

2D vector calculus and forms on 4D space-time

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We will discuss:

- 3D vector calculus and forms on E^3

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- Coordinate free approach:
- How 4D forms can be 1+3 decomposed w.r.t. V
- How 4D forms can be 1+1+2 decomposed w.r.t. V, W

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3D vector calculus and forms on E^3

On **any** (M, g, o) we have **canonical isomorphisms** $\sharp_g$, $\flat_g$ and $*_g$.
 Happy coincidence on (M^3, g, o) : We can **"forget about forms"**:

$$\begin{array}{ccccccc}
 \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\
 \uparrow id & & \uparrow \flat & & \uparrow *_\flat & & \uparrow * \\
 \mathcal{F} & \xrightarrow{\text{grad}} & \mathfrak{X} & \xrightarrow{\text{curl}} & \mathfrak{X} & \xrightarrow{\text{div}} & \mathcal{F}
 \end{array}$$

We can make do with just **functions** and **vector fields**.

Then grad, curl and div represent **effective exterior derivative**.

3D vector calculus and forms on E^3 (2)

There is a **convenient notation** for the action of the three canonical isomorphisms:

$$\begin{array}{ll} \flat : \mathfrak{X} \rightarrow \Omega^1 & \mathbf{A} \mapsto \mathbf{A} \cdot d\mathbf{r} \\ * \circ \flat : \mathfrak{X} \rightarrow \Omega^2 & \mathbf{A} \mapsto \mathbf{A} \cdot d\mathbf{S} \\ * : \mathcal{F} \rightarrow \Omega^3 & h \mapsto h dV \end{array}$$

Directly **from** the **diagram** we can read **differential** identities

$$\begin{aligned} df &= \text{grad } f \cdot d\mathbf{r} \\ d(\mathbf{A} \cdot d\mathbf{r}) &= (\text{curl } \mathbf{A}) \cdot d\mathbf{S} \\ d(\mathbf{A} \cdot d\mathbf{S}) &= (\text{div } \mathbf{A}) dV \end{aligned}$$

3D vector calculus and forms on E^3 (3)

Integrating and using Stokes theorem (**for forms**) we get the following (well-known) **integral** identities

gradient theorem $\int_c \text{grad } f \cdot d\mathbf{r} := f(B) - f(A)$

Stokes theorem $\int_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} := \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}$

Gauss's theorem $\int_V (\text{div } \mathbf{A}) dV := \oint_{\partial V} \mathbf{A} \cdot d\mathbf{S}$

2D vector calculus and forms on E^2

Again we have **canonical isomorphisms** $\sharp_g$, $\flat_g$ and $*_g$.

Also on (M^2, g, o) we **can "forget about forms"**,

we can **still** make do with just **functions** and **vector fields**.

But now **a new feature** arises: $* : \Omega^1 \rightarrow \Omega^1$

$\Rightarrow$ we have **two** commutative diagrams (see Fecko 2021):

$$\begin{array}{ccccc}
 \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 \\
 \text{id} \downarrow & & \downarrow \sharp & & \downarrow * \\
 \mathcal{F} & \xrightarrow{\text{grad}} & \mathfrak{X} & \xrightarrow{\text{curl}_3} & \mathcal{F}
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 \\
 \text{id} \downarrow & & \downarrow \sharp^* & & \downarrow * \\
 \mathcal{F} & \xrightarrow{\text{Ham}} & \mathfrak{X} & \xrightarrow{-\text{div}} & \mathcal{F}
 \end{array}$$

So there are as many as **four effective exterior derivative operators**.

2D vector calculus and forms on E^2 (2)

Now we can read off **four differential** identities

$$df = \text{grad } f \cdot d\mathbf{r}$$

$$d(\mathbf{A} \cdot d\mathbf{r}) = (\text{curl}_3 \mathbf{A}) dS$$

$$df = - * (\text{Ham } f \cdot d\mathbf{r})$$

$$d * (\mathbf{A} \cdot d\mathbf{r}) = (\text{div } \mathbf{A}) dS$$

and obtain corresponding **four integral** identities.

$$\int_C \text{grad } f \cdot d\mathbf{r} = f(B) - f(A) \quad \int_S (\text{curl}_3 \mathbf{A}) dS = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}$$

$$\int_C * (\text{Ham } f \cdot d\mathbf{r}) = f(A) - f(B) \quad \int_S (\text{div } \mathbf{A}) dS = \oint_{\partial S} * (\mathbf{A} \cdot d\mathbf{r})$$

2D vector calculus and forms on E^2 (3)

A **useful observation** about $* : \Omega^1 \rightarrow \Omega^1$:

$$*(\mathbf{A} \cdot d\mathbf{r}) =: \mathbf{A}_{\pi/2} \cdot d\mathbf{r}$$

where $\mathbf{A}_{\pi/2}$ is the vector field **rotated by** $\pi/2$ (in positive sense):

$$\mathbf{A} \mapsto \mathbf{A}_{\pi/2} \quad \leftrightarrow \quad (A_x, A_y) \mapsto (-A_y, A_x)$$

So in E^2 , **contrary** to E^3 , there is a **canonical** operation on **vector fields**

$$(\)_{\pi/2} : \mathfrak{X} \rightarrow \mathfrak{X} \quad \mathbf{A} \mapsto \mathbf{A}_{\pi/2}$$

(In E^3 **no canonical axis** - or a **2-plane** - is available.)

2D vector calculus and forms on E^2 (4)

This observation also provides an **alternative way** how to “understand” the **two new** (and strange looking) differential operations present in 2D vector calculus:

$$\text{Ham } f = (\text{grad } f)_{\pi/2} \quad \text{curl}_3 \mathbf{A} = -\text{div } \mathbf{A}_{\pi/2}$$

So **Ham** f can be produced in **two steps**:

- compute “just” the good old **gradient** of f
- **rotate** the resulting vector field, in each point, by $\pi/2$.

Similarly **curl**₃ $\mathbf{A}$:

- **first rotate** $\mathbf{A}$, in each point, by $\pi/2$, then
- compute “just” the good old **divergence** of the resulting field.

3D vector calculus and forms on Minkowski space-time $E^{1,3}$

Consider now **Minkowski** space-time $E^{1,3}$

with **Cartesian** coordinates $x^\mu = (t, x, y, z)$.

So 1-form **coordinate basis** is $dx^\mu = (dt, dx, dy, dz)$.

Then, in coordinate expression of **any p -form** α on $E^{1,3}$,

any term contains **dt** exactly **once or not at all**

(since $dt \wedge dt = 0$:-).

Then, we can write

$$\alpha = dt \wedge \hat{s} + \hat{r}$$

where $\hat{s}$ and $\hat{r}$ **only** contain **dx, dy, dz** (but not dt).

Such forms are called **spatial** forms.

Spatial forms and 3D vector calculus

Since dx, dy, dz may be also treated as a basis of 1-forms on E^3 , each **spatial** form may be written in the **notation from** E^3 .

As an example, a **two**-form α on Minkowski space-time reads

$$\alpha = dt \wedge \mathbf{a} \cdot d\mathbf{r} + \mathbf{b} \cdot d\mathbf{S}$$

Notice that $\mathbf{a} \cdot d\mathbf{r}$ and $\mathbf{b} \cdot d\mathbf{S}$ are **not literally** forms on E^3 , since they are in general **depend on time** t .

So, when working with **forms** on **Minkowski** space-time $E^{1,3}$, we can instead **switch** to work with **time-dependent** objects from good-old **3D vector calculus**.

Operations d , $*$ and δ on forms on $E^{1,3}$

Now a computation shows that

$$\begin{aligned} d(dt \wedge \hat{s} + \hat{r}) &= dt \wedge (\mathcal{L}_{\partial_t} \hat{r} - \hat{d}\hat{s}) + \hat{d}\hat{r} \\ *(dt \wedge \hat{s} + \hat{r}) &= dt \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \\ \delta(dt \wedge \hat{s} + \hat{r}) &= dt \wedge \hat{\delta}\hat{s} + (-\mathcal{L}_{\partial_t} \hat{s} - \hat{\delta}\hat{r}) \end{aligned}$$

where

- $\delta = *^{-1}d*$ is the **codifferential**
- **hatted** operations ($\hat{d}, \hat{*}, \hat{\delta}$) only act on **hatted** forms

That means, we get **operations** from **3D vector calculus**.

d , $*$ and δ and 3D vector calculus

The **only** operation which **is not** from 3D vector calculus is the **Lie derivative** $\mathcal{L}_{\partial_t}$ along the **time axis** (it comes from 4D d). This operation is **completely foreign** to the **proper** 3D vector calculus!

So, **all** operations on **forms on Minkowski space** boil down to

- standard operations of **3D vector calculus**
- **plus** $\mathcal{L}_{\partial_t}$ playing role of **time derivative**.

Example: Electromagnetism on Minkowski space-time $E^{1,3}$

The best known example:

$$\begin{aligned} F &= dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S} \\ j &= \rho dt - \mathbf{j} \cdot d\mathbf{r} \end{aligned}$$

The general formulas lead to

$$\begin{aligned} d(dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) &= dt \wedge (-\partial_t \mathbf{B} - \text{curl } \mathbf{E}) \cdot d\mathbf{S} - (\text{div } \mathbf{B}) dV \\ \delta(dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) &= (-\text{div } \mathbf{E}) dt + (-\partial_t \mathbf{E} + \text{curl } \mathbf{B}) \cdot d\mathbf{r} \end{aligned}$$

Then, the two Maxwell equations

$$\delta F = -j \quad dF = 0$$

translate to four equations from electromagnetism course

$$\begin{aligned} \text{div } \mathbf{E} &= \rho & -\partial_t \mathbf{E} + \text{curl } \mathbf{B} &= \mathbf{j} \\ \text{div } \mathbf{B} &= 0 & -\partial_t \mathbf{B} - \text{curl } \mathbf{E} &= 0 \end{aligned}$$

2D vector calculus and forms on Minkowski space-time $E^{1,3}$

Consider again Minkowski space-time $E^{1,3}$
 with Cartesian coordinates $x^\mu = (t, x, y, z)$.
 We already know that

$$\alpha = dt \wedge \hat{s} + \hat{r}$$

where $\hat{s}$ and $\hat{r}$ only contain dx, dy, dz (but not dt).

Now we can iterate the procedure with (say) dz present in spatial forms:

$$\hat{s} = dz \wedge \check{A} + \check{B} \quad \hat{r} = dz \wedge \check{C} + \check{D}$$

Here $\check{A}, \check{B}, \check{C}, \check{D}$ only contain dx, dy (but neither dz nor dt).
 We can call them 2D-spatial forms.

2D-spatial forms and 2D vector calculus

So, for a **general** form on **Minkowski** space
we have a **finer** decomposition (with **four 2D-spatial** terms)

$$\alpha = dt \wedge dz \wedge \check{A} + dt \wedge \check{B} + dz \wedge \check{C} + \check{D}$$

Each **2D-spatial** form may be written in the **notation from E^2** .

So, when working with **forms** on **Minkowski** space-time $E^{1,3}$,
we can **also** decide to **switch** to work
with **z-and-time-dependent** objects from **2D vector calculus**.

2D-spatial forms and 2D vector calculus

As an example, in electromagnetism, it turns out that

$$F = dt \wedge dz \wedge E_z + dt \wedge \mathbf{E} \cdot d\mathbf{r} + dz \wedge \star(\mathbf{B} \cdot d\mathbf{r}) - B_z dS$$

where

$$\mathbf{E} \cdot d\mathbf{r} = E_x dx + E_y dy$$

$$\mathbf{B} \cdot d\mathbf{r} = B_x dx + B_y dy$$

$$\star(\mathbf{B} \cdot d\mathbf{r}) = B_x dy - B_y dx$$

$$B_z dS \equiv \star B_z = B_z dx \wedge dy$$

2D-vector calculus version of Maxwell equations

The Maxwell equations

$$\delta F = -j \quad dF = 0$$

translate to their **not so well** known 2D-version

$$-\nabla_{\pi/2} B_z + \partial_z \mathbf{B}_{\pi/2} - \partial_t \mathbf{E} = \mathbf{j}$$

$$-\operatorname{div} \mathbf{B}_{\pi/2} - \partial_t E_z = j_3$$

$$\operatorname{div} \mathbf{E} + \partial_z E_z = \rho$$

$$\nabla_{\pi/2} E_z - \partial_z \mathbf{E}_{\pi/2} - \partial_t \mathbf{B} = 0$$

$$\operatorname{div} \mathbf{E}_{\pi/2} - \partial_t B_z = 0$$

$$\operatorname{div} \mathbf{B} + \partial_z B_z = 0$$

Dimensional reduction of electrodynamics w.r.t. ∂_t

Return to description of electrodynamics in the language of **3D-spatial** objects, i.e. in terms of **3D vector calculus**.

Consider the situation when **all quantities** happen to be **invariant** w.r.t. **time** direction

$$\mathcal{L}_{\partial_t}(\dots) = 0$$

This means that the **time** direction becomes in a sense **superfluous**, the quantities become “true” objects from **3D-vector calculus** rather than **time-dependent** objects from 3D-vector calculus, as was the case before.

The procedure is known as **dimensional reduction** (from 4D to 3D) or (sometimes) as **Hadamard method of descent**.

Dimensional reduction of electrodynamics w.r.t. ∂_t (2)

Then in the “full” set of Maxwell equations

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho & -\partial_t \mathbf{E} + \operatorname{curl} \mathbf{B} &= \mathbf{j} \\ \operatorname{div} \mathbf{B} &= 0 & -\partial_t \mathbf{B} - \operatorname{curl} \mathbf{E} &= 0 \end{aligned}$$

the highlighted terms **vanish** and the following equations survive

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho & \operatorname{curl} \mathbf{B} &= \mathbf{j} \\ \operatorname{curl} \mathbf{E} &= 0 & \operatorname{div} \mathbf{B} &= 0 \end{aligned}$$

Notice that variables $\mathbf{E}$ and $\mathbf{B}$ **no longer interact**.

We get two **separate sectors**: $(\mathbf{E}, \rho)$ and $(\mathbf{B}, \mathbf{j})$.

What we get physically is (historically) **a step back** when separate disciplines - **electrostatics** and **magnetostatics** - existed.

Dimensional reduction of electrodynamics w.r.t. ∂_z

Now, consider the situation when **all quantities** happen to be **invariant** w.r.t. **z** direction

$$\mathcal{L}_{\partial_z}(\dots) = 0$$

This means that **z** direction becomes in a sense **superfluous**, the quantities become **time-dependent** objects from **2D-vector calculus**, namely living in the **(x, y) -plane**.

Dimensional reduction of electrodynamics w.r.t. ∂_z (2)

Then in the **full** set of (2D-version!) of Maxwell equations

$$-\nabla_{\pi/2} B_z + \partial_z \mathbf{B}_{\pi/2} - \partial_t \mathbf{E} = \mathbf{j}$$

$$-\operatorname{div} \mathbf{B}_{\pi/2} - \partial_t E_z = j_3$$

$$\operatorname{div} \mathbf{E} + \partial_z E_z = \rho$$

$$\nabla_{\pi/2} E_z - \partial_z \mathbf{E}_{\pi/2} - \partial_t \mathbf{B} = 0$$

$$\operatorname{div} \mathbf{E}_{\pi/2} - \partial_t B_z = 0$$

$$\operatorname{div} \mathbf{B} + \partial_z B_z = 0$$

the highlighted terms **vanish**.

Dimensional reduction of electrodynamics w.r.t. ∂_z (3)

The following equations **survive**:

$$\begin{aligned}
 \operatorname{div} \mathbf{E} &= \rho \\
 -\nabla_{\pi/2} B_z - \partial_t \mathbf{E} &= \mathbf{j} \\
 \operatorname{div} \mathbf{E}_{\pi/2} - \partial_t B_z &= 0 \\
 \\
 -\operatorname{div} \mathbf{B}_{\pi/2} - \partial_t E_z &= j_3 \\
 \nabla_{\pi/2} E_z - \partial_t \mathbf{B} &= 0 \\
 \operatorname{div} \mathbf{B} &= 0
 \end{aligned}$$

Again degrees of freedom separate to two **distinct sectors**:

$(E_x, E_y, B_z \leftrightarrow \rho, j_x, j_y)$ and $(B_x, B_y, E_z \leftrightarrow j_3)$.

Dimensional reduction of electrodynamics w.r.t. ∂_z (4)

The **first** half

$$\begin{aligned}\operatorname{div} \mathbf{E} &= \rho \\ -\nabla_{\pi/2} B_z - \partial_t \mathbf{E} &= \mathbf{j} \\ \operatorname{div} \mathbf{E}_{\pi/2} - \partial_t B_z &= 0\end{aligned}$$

turns out to describe **standard 1+2 electrodynamics**
(see Lapidus 1982, Moreno, Rivas 1984, Boito et.al 2020, ...)

Dimensional reduction of electrodynamics w.r.t. ∂_z (5)

The **second** half

$$\begin{aligned} -\operatorname{div} \mathbf{B}_{\pi/2} - \partial_t E_z &= j_3 \\ \nabla_{\pi/2} E_z - \partial_t \mathbf{B} &= 0 \\ \operatorname{div} \mathbf{B} &= 0 \end{aligned}$$

describes **another possible 1+2 electrodynamics**.

(See Maggi, Ercolessi, Facchi, Marmo, Pascazio, Pepe 2022).

Decomposition w.r.t. observer field

Decomposition of forms on Minkowski space-time used above

$$\alpha = dt \wedge \hat{s} + \hat{r}$$

is just the simplest version of a much more general decomposition

$$\alpha = \tilde{V} \wedge \hat{s} + \hat{r}$$

w.r.t. a general observer field V on a Lorentzian manifold (M, g) .

Here

$$\tilde{V} := g(V, \cdot) \quad g(V, V) \equiv ||V||^2 = 1$$

The first formula comes from Minkowski space-time $(\mathbb{R}^4, \eta)$ with

$$V = \partial_t \quad \tilde{V} = \eta(\partial_t, \cdot) = dt$$

Decomposition w.r.t. **observer field** (2)

The decomposition is based on **elementary identity**

$$i_V j_V + j_V i_V = g(V, V) \hat{1} = \hat{1} \quad j_V \alpha := \tilde{V} \wedge \alpha$$

So

$$\alpha = j_V i_V \alpha + i_V j_V \alpha \equiv \tilde{V} \wedge \hat{s} + \hat{r}$$

The field V encodes important **kinematical characteristics** of the **reference frame**. In general (see e.g. Fecko 1997)

$$d\tilde{V} = \tilde{V} \wedge \hat{a} + \hat{y}$$

where $\hat{a}$ is **acceleration** 1-form and $\hat{y}$ is **vorticity** 2-form ($\hat{y} \neq 0$ means **time synchronization** is **not possible**).

Operations d , $*$ and δ on forms on $E^{1,3}$

Recall that for $V = \partial_t$ we had

$$\begin{aligned} d(dt \wedge \hat{s} + \hat{r}) &= dt \wedge (\mathcal{L}_{\partial_t} \hat{r} - \hat{d}\hat{s}) + \hat{d}\hat{r} \\ \delta(dt \wedge \hat{s} + \hat{r}) &= dt \wedge \hat{\delta}\hat{s} + (-\mathcal{L}_{\partial_t} \hat{s} - \hat{\delta}\hat{r}) \end{aligned}$$

Now (for general V) life becomes more complex and we get

$$\begin{aligned} d(\tilde{V} \wedge \hat{s} + \hat{r}) &= \tilde{V} \wedge (\mathcal{L}_V \hat{r} - \hat{d}\hat{s} + \hat{a} \wedge \hat{s}) + \hat{d}\hat{r} + \hat{y} \wedge \hat{s} \\ \delta(\tilde{V} \wedge \hat{s} + \hat{r}) &= \tilde{V} \wedge (\hat{\delta}\hat{s} + \hat{*}(\hat{y} \wedge \hat{*}\hat{r})) + (-\hat{*}\mathcal{L}_V \hat{*}\hat{s} - \hat{\delta}\hat{r} + \hat{*}(\hat{a} \wedge \hat{*}\hat{\eta}\hat{r})) \end{aligned}$$

Example: Rotating reference frame in Minkowski space

If (t, r, φ, z) is **cylindrical** coordinate system in Minkowski space-time, then

$$\tau \mapsto \gamma(\tau) = (t_0 + \tau, r_0, \varphi_0 + \omega\tau, z_0)$$

is a world-line describing **uniform rotation** around z-axis.

All such world-lines happen to be **integral curves** of the vector field

$$\begin{aligned} v &= \partial_t + \omega \partial_\varphi && \text{(in cylindrical coordinates)} \\ &= \partial_t + \omega(x\partial_y - y\partial_x) && \text{(in Cartesian coordinates)} \end{aligned}$$

Its norm is not unity,

$$||v||^2 = g(v, v) = 1 - (\omega r)^2$$

Example: Rotating reference frame in Minkowski space (2)

So, the vector field V which we look for is just the **normalized** vector

$$V := \frac{v}{||v||} = \gamma(\partial_t + \omega\partial_\varphi) \quad \gamma(r) := \frac{1}{\sqrt{1 - (\omega r)^2}}$$

Thus **normalization** just adds overall, position dependent, **relativistic factor** γ .

Here we get

$$\hat{a} = \gamma^2(\omega^2 r)dr \quad \hat{y} = -2\gamma^3\omega r dr \wedge (d\varphi - \omega dt)$$

So, in rotating reference frame we feel **radial acceleration** (clear :-)
and we **cannot** perform time synchronization (known).

Example: Rotating reference frame in Minkowski space (3)

For **electromagnetic** field 2-form F we have

$$F = j_V i_V F + i_V j_V F \equiv \tilde{V} \wedge \hat{E} - \hat{B}$$

Here $\hat{E}$ and $\hat{B}$ describe **electric** and **magnetic** fields, respectively, **as detected in rotating** reference frame.

One can, as the “proof of the concept” (or just for fun :-), compute **in this way** the **induced voltage** for a wire rim uniformly rotating in a static magnetic field **from the perspective** of the **observer rotating with the rim** (see Fecko 2013).

Maxwell equations w.r.t. general V

The 4D Maxwell equations

$$\delta F = -j \quad dF = 0$$

translate, for general decomposition w.r.t. V , to 3D system

$$\begin{aligned} \hat{d}\hat{*}\hat{E} + \hat{y} \wedge \hat{*}\hat{B} &= \rho\hat{\omega} \\ \hat{d}\hat{*}\hat{B} - \mathcal{L}_V\hat{*}\hat{E} - \hat{a} \wedge \hat{*}\hat{B} &= \hat{*}\hat{j} \end{aligned}$$

$$\begin{aligned} \hat{d}\hat{E} + \mathcal{L}_V\hat{B} - \hat{a} \wedge \hat{E} &= 0 \\ \hat{d}\hat{B} - \hat{a} \wedge \hat{E} &= 0 \end{aligned}$$

Notice the “non-inertial” terms (containing $\hat{a}$ and $\hat{y}$), which are usually **not** present (for $\tilde{V} = dt$ we have $d\tilde{V} = 0$).

Decomposition w.r.t. two fields V and W

More detailed decomposition of forms on Minkowski space-time used above

$$\alpha = dt \wedge dz \wedge \check{A} + dt \wedge \check{B} + dz \wedge \check{C} + \check{D}$$

is again just the simplest version of a general decomposition

$$\alpha = \tilde{V} \wedge \tilde{W} \wedge \check{A} + \tilde{V} \wedge \check{B} + \tilde{W} \wedge \check{C} + \check{D}$$

w.r.t. a pair of fields V, W on a Lorentzian manifold (M, g) .

Namely we get the first formula for

$$V = \partial_t \quad W = -\partial_z \quad \tilde{V} = \eta(\partial_t, \cdot) = dt \quad \tilde{W} = \eta(-\partial_z, \cdot) = dz$$

Computations (in particular exterior derivative)
become fairly complex :-)

For Further Reading (1)



M. Fecko.

Differential geometry and Lie groups for physicists.

Cambridge University Press 2006 (2011)

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On 3+1 decompositions w.r.t an observer field via diff. forms.

Journal of Mathematical Physics. 38, 4542-4560 (1997)

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Acta Physica Slovaca 63, No.5, 261-359 (2013)

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Vector calculus in two-dimensional space.

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The End :-)

Thank You for Your attention!