p)$-forms and vice versa,

$$\Omega^p \underset{*}{\overset{*}{\rightleftarrows}} \Omega^{d-p}. \tag{15}$$

A straightforward consequence of (11) and (12) is a striking feature of the new Hodge stars – they both square to *zero*[2],

$$** = 0, \qquad ** : \Omega^p \to \Omega^p, \qquad 0 \neq p \neq d. \tag{16}$$

This is in sharp contrast with the standard Hodge operator $*_g$ from (2) which, as is well known, squares to (plus or minus) the *unit* operator,

$$*_g *_g = \pm \hat{1}, \qquad *_g *_g : \Omega^p \to \Omega^p, \qquad \text{for } any \ p. \tag{17}$$

So, contrary to the standard case, neither the *Galilean*–Hodge star nor the *Carrollian*–Hodge star are *isomorphisms* between the two spaces of equal dimensions (spaces of p-forms and $(d-p)$-forms, respectively; exceptions are $p=0$ and $p=d$). Their (nonzero) kernels are

$$*(e^0 \wedge \hat{s}) = 0 \qquad \textit{Galilean} \text{ case} \qquad \deg \hat{s} \neq n, \tag{18}$$

$$* \ \hat{r} = 0 \qquad \textit{Carrollian} \text{ case} \qquad \deg \hat{r} \neq 0. \tag{19}$$

One can also rephrase the statement as that we can *no longer* speak of Galilean-Hodge *duality* as well as Carrollian-Hodge *duality* (with the above mentioned exceptions).

That is the price to pay for replacing *Lorentzian* spacetime with *Galilean* and *Carrollian* spacetimes, while still insisting that the concept of “Hodge star operator” remains *canonical*.

3.3. Behaviour with respect to pull-back

From the component expression (2) for the *standard* Hodge star operator one can see that the corresponding component-free expression reads

$$*_g \alpha \ \sim \ C \dots C(g^{-1} \otimes \cdots \otimes g^{-1} \otimes \omega_g \otimes \alpha), \tag{20}$$

where g^{-1} denotes cometric and C stands for contraction.

[2]Except for the “extremal” degrees $p=0$ and $p=d$ when, as is mentioned at the end of Section 3.1, we prefer (102) to (11) and (110) to (12). In these exceptional cases it squares to (plus or minus) the unit operator, exactly as we are accustomed to in the standard case.

Now, let

$$f : M \to M \tag{21}$$

be a diffeomorphism of M. Then, taking into account

$$f^*(g^{-1}) = (f^*g)^{-1}, \qquad f^*\omega_g = \omega_{f^*g}, \tag{22}$$

we get a well-known (and useful) property of the (standard) Hodge star:

$$f^*(*_g\alpha) = *_{f^*g}(f^*\alpha), \qquad \text{i.e.} \qquad f^* \circ *_g = *_{f^*g} \circ f^*. \tag{23}$$

Now, consider replacing the (peudo)Riemannian manifold (M, g) with a Galilean manifold (M, h, ξ) or a Carrollian manifold $(M, \tilde{h}, \tilde{\xi})$. The *structure* (20) of the Hodge star remains the same, we just make substitutions of tensor fields (depending on particular version, out of four possibilities, of the star, discussed in Section 2),

$$\textit{Galilean}\text{ case:} \qquad 1. \qquad g^{-1} \mapsto h, \qquad \omega_g \mapsto \omega_{h,\xi}, \tag{24}$$

$$2. \qquad g^{-1} \mapsto k, \qquad \omega_g \mapsto \tilde{\omega}_{h,\xi}. \tag{25}$$

$$\textit{Carrollian}\text{ case:} \qquad 1. \qquad g^{-1} \mapsto \tilde{h}, \qquad \omega_g \mapsto \tilde{\omega}_{\tilde{h},\tilde{\xi}}, \tag{26}$$

$$2. \qquad g^{-1} \mapsto \tilde{k}, \qquad \omega_g \mapsto \omega_{\tilde{h},\tilde{\xi}}. \tag{27}$$

(where $\omega_{h,\xi}$, $\tilde{\omega}_{h,\xi}$, $\omega_{\tilde{h},\tilde{\xi}}$ and $\tilde{\omega}_{\tilde{h},\tilde{\xi}}$ are defined in (65) and k and $\tilde{k}$ are introduced in Section 2). Then, taking into account

$$f^*\omega_{h,\xi} = \omega_{f^*h,f^*\xi}, \qquad f^*\tilde{\omega}_{\tilde{h},\tilde{\xi}} = \tilde{\omega}_{f^*\tilde{h},f^*\tilde{\xi}}, \tag{28}$$

(and similarly for all other "derived" canonical tensor fields mentioned above) we get corresponding useful property of the "new" Hodge stars:

$$f^* \circ *_{h,\xi} = *_{f^*h,f^*\xi} \circ f^* \qquad \text{and} \qquad f^* \circ *_{\tilde{h},\tilde{\xi}} = *_{f^*\tilde{h},f^*\tilde{\xi}} \circ f^* \tag{29}$$

(where $*_{h,\xi}$ denotes *any* of the two Galilean stars and similarly $*_{\tilde{h},\tilde{\xi}}$ denotes *any* of the two Carrolian stars). In particular, for *structure-preserving* diffeomorphisms, i.e. such that

$$\text{in } \textit{Lorentzian}\text{ case,} \qquad f^*g = g, \tag{30}$$

$$\text{in } \textit{Galilean}\text{ case,} \qquad f^*h = h, \quad f^*\xi = \xi, \tag{31}$$

$$\text{in } \textit{Carrollian}\text{ case,} \qquad f^*\tilde{h} = \tilde{h}, \quad f^*\tilde{\xi} = \tilde{\xi}. \tag{32}$$

hold, the *corresponding* Hodge star *commutes* with pull-back,

$$f^* \circ * = * \circ f^*. \tag{33}$$

So, as an example, for the simplest versions of the two non-Lorentzian spacetimes, *Galilei* and *Carroll* spacetimes, the corresponding star operators commute with (translations, rotations plus)

$$\text{Galilei boost}: \quad t' = t, \qquad \text{Carroll boost}: \quad t' = t + \mathbf{v} \cdot \mathbf{r}, \tag{34}$$

$$\mathbf{r}' = \mathbf{r} + \mathbf{v}t, \qquad \mathbf{r}' = \mathbf{r}. \tag{35}$$

(As for the Carroll boost, see e.g. (9) in [24], II.10 in [8] or Appendix A.1.2.) Put it differently, the new operators happen to be *invariant* w.r.t. Galilei and Carroll transformations, respectively.

More generally, the new "Hodge stars" are invariant w.r.t. *local Galilean* and *local Carrollian* transformations, respectively, in the same way as, say, the standard Hodge star on a Lorentzian spacetime is known to be invariant w.r.t. *local Lorentzian* transformations.

This property makes the Galilean and Carrollian Hodge star operators potentially interesting from the point of view of writing important *physics equations* in terms of *differential forms* on corresponding *spacetimes* (see an example in Section 4.2).

4. An application: Electrodynamics

As a simple application of Galilean/Carrollian Hodge star operator in physics, let us have just a brief look at Galilei and Carroll *electrodynamics*. In these two versions of electrodynamics (particular limits of the standard Minkowski one, see e.g. [8, 20]), the corresponding equations of motion for the fields **E** and **B** are Galilei/Carroll (rather than Poincaré) invariant.

4.1. Standard electrodynamics

In *Minkowski* spacetime, the (source-less, for simplicity) Maxwell equations

$$\operatorname{div}\mathbf{E} = 0, \tag{36}$$

$$\operatorname{curl}\mathbf{B} - \partial_t\mathbf{E} = 0, \tag{37}$$

$$\operatorname{curl}\mathbf{E} + \partial_t\mathbf{B} = 0, \tag{38}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{39}$$

may be neatly written in terms of 2-form of electromagnetic field,

$$F = dt \wedge \mathbf{E}\cdot d\mathbf{r} - \mathbf{B}\cdot d\mathbf{S}, \tag{40}$$

as follows:

$$d * F = 0, \qquad * = *_\eta \ \text{ here}, \tag{41}$$

$$dF = 0. \tag{42}$$

As already mentioned in Section 3.3, just because of properties of (d and) $*_\eta$, this way of presentation of the Maxwell equations makes their *Poincaré* invariance evident.

In more detail, the assignment reads

$$d * F = 0 \qquad \leftrightarrow \qquad \operatorname{div}\mathbf{E} = 0, \tag{43}$$

$$\operatorname{curl}\mathbf{B} - \partial_t\mathbf{E} = 0, \tag{44}$$

$$dF = 0 \qquad \leftrightarrow \qquad \operatorname{curl}\mathbf{E} + \partial_t\mathbf{B} = 0, \tag{45}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{46}$$

(see, e.g. Section 16.1 in [7]). Notice that each spacetime equation on the l.h.s. corresponds to as many as *two* spatial equations on the r.h.s.

4.2. Galilei and Carroll electrodynamics

Now it looks natural, in an effort to switch to Galilei (Carroll) electrodynamics, to just repeat the complete story with *Galilei* (*Carroll*) Hodge star operator. So again, introduce (see (117)) the 2-form of electromagnetic field exactly as in (40) and write down equations of motion à la (41) and (42), the only change being the replacement of the Minkowski Hodge star with the Galilei (Carroll) one (so with $*$ from (11) or (12); more explicitly from (132) or (137)):

$$d * F = 0, \qquad * = *_{h,\xi} \text{ or } *_{\tilde{h},\tilde{\xi}}, \text{ here,} \tag{47}$$

$$dF = 0. \tag{48}$$

Then it is clear that, again just because of the properties of d and $*$, this is a system of 1-st order partial differential equations for the fields $\mathbf{E}$ and $\mathbf{B}$, which is *Galilei* or *Carroll* (rather than Poincaré) *invariant*.

All the items listed above provide a promising signal that Eqs. (47)–(48) are probably closely associated with the desired Galilei (Carroll) electrodynamics.

How the system (47)–(48) actually looks like in terms of the fields $\mathbf{E}$ and $\mathbf{B}$?

First, it is clear that (48) looks *the same* for all three cases (standard Minkowski, Galilei as well as Carroll), namely it always corresponds to (45) and (46). So, both the Faraday's law and nonexistence of magnetic monopoles hold in all three versions of electrodynamics.

What really makes a difference is Eq. (47). When (132) and (137) – rather than (127) – is used for computation of $*$, we get

$$d * F = 0 \qquad \leftrightarrow \qquad \operatorname{curl} \mathbf{B} = 0 \qquad \text{in Galilei case,} \tag{49}$$

$$d * F = 0 \qquad \leftrightarrow \qquad \partial_t \mathbf{E} = 0, \ \operatorname{div} \mathbf{E} = 0 \qquad \text{in Carroll case.} \tag{50}$$

So, what we get when *all* spatial equations, resulting from

$$d * F = 0, \qquad dF = 0, \tag{51}$$

are displayed side by side for *all three* cases is

$$\begin{array}{ccc} \text{Minkowski} & \text{Galilei} & \text{Carroll} \end{array} \tag{52}$$

$$\begin{array}{rrr} \operatorname{div} \mathbf{E} = 0, & & \operatorname{div} \mathbf{E} = 0, \end{array} \tag{53}$$

$$\begin{array}{rrr} \operatorname{curl} \mathbf{B} - \partial_t \mathbf{E} = 0, & \operatorname{curl} \mathbf{B} \qquad = 0, & \partial_t \mathbf{E} = 0, \end{array} \tag{54}$$

$$\begin{array}{rrr} \operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0, & \operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0, & \operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0, \end{array} \tag{55}$$

$$\begin{array}{rrr} \operatorname{div} \mathbf{B} = 0, & \operatorname{div} \mathbf{B} = 0, & \operatorname{div} \mathbf{B} = 0. \end{array} \tag{56}$$

Checking equations in Galilei and Carroll columns versus *standard* Maxwell equations (left column) shows that there are some objects *missing* in Galilei as well as Carroll versions of "Maxwell equations" (51), namely

– *displacement current* $\partial_t \mathbf{E}$ in Ampère's law in the *Galilei* case,
– "*Ampère* term" $\operatorname{curl} \mathbf{B}$ in Ampère's law in the *Carroll* case,
– Eq. (53), i.e. *Gauss's law*, in the *Galilei* case.

Now, while the absence of the displacement current in Galilei electrodynamics and the "basic Ampère term" in Carroll electrodynamics are well-known (so expected and desired) facts (namely in the so-called "magnetic limit" version of Galilei electrodynamics, see [20], and "electric limit" version of Carroll electrodynamics, see [8]), the Gauss' law is usually present in all three versions of electrodynamics. In particular, the Galilei system usually contains the equations from the central column plus the Eq. (43). (The fact that the Gauss' law is not present in the spacetime differential forms version of the Galilei electrodynamics needs further study. We *do get*, by the way, the Gauss' law with the help of Galilei *connection*.)

So, to summarize, (51) *do* represent *correct* equations in *all three* versions of electrodynamics, albeit they do not provide, in the Galilei case, the whole story.

[Technically, the reason why just a *single* equation (49) corresponds to (47) (so that, at the end of the day, one equation is missing), contrary to *two* equations (43) and (44) in Minkowski case, looks to be simple: The Galilei star is no longer an isomorphism on 2-forms, it *kills* once and for all the "electric" part of F and then the action of d only produces a single term. Notice, however, that neither the *Carroll* star is an isomorphism on 2-forms (it kills once and for all the "magnetic" part of F). And here action of d produces *two* terms. So, the number of resulting equations *alone* actually depends on more subtle *details* of actions of the two star operators, rather than on their general common property of not being isomorphism.]

5. Summary and conclusions

The Hodge star operator belongs to essential tools in applications of differential forms on (pseudo)Riemannian manifolds. In particular, this is true for standard Lorentzian spacetimes.

However, Galilean and Carrollian spacetimes are not (pseudo)Riemannian manifolds. There is no (distinguished, nondegenerate) metric tensor available on them. Consequently, there is no standard "full-fledged" Hodge star operator on them.

Nevertheless, it turns out that one can *mimic* the usual construction of the Hodge star using specific (well-known) canonical tensors available. Since algebra of canonical tensors depends on particular spacetime, particular constructions and the resulting "Hodge stars" depend on particular spacetime as well.

The new "Hodge stars" retain some important properties of the standard one, while losing others.

In particular, they remain to be *invariant* w.r.t. structure preserving diffeomorphisms (in particular w.r.t. Galilei or Carroll transformations, respectively). More generally, w.r.t. *local* Galilean and *local* Carrollian transformations, respectively. This makes them potentially interesting from the point of view of writing *physics equations* in terms of differential forms on Galilean and Carrollian spacetimes.

Both new “Hodge stars”, however, cease to be linear *isomorphisms* (in general). One can no longer speak of Hodge *duality* (modulo exceptions).

In the three spacetimes under consideration (Lorentzian, Galilean and Carrollian) one can express differential forms in terms of a *pair* of *spatial* forms. Then the three Hodge star operators act according to (10)–(12) or (13)–(14).

For example, on Minkowski, Galilei and Carroll $(1+3)$-dimensional spacetimes, the action on the 2-form F of electromagnetic field reads

$$*_M(dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = dt \wedge (-\mathbf{B}) \cdot d\mathbf{r} - \mathbf{E} \cdot d\mathbf{S}, \tag{57}$$

$$*_G(dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = dt \wedge (-\mathbf{B}) \cdot d\mathbf{r}, \tag{58}$$

$$*_C(dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = \qquad\qquad - \mathbf{E} \cdot d\mathbf{S}. \tag{59}$$

Consequently, single (universal) *manifestly invariant* equation on spacetime

$$d * F = 0, \tag{60}$$

has substantially different 3-dimensional content (expression in terms of fields **E** and **B**), depending on *which* particular Hodge star operator ($*_M$, $*_G$ or $*_C$) is used (see (53) and (54).

A. Appendices

A.1. Galilean and Carrollian manifolds

Here, for convenience of the reader, basic definitions and facts are collected. In order to learn more details, one should consult references at the end of the paper, e.g. [8–17, 21–24].

A.1.1. Galilean manifold

Oriented Galilean manifold is a 4-tuple (M, ξ, h, o), where

- (M, o) is a d-dimensional oriented manifold, $d = 1 + n$,
- ξ is an everywhere nonzero covector (i.e. a $\binom{0}{1}$-tensor) field on M,
- h is an everywhere rank-n symmetric type-$\binom{2}{0}$-tensor field on M,
- such that $h(\xi, \ \cdot\ ) = 0$.

We call a (local) right-handed frame field $e_a = (e_0, e_i)$ and the (dual) coframe field $e^a = (e^0, e^i)$, $i = 1, \dots, n$ on (M, ξ, h, o) *adapted* (or distinguished) if

- $e^0 = \xi$,
- $h = \delta^{ij} e_i \otimes e_j$,

so that, in (any) adapted frame field, the components of the two tensor fields have “canonical form”

$$\xi_a \leftrightarrow \begin{pmatrix} \xi_0 \\ \xi_i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad h^{ab} \leftrightarrow \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{61}$$

The (point-dependent) change-of-basis matrix A between any pair $\hat{e}_a$, e_a of adapted frame fields, given by $\hat{e}_a = A^b_a e_b$, has the structure

$$A^b_a \leftrightarrow \begin{pmatrix} A^0_0 & A^0_i \\ A^i_0 & A^i_j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v^i & R^i_j \end{pmatrix}, \quad \text{i.e.} \quad A \leftrightarrow \begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix}, \tag{62}$$

where R is an n-dimensional rotation matrix.

In each point, such matrices form a Lie group G, subgroup of $\mathrm{GL}(d,\mathbb{R})$, the (homogeneous) *Galilei group* (R parametrizes rotations and v boosts, respectively).

A.1.2. Carrollian manifold

Oriented Carrollian manifold is a 4-tuple $(M, \tilde{\xi}, \tilde{h}, o)$, where

- (M, o) is a d-dimensional oriented manifold, $d = 1 + n$,
- $\tilde{\xi}$ is an everywhere nonzero vector (i.e. a $\binom{1}{0}$-tensor) field on M,
- $\tilde{h}$ is an everywhere rank-n symmetric type-$\binom{0}{2}$-tensor field on M,
- such that $\tilde{h}(\tilde{\xi}, \ \cdot \ ) = 0$.

We call a (local) right-handed frame field $e_a = (e_0, e_i)$ and the (dual) coframe field $e^a = (e^0, e^i)$, $i = 1, \dots, n$, on $(M, \tilde{\xi}, \tilde{h}, o)$ *adapted* (or distinguished) if

- $e_0 = \tilde{\xi}$,
- $\tilde{h} = \delta_{ij} e^i \otimes e^j$,

so that, in (any) adapted frame field, the components of the two tensors have "canonical form"

$$\xi^a \leftrightarrow \begin{pmatrix} \xi^0 \\ \xi^i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad h_{ab} \leftrightarrow \begin{pmatrix} h_{00} & h_{0i} \\ h_{i0} & h_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{63}$$

The (point-dependent) change-of-basis matrix A between any pair $\hat{e}_a$, e_a of adapted local frame fields, given by $\hat{e}_a = A^b_a e_b$, has the structure

$$A^b_a \leftrightarrow \begin{pmatrix} A^0_0 & A^0_i \\ A^i_0 & A^i_j \end{pmatrix} = \begin{pmatrix} 1 & v_i \\ 0 & R^i_j \end{pmatrix}, \quad \text{i.e.} \quad A \leftrightarrow \begin{pmatrix} 1 & v^T \\ 0 & R \end{pmatrix}, \tag{64}$$

where R is an n-dimensional rotation matrix.

In each point, such matrices form a Lie group G, subgroup of $\mathrm{GL}(d,\mathbb{R})$, the (homogeneous) *Carroll group* (R parametrizes rotations and v boosts, respectively).

A.1.3. Top-degree forms and polyvectors

Consider, on a patch $\mathcal{O}$ of a $(1+n)$-dimensional manifold M, *arbitrary local* frame field $e_a = (e_0, e_i)$ and the dual coframe field $e^a = (e^0, e^i)$. In terms of the frame/coframe fields, we introduce (local, everywhere nonvanishing) top-degree form $\omega(e)$ and top-degree polyvector $\tilde{\omega}(e)$,

$$\omega(e) := e^0 \wedge e^1 \wedge \dots \wedge e^n, \qquad \tilde{\omega}(e) := e_0 \wedge e_1 \wedge \dots \wedge e_n. \tag{65}$$

Introduce, similarly, a primed frame/coframe on primed $\mathcal{O}'$. Then, on the overlap $\mathcal{O} \cap \mathcal{O}'$, one has

$$e'_a = A^b_a e_b, \qquad e'^a = (A^{-1})^a_b e^b \tag{66}$$

(where $A \equiv A(x)$ is a unique position-dependent regular matrix) and we standardly get

$$\omega(e') = (\det A)^{-1}\ \omega(e), \qquad \tilde{\omega}(e') = (\det A)\ \tilde{\omega}(e). \tag{67}$$

Now consider restriction of the above situation to

- oriented *Galilean/Carrollian* manifold,
- *adapted* frame/coframe fields.

Then the change-of-basis matrices A have the form (62) or (64), depending on whether we speak of Galilean or Carrollian manifold. In both cases, however, we have clearly

$$\det A = 1 \tag{68}$$

and, therefore, on the overlap $\mathcal{O} \cap \mathcal{O}'$,

$$\omega(e') = \omega(e), \qquad \tilde{\omega}(e') = \tilde{\omega}(e). \tag{69}$$

This means that both objects, ω and $\tilde{\omega}$, are well defined on the union $\mathcal{O} \cup \mathcal{O}'$ and, consequently, that *canonical global* objects are actually defined in terms of (local) expressions (65).

Put another way,

- on *both* Galilean *and* Carrollian (oriented) manifolds,
- *both* canonical *volume form* and canonical *top-degree polyvector* exist.[3]

In components (w.r.t. arbitrary local *adapted* frame/coframe fields) this may be written in terms of the Levi-Civita symbols as

$$\omega_{a\dots b} = \epsilon_{a\dots b}, \qquad \tilde{\omega}^{a\dots b} = \epsilon^{a\dots b}. \tag{70}$$

A.2. Details of computation of the four analogs of Hodge star

Here, detailed calculations of (analogs of) Hodge star operators, leading to results collected in Section 3.1, are shown.

Component definition (2) is equivalent to an expression for action of $*$ on *basis* p-forms (on d-dimensional manifold)

$$*(e^a \wedge \dots \wedge e^b) = \frac{1}{(d-p)!}\omega^{a\dots b}{}_{c\dots d}\ e^c \wedge \dots \wedge e^d. \tag{71}$$

What we need to calculate from (71) is action of $*$ on a *general decomposed* p-form, i.e.

$$*\alpha = *(e^0 \wedge \hat{s} + \hat{r}) = \ ? \qquad \hat{s} = \hat{s}_{p-1},\ \ \hat{r} = \hat{r}_p. \tag{72}$$

[3]It is clear, that *any constant multiples* of the tensor fields discussed above are canonical as well.

In *Lorentzian* case, the answer reads

$$*(e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \tag{73}$$

(see e.g. [25, 26] or, in the simplest case, Section 16.1 in [7]), where $\hat{*}$ is the Euclidean Hodge star and $\hat{\eta}$ is the main automorphism on forms, i.e. just multiplication by $(-1)^p$. What we want here are similar formulae for all four new cases, Galilean versus Carrollian case and $h^{ab}/\tilde{h}_{ab}$ versus $k_{ab}/\tilde{k}^{ab}$ case.

Because of specific features of the tensors $h^{ab}, \tilde{h}_{ab}, k_{ab}, \tilde{k}^{ab}$, it is convenient to discuss the two parts ($e^0 \wedge \hat{s}$ and $\hat{r}$) separately, i.e. to compute

$$*(e^0 \wedge e^i \wedge \cdots \wedge e^j), \qquad\qquad *(e^i \wedge \cdots \wedge e^j), \tag{74}$$

and then combine the two results into the answer to (72).

Note: In what follows, we multiply the canonical tensors $h^{ab}, \tilde{h}_{ab}, k_{ab}, \tilde{k}^{ab}$ by a *free parameter* λ (and sometimes also ω and $\tilde{\omega}$ by μ; the tensors clearly remain to be canonical) and choose particular value of λ (and μ) at the end of computation so that the result resembles the standard Lorentzian case *as much as possible*.

A.2.1. Hodge stars based on tensors $h^{ab}/\tilde{h}_{ab}$

Galilean case – based on h^{ab}:

Here, w.r.t. an *adapted* frame (see (61) and (70)),

$$\omega^{a\dots b}{}_{c\dots d} := h^{a\hat{a}} \dots h^{b\hat{b}} \omega_{\hat{a}\dots\hat{b}c\dots d} = h^{a\hat{a}} \dots h^{b\hat{b}} \epsilon_{\hat{a}\dots\hat{b}c\dots d}. \tag{75}$$

From $h^{00} = h^{0i} = 0$ we can immediately deduce that

$$\omega^{0i\dots j}{}_{k\dots l} = 0, \qquad \text{i.e.} \qquad *\,(e^0 \wedge \hat{s}) = 0. \tag{76}$$

On strictly *spatial* p-form $\hat{r}$ it is more interesting. For $h^{ij} = \lambda\delta^{ij}$ we get from (71) on *basis* spatial p-form

$$\begin{aligned}
*(\underbrace{e^i \wedge \cdots \wedge e^j}_{p}) &= \tfrac{1}{(1+n-p)!}\omega^{i\dots j}{}_{c\dots d}\; e^c \wedge \cdots \wedge e^d \\
&= \tfrac{(1+n-p)}{(1+n-p)!}\omega^{i\dots j}{}_{0k\dots l}\; e^0 \wedge e^k \wedge \cdots \wedge e^l \\
&= \lambda^p e^0 \wedge \tfrac{1}{(n-p)!}\delta^{i\hat{\imath}} \dots \delta^{j\hat{\jmath}}\omega_{\hat{\imath}\dots\hat{\jmath}0k\dots l}\; e^k \wedge \cdots \wedge e^l \\
&= \lambda^p e^0 \wedge \tfrac{1}{(n-p)!}\epsilon_{i\dots j0k\dots l}\; e^k \wedge \cdots \wedge e^l \\
&= (-\lambda)^p e^0 \wedge \tfrac{1}{(n-p)!}\epsilon_{0i\dots jk\dots l}\; e^k \wedge \cdots \wedge e^l \\
&= (-\lambda)^p e^0 \wedge \tfrac{1}{(n-p)!}\epsilon_{i\dots jk\dots l}\; e^k \wedge \cdots \wedge e^l \\
&= (-\lambda)^p e^0 \wedge \hat{*}\left(e^i \wedge \cdots \wedge e^j\right),
\end{aligned}$$

i.e. on *complete* spatial p-form $\hat{r}$,

$$*\hat{r} = (-\lambda)^p e^0 \wedge \hat{*}\hat{r}. \tag{77}$$

Combining with (76), we have

$$*(e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge (-\lambda)^p \hat{*}\hat{r}. \tag{78}$$

In order to make this formula as similar as possible to (73), we have to choose

$$\lambda = -1, \qquad \text{i.e.} \qquad h^{ij} = -\delta^{ij}. \tag{79}$$

For this fixation of the arbitrary constant, we get

$$*(e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge \hat{*}\hat{r}. \tag{80}$$

Carrollian case – based on $\tilde{h}_{ab}$:

Here

$$\omega^{a\dots b}{}_{c\dots d} := \tilde{\omega}^{a\dots b\hat{c}\dots\hat{d}}\tilde{h}_{c\hat{c}}\dots\tilde{h}_{d\hat{d}} = \epsilon_{a\dots b\hat{c}\dots\hat{d}}\tilde{h}_{c\hat{c}}\dots\tilde{h}_{d\hat{d}} = \epsilon_{a\dots bk\dots l}\tilde{h}_{ck}\dots\tilde{h}_{dl}. \tag{81}$$

From $\tilde{h}_{00} = \tilde{h}_{0i} = 0$ (see (7)) we can immediately deduce that

$$\omega^{i\dots j}{}_{c\dots d} = 0, \qquad \text{i.e.} \qquad *\,\hat{r} = 0. \tag{82}$$

On *combined* p-form $e^0 \wedge \hat{s}$ it is more interesting. For $\tilde{h}_{ij} = \lambda\delta_{ij}$ we get (see (71)) on *basis* combined p-form

$$\begin{aligned}
*(e^0 \wedge \overbrace{e^i \wedge \dots \wedge e^j}^{p-1}) &= \tfrac{1}{(1+n-p)!}\omega^{0i\dots j}{}_{c\dots d}\;\; e^c \wedge \dots \wedge e^d \\
&= \tfrac{1}{(1+n-p)!}\omega^{0i\dots j}{}_{k\dots l}\;\; e^k \wedge \dots \wedge e^l \\
&= \tfrac{\lambda^{1+n-p}}{(1+n-p)!}\epsilon_{0i\dots jk\dots l}\;\; e^k \wedge \dots \wedge e^l \\
&= \lambda^{1+n-p}\;\; \tfrac{1}{(n-(p-1))!}\epsilon_{i\dots jk\dots l}\;\; e^k \wedge \dots \wedge e^l \\
&= \lambda^{n-(p-1)}\;\; \hat{*}(e^i \wedge \dots \wedge e^j),
\end{aligned}$$

i.e.

$$*(e^0 \wedge \hat{s}) = \lambda^{n-(p-1)}\;\; \hat{*}\hat{s}. \tag{83}$$

Combining with (82), we have

$$*(e^0 \wedge \hat{s} + \hat{r}) = \lambda^{n-(p-1)}\;\; \hat{*}\hat{s}. \tag{84}$$

In order to make this formula as similar as possible to (73), we need to fulfill

$$\lambda^{n-(p-1)} = (-1)^{p-1}. \tag{85}$$

This is not possible for any choice of λ. But we can use freedom in *both* $\tilde{\omega}$ *and* $\tilde{h}$ and take, from the very beginning,

$$\tilde{\omega}^{0i\dots jk\dots l} = \mu(n)\epsilon_{0i\dots jk\dots l}, \qquad \tilde{h}_{ij} = \lambda(n,p)\delta_{ij}. \tag{86}$$

Then we need to fulfill (instead of (85))

$$\mu\lambda^{n-(p-1)} \equiv (\mu\lambda^n)\lambda^{-(p-1)} = (-1)^{p-1}, \tag{87}$$

with solution

$$\mu = (-1)^n \qquad \lambda = -1. \tag{88}$$

Then we get from (84) the final expression

$$*(e^0 \wedge \hat{s} + \hat{r}) = \hat{*}\hat{\eta}\hat{s}. \tag{89}$$

A.2.2. Hodge stars based on tensors $k_{ab}/\tilde{k}^{ab}$

Galilean case – based on k_{ab}:

Here

$$\omega^{a\dots b}{}_{c\dots d} := \tilde{\omega}^{a\dots b\hat{c}\dots\hat{d}} k_{c\hat{c}} \dots k_{d\hat{d}} = \epsilon_{a\dots b\hat{c}\dots\hat{d}} k_{c\hat{c}} \dots k_{d\hat{d}}. \tag{90}$$

From $k_{00} = \lambda$ and all other possibilities vanishing (see (6)) we have

$$\omega^{a\dots b}{}_{c\dots d} = \epsilon_{a\dots b0\dots 0} k_{c0} \dots k_{d0}, \tag{91}$$

and, consequently,

$$*(e^a \wedge \dots \wedge e^b) = \frac{1}{(1+n-p)!} \lambda^{1+n-p} \epsilon_{a\dots b0\dots 0} \overbrace{e^0 \wedge \dots \wedge e^0}^{1+n-p}. \tag{92}$$

So there are just two *nonzero* cases (values of p):

1) $1 + n - p = 0$ (i.e. $p = 1 + n$) and then

$$*(e^0 \wedge e^1 \wedge \dots \wedge e^n) = \epsilon_{01\dots n} = 1, \tag{93}$$

irrespective of the choice of λ.

2) $1 + n - p = 1$ (i.e. $p = n$) and then

$$*(e^1 \wedge \dots \wedge e^n) = \lambda \epsilon_{1\dots n0} e^0 = (-1)^n \lambda e^0. \tag{94}$$

Here we can obtain, by appropriate choice of λ, coincidence with formula (80). Indeed, since (in n-dimensional *Euclidean* space)

$$\hat{*}(e^1 \wedge \dots \wedge e^n) = \operatorname{sgn} \hat{g} = 1, \tag{95}$$

we can rewrite (94) as

$$*(e^1 \wedge \dots \wedge e^n) = (-1)^n \lambda e^0 \wedge \hat{*}(e^1 \wedge \dots \wedge e^n), \tag{96}$$

and this becomes, for the choice $\lambda = (-1)^n$,

$$*(e^1 \wedge \dots \wedge e^n) = e^0 \wedge \hat{*}(e^1 \wedge \dots \wedge e^n), \tag{97}$$

coinciding with what (80) says for this case.

If we denote the "spatial volume element" as

$$\hat{\omega} := e^1 \wedge \dots \wedge e^n, \tag{98}$$

we can summarize (93) and (97) as

$$* \ \hat{\omega} = e^0 \wedge \hat{*}\hat{\omega} \equiv e^0, \qquad *(e^0 \wedge \hat{\omega}) = 1, \tag{99}$$

or, make complete summarizing as

$$\begin{aligned} *(e^0 \wedge \hat{s} + \hat{r}) &= 0 & & \text{for} \quad p < n, & (100)\\ *\hat{\omega} &= e^0 \qquad *\,(e^0 \wedge \hat{s}) = 0 & & \text{for} \quad p = n, & (101)\\ *(e^0 \wedge \hat{\omega}) &= 1 & & \text{for} \quad p = 1 + n. & (102)\end{aligned}$$

Carrollian case – based on $\tilde{k}^{ab}$:

Here

$$\omega^{a\dots b}{}_{c\dots d} := \tilde{k}^{a\hat{a}} \dots \tilde{k}^{b\hat{b}} \omega_{\hat{a}\dots\hat{b}c\dots d} = \tilde{k}^{a\hat{a}} \dots \tilde{k}^{b\hat{b}} \epsilon_{\hat{a}\dots\hat{b}c\dots d}. \tag{103}$$

From $\tilde{k}^{00} = \lambda$ and all other possibilities vanishing (see (8)) we have

$$\omega^{a\dots b}{}_{c\dots d} = \tilde{k}^{a0} \dots \tilde{k}^{b0} \epsilon_{0\dots 0c\dots d}, \tag{104}$$

and, consequently,

$$*(e^a \wedge \dots \wedge e^b) = \frac{1}{(1+n-p)!} \tilde{k}^{a0} \dots \tilde{k}^{b0} \epsilon_{0\dots 0c\dots d} \overbrace{e^c \wedge \dots \wedge e^d}^{1+n-p}. \tag{105}$$

So there are just two *nonzero* cases (values of p):

1) $p = 0$, and then

$$*1 \;=\; \frac{1}{(1+n)!} \epsilon_{c\dots d} \overbrace{e^c \wedge \dots \wedge e^d}^{1+n} \;=\; e^0 \wedge e^1 \wedge \dots \wedge e^n \;\equiv\; e^0 \wedge \hat{\omega}, \tag{106}$$

irrespective of the choice of λ.

2) $p = 1$, and then

$$*e^0 \;=\; \frac{1}{n!} \tilde{k}^{00} \epsilon_{0i\dots j} \overbrace{e^i \wedge \dots \wedge e^j}^{n} \;=\; \lambda e^1 \wedge \dots \wedge e^n \;\equiv\; \lambda \hat{\omega}. \tag{107}$$

Here we can obtain, by the choice $\lambda = 1$, coincidence with formula (89). So

$$*e^0 = \hat{\omega}. \tag{108}$$

We can summarize (106) and (97) as

$$*1 = e^0 \wedge \hat{\omega}, \qquad *e^0 = \hat{\omega}, \tag{109}$$

or, make complete summarizing as

$$\begin{aligned} *1 &= e^0 \wedge \hat{\omega} & & \text{for} \quad p = 0, & (110)\\ *e^0 &= \hat{\omega} \qquad *\, e^i = 0 & & \text{for} \quad p = 1, & (111)\\ *(e^0 \wedge \hat{s} + \hat{r}) &= 0 & & \text{for} \quad p > 1. & (112)\end{aligned}$$

A.3. Explicit formulae for $1+3$

In the usual $(1+3)$-dimensional Minkowski, Galilei and Carroll spacetimes, we can use standard Cartesian coordinate basis as an adapted frame

$$(e^0, e^i) = (dt, dx^i) = (dt, dx, dy, dz). \tag{113}$$

Then the general decomposition formula from (9) reads

$$\alpha = dt \wedge \hat{s} + \hat{r}, \tag{114}$$

and may be further specified, for the five relevant degrees of differential forms, as follows:

$$\Omega^0: \quad \alpha = f, \tag{115}$$
$$\Omega^1: \quad \alpha = f dt + \mathbf{a} \cdot d\mathbf{r}, \tag{116}$$
$$\Omega^2: \quad \alpha = dt \wedge \mathbf{a} \cdot d\mathbf{r} + \mathbf{b} \cdot d\mathbf{S}, \tag{117}$$
$$\Omega^3: \quad \alpha = dt \wedge \mathbf{a} \cdot d\mathbf{S} + f dV, \tag{118}$$
$$\Omega^4: \quad \alpha = f dt \wedge dV. \tag{119}$$

where

$$\mathbf{a} \cdot d\mathbf{r} = a_x dx + a_y dy + a_z dz, \tag{120}$$
$$\mathbf{a} \cdot d\mathbf{S} = a_x dS_x + a_y dS_y + a_z dS_z, \tag{121}$$
$$\equiv a_x dy \wedge dz + a_y dz \wedge dx + a_z dx \wedge dy, \tag{122}$$
$$dV = dx \wedge dy \wedge dz. \tag{123}$$

So, concerning this presentation of forms alone, there is no difference between the three spacetimes (for the Minkowski case, see e.g. Section 16.1 in [7]).

Due to (10)–(12) and well-known Euclidean 3D-results,

$$\hat{*} f = f dV, \qquad \hat{*}(\mathbf{a} \cdot d\mathbf{r}) = \mathbf{a} \cdot d\mathbf{S}, \qquad \hat{*}(\mathbf{a} \cdot d\mathbf{S}) = \mathbf{a} \cdot d\mathbf{r}, \qquad \hat{*}(f dV) = f \tag{124}$$

(see e.g. Section 8.5 in [7]) we get, in this language, the following explicit results for the action of the three Hodge star operators:[4]

Minkowski Hodge star:

$$*f = f dt \wedge dV, \tag{125}$$
$$*(f dt + \mathbf{a} \cdot d\mathbf{r}) = dt \wedge \mathbf{a} \cdot d\mathbf{S} + f dV, \tag{126}$$
$$*(dt \wedge \mathbf{a} \cdot d\mathbf{r} + \mathbf{b} \cdot d\mathbf{S}) = dt \wedge \mathbf{b} \cdot d\mathbf{r} - \mathbf{a} \cdot d\mathbf{S}, \tag{127}$$
$$*(dt \wedge \mathbf{a} \cdot d\mathbf{S} + f dV) = f dt + \mathbf{a} \cdot d\mathbf{r}, \tag{128}$$
$$*(f dt \wedge dV) = -f. \tag{129}$$

[4]The signs in the Galilei and Carroll cases were chosen so that the results resembled the standard Minkowski results as much as possible.

Galilei Hodge star:

$$*f = f dt \wedge dV, \tag{130}$$

$$*(f dt + \mathbf{a} \cdot d\mathbf{r}) = dt \wedge \mathbf{a} \cdot d\mathbf{S}, \tag{131}$$

$$*(dt \wedge \mathbf{a} \cdot d\mathbf{r} + \mathbf{b} \cdot d\mathbf{S}) = dt \wedge \mathbf{b} \cdot d\mathbf{r}, \tag{132}$$

$$*(dt \wedge \mathbf{a} \cdot d\mathbf{S} + f dV) = f dt, \tag{133}$$

$$*(f dt \wedge dV) = -f. \tag{134}$$

Carroll Hodge star:

$$*f = f dt \wedge dV, \tag{135}$$

$$*(f dt + \mathbf{a} \cdot d\mathbf{r}) = f dV, \tag{136}$$

$$*(dt \wedge \mathbf{a} \cdot d\mathbf{r} + \mathbf{b} \cdot d\mathbf{S}) = -\mathbf{a} \cdot d\mathbf{S}, \tag{137}$$

$$*(dt \wedge \mathbf{a} \cdot d\mathbf{S} + f dV) = \mathbf{a} \cdot d\mathbf{r}, \tag{138}$$

$$*(f dt \wedge dV) = -f. \tag{139}$$

Notice that in expressions (134) and (135), Hodge stars based on tensors $k_{ab}/\tilde{k}^{ab}$ were used (i.e. (102) and (110) rather than (11) and (12), leading to zero result). See the discussion at the end of Section 3.1.

As for general Galilean and Carrollian (rather than Galilei and Carroll) spacetimes, the same formulae are true if we replace interpretation of the l.h.s. of (120)–(123) with more general expressions

$$\mathbf{a} \cdot d\mathbf{r} = a_1 e^1 + a_2 e^2 + a_3 e^3, \tag{140}$$

$$\mathbf{a} \cdot d\mathbf{S} = a_1 e^2 \wedge e^3 + a_2 e^3 \wedge e^1 + a_3 e^1 \wedge e^2, \tag{141}$$

$$dV = e^1 \wedge e^2 \wedge e^3. \tag{142}$$

where $e^a \equiv (e^0, e^i)$, $i = 1, 2, 3$, represent any *adapted* coframe fields.

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