On symmetries and conserved quantities in Nambu mechanics

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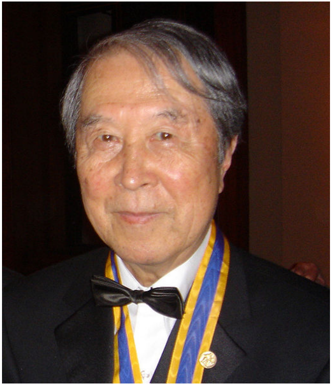
- Where it differs geometrically from Hamiltonian one
- Why action integral differs so much
- Recall: symmetries in Hamiltonian mechanics
- Recall: corresponding conserved quantities
- What the same algorithm gives in Nambu mechanics

Introduction

Yoichiro Nambu - a few facts

1921: Born in Tokio (Japan).
1950: Professor at Osaka City University
. (he was 29 then).
1958: Professor at University of
. Chicago.
1970: USA citizen.
2008: Nobel prize for Physics.
2015: Died (aged 94).



His paper on Nambu mechanics (520 citations)

PHYSICAL REVIEW D VOLUME 7, NUMBER 8 15 APRIL 1973

Generalized Hamiltonian Dynamics*

Yoichiro Nambu
The Enrico Fermi Institute and the Department of Physics, The University of Chicago, Chicago, Illinois 60637
(Received 26 December 1972)

Taking the Liouville theorem as a guiding principle, we propose a possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space. The equation of motion involves two Hamiltonians and three canonical variables. The fact that the Euler equations for a rotator can be cast into this form suggests the potential usefulness of the formalism. In this article we study its general properties and the problem of quantization.

I. INTRODUCTION

A notable feature of the Hamiltonian description of classical dynamics is the Liouville theorem, which states that the volume of phase space occupied by an ensemble of systems is conserved. The theorem plays, among other things, a fundamental role in statistical mechanics. On the other hand, Hamiltonian dynamics is not the only formalism that makes a statistical mechanics possible. Any

$[F,H,G]$. Obviously a PB is antisymmetric under interchange of any pair of its components. As a result we have $\dot{H}=\dot{F}=0$, i.e., both H and G are constants of motion. The orbit of a system in phase space is thus determined as the intersection of two surfaces, $H=$const. and $G=$const.

Equation (1) or (1′) also shows that the velocity field $d\vec{r}/dt$ is divergenceless,

$$\vec{\nabla}\cdot(\vec{\nabla}H\times\vec{\nabla}G)\equiv 0, \qquad (3)$$

How exactly Hamilton mechanics is "generalized" (1)

Hamilton equations for a single canonical pair (q, p) read

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Denote $(q, p) = (x_1, x_2)$; then

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}$$

or

$$\dot{x}_i = \epsilon_{ij} \frac{\partial H}{\partial x_j}$$

How exactly Hamilton mechanics is "generalized" (2)

Y.Nambu felt that two is not enough.
He introduced canonical triplet (x_1, x_2, x_3) and postulated equations

$$\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}$$

The same system in vector notation

$$\dot{\mathbf{r}} = \nabla H_1 \times \nabla H_2$$

How exactly Hamilton mechanics is "generalized" (3)

He also generalized the idea in several ways;
e.g. to canonical n-tuple $(x_1, \dots, x_n)$ and equations

$$\dot{x}_i = \epsilon_{ij\dots k} \frac{\partial H_1}{\partial x_j} \dots \frac{\partial H_{n-1}}{\partial x_k}$$

where $\epsilon_{ij\dots k}$ is (n-dimensional) Levi-Civita symbol.

How exactly Hamilton mechanics is "generalized" (4)

The dynamics may also be written in terms of Nambu bracket:

$$\dot{f} = \{H_1, \dots, H_{n-1}, f\} \qquad (n \text{ entries})$$

For $n = 2$ we get back good old Poisson bracket

$$\dot{f} = \{H, f\} \qquad \{f, g\} \equiv \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$$

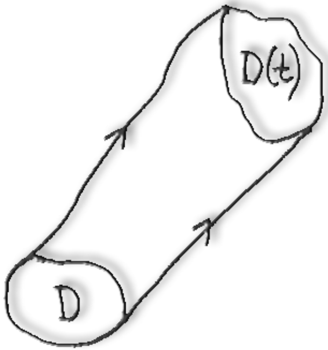
Liouville theorem - still true

One easily shows that
Liouville theorem still holds:
if phase volume is introduced

$$\text{volume of } D \equiv \int_D dx_1 \dots dx_n$$

then it is conserved by time development

$$\text{volume of } D = \text{volume of } D(t)$$



Recall ...

Generalized Hamiltonian Dynamics*

Yoichiro Nambu

The Enrico Fermi Institute and the Department of Physics, The University of Chi

(Received 26 December 1972)

Taking the Liouville theorem as a guiding principle, we propose a possible g

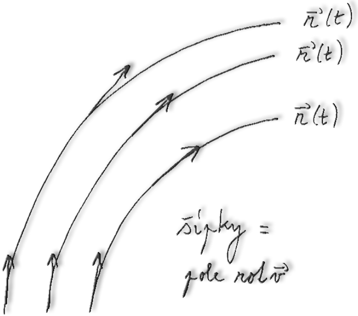
Vortex lines equations in hydrodynamics

In hydrodynamics:

$\mathbf{v}$	velocity field
$\mathrm{curl}\,\mathbf{v}$	vorticity field

Lines $\mathbf{r}(t)$, which are at each point tangent to vorticity vector, i.e. for which $(\mathrm{curl}\,\mathbf{v}) \parallel \dot{\mathbf{r}}$ holds, are vortex lines. So they satisfy differential equations

$$\dot{\mathbf{r}} \times \mathrm{curl}\,\mathbf{v} = \mathbf{0}$$



The same in the language of differential forms (1)

Velocity field may be encoded into 1-form

$$\theta = \mathbf{v} \cdot d\mathbf{r}$$

Its exterior derivative is 2-form

$$d\theta = (\mathrm{curl}\, \mathbf{v}) \cdot d\mathbf{S}$$

Interior product with the vector $\dot{\mathbf{r}}$ gives 1-form

$$i_{\dot{\mathbf{r}}} d\theta = (\mathrm{curl}\, \mathbf{v} \times \dot{\mathbf{r}}) \cdot d\mathbf{r}$$

The same in the language of differential forms (2)

This means that differential equations for finding vortex lines $\mathbf{r}(t)$

$$\dot{\mathbf{r}} \times \operatorname{curl} \mathbf{v} = \mathbf{0}$$

may also be written in the form

$$i_{\dot{\mathbf{r}}} d\theta = 0$$

Hamilton equations and "vortex lines" (1)

In extended phase space (coordinates q^a, p_a, t) introduce 1-form

$$\sigma = p_a dq^a - H dt$$

Its exterior derivative is 2-form

$$d\sigma = dp_a \wedge dq^a - dH \wedge dt$$

If $\gamma(t)$ is a curve and $\dot{\gamma}$ its tangent vector

$$\dot{\gamma} = \dot{q}^a \frac{\partial}{\partial q^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \frac{\partial}{\partial t}$$

Hamilton equations and "vortex lines" (2)

then its interior product with $d\sigma$ gives 1-form

$$\begin{aligned} i_{\dot\gamma} d\sigma &= \left(\dot p_a + \frac{\partial H}{\partial q^a}\right) dq^a + \left(-\dot q^a + \frac{\partial H}{\partial p_a}\right) dp_a \\ &- \left(\dot q^a \frac{\partial H}{\partial q^a} + \dot p_a \frac{\partial H}{\partial p_a}\right) dt \end{aligned}$$

If the first two brackets vanish, the third one vanishes, too. But making the first two brackets vanish is writing Hamilton equations!

Hamilton equations and "vortex lines" (3)

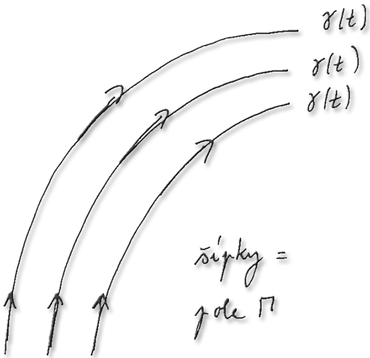
This means that Hamilton equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a}$$

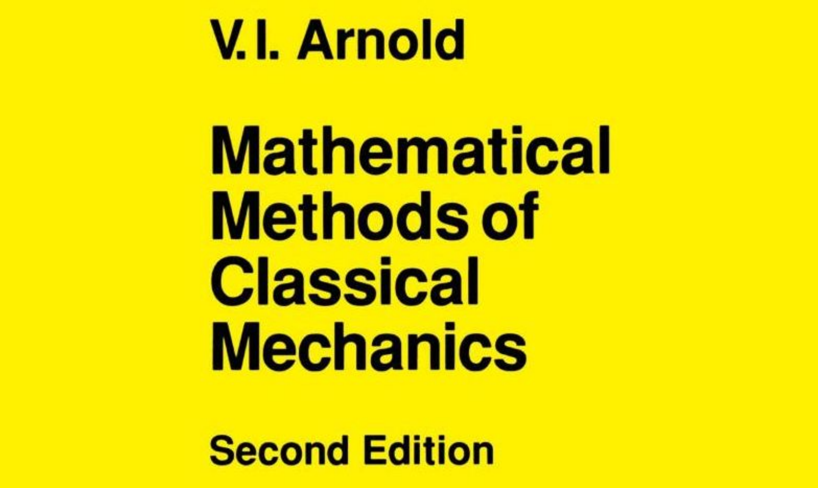
may also be written in the form

$$i_{\dot{\gamma}} d\sigma = 0$$

i.e. they are formally vortex lines equations.
Solutions of Hamilton equations are vortex lines.
(In appropriate space.)



One can read this stuff here (e.g.)



A small piece from inside

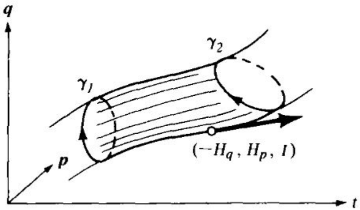


Figure 182 Hamiltonian field and vortex lines of the form $\mathbf{p}\, d\mathbf{q} - H\, dt$.

Theorem. *The vortex lines of the form* $\omega^1 = \mathbf{p}\, d\mathbf{q} - H dt$ *on the* $2n + 1$-*dimensional extended phase space* $\mathbf{p}, \mathbf{q}, t$ *have a one-to-one projection onto the* t *axis, i.e., they are given by functions* $\mathbf{p} = \mathbf{p}(t), \mathbf{q} = \mathbf{q}(t)$. *These functions satisfy the system of canonical differential equations with hamiltonian function* H:

(1)
$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}.$$

A small piece from inside of the original

Т е о р е м а. *Линии ротора формы $\omega^1 = p\,dq - H\,dt$ в $2n+1$-мерном расширенном фазовом пространстве p, q, t однозначно проектируются на ось t, т. е. задаются функциями $p = p(t)$, $q = q(t)$. Эти функции удовлетворяют системе канонических дифференциальных уравнений с функцией Гамильтона H:*

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}. \tag{1}$$

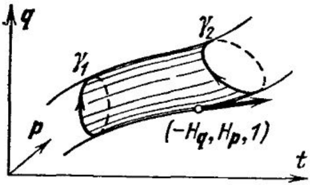


Рис. 182. Гамильтоново поле и линии ротора формы $p\,dq - H\,dt$

Иными словами, *линии ротора формы $p\,dq - H\,dt$ суть траектории фазового потока в расширенном фазовом пространстве, т. е. интегральные кривые канонических уравнений* (1).

Nambu equations and "vortex lines" (1)

Replace the "Hamiltonian" 1-form $\sigma = pdq - Hdt$ with 2-form

$$\sigma = x_3 dx_1 \wedge dx_2 - H_1 dH_2 \wedge dt$$

Its exterior derivative is 3-form

$$d\sigma = dx_1 \wedge dx_2 \wedge dx_3 - dH_1 \wedge dH_2 \wedge dt$$

If $\gamma(t)$ is a curve and $\dot{\gamma}$ its tangent vector

$$\dot{\gamma} = \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{x}_3 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial t}$$

Nambu equations a "vortex lines" (2)

then its interior product with $d\sigma$ gives 2-form

$$\begin{aligned} i_{\dot\gamma} d\sigma &= (\dot{\mathbf{r}} - \nabla H_1 \times \nabla H_2) \cdot d\mathbf{S} \\ &- ((\nabla H_1 \times \nabla H_2) \times \dot{\mathbf{r}}) \cdot d\mathbf{r} \wedge dt \end{aligned}$$

If the first bracket is zero, the second one vanishes as well.
But making the first bracket zero is writing Nambu equations!

Nambu equations and "vortex lines" (3)

This means that also Nambu equations

$$\dot{\mathbf{r}} = \nabla H_1 \times \nabla H_2$$

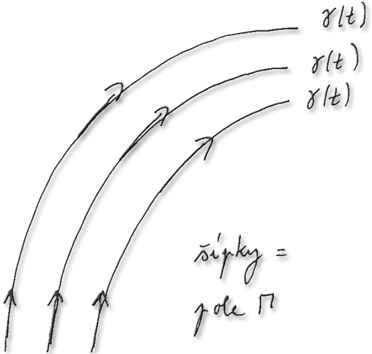
may be written in the form

$$i_{\dot\gamma} d\sigma = 0$$

i.e. again formally as vortex lines equations.

Also solutions of Nambu equations are "vortex lines".

(But σ becomes 2-form now!)



One can read more details here

On a variational principle for the Nambu dynamics

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(Received 17 May 1991; accepted for publication 4 October 1991)

A variational principle for the Nambu dynamics is analyzed. Since the equations of motion single out a distinguished two-form rather then a one-form, the usual construction of the action $S[\gamma]$ as an integral of a one-form along the curve γ on the extended phase space has to be modified.

I. INTRODUCTION

In our previous paper[1] we discussed a geometrical

integral and the solutions of the dynamical equations is absent in the standard (nonsingular) Lagrangian dynamics as well as in the Hamiltonian one.

...

930 J. Math. Phys. 33 (3), March 1992 0022-2488/92/030930-04$03.00

and also here (> 300 citations)

Commun. Math Phys 160, 295–315 (1994)

Communications in Mathematical Physics

On Foundation of the Generalized Nambu Mechanics

Leon Takhtajan
Department of Mathematics, State University of New York at Stony Brook, Stony Brook, NY 11794-3651, USA

Received: 1 April 1993

Abstract: We outline basic principles of a canonical formalism for the Nambu mechanics – a generalization of Hamiltonian mechanics proposed by Yoichiro Nambu in 1973. It is based on the notion of a Nambu bracket, which generalizes the Poisson bracket – a "binary" operation on classical observables on the phase

Action integral for Hamilton equations (1)

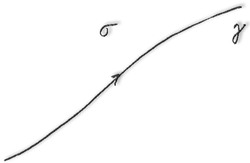
In Hamilton equations 1-form

$$\sigma \equiv p_a dq^a - H dt$$

occurs. Its integral along curve γ

$$S[\gamma] = \int_\gamma \sigma \equiv \int_{t_1}^{t_2} (p_a \dot{q}^a - H) dt$$

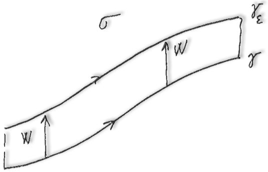
serves as the action for Hamilton equations. So Ham.eq. can be derived via variation of the latter $S \mapsto S + \delta S$ and requirement $\delta S = 0$.



Action integral for Hamilton equations (2)

If the variation is performed with the help of (arbitrary) vector field W, one gets

$$\begin{aligned}\delta S &= \epsilon \int_{t_1}^{t_2} \langle - i_{\dot\gamma} d\sigma, W\rangle dt + \epsilon\langle\sigma, W\rangle_{\gamma(t_1)}^{\gamma(t_2)} \\ &= \epsilon \int_{t_1}^{t_2} \langle - i_{\dot\gamma} d\sigma, W\rangle dt + \epsilon[p_a \delta q^a]_{\gamma(t_1)}^{\gamma(t_2)}\end{aligned}$$



Requiring vanishing of variations of coordinates at the ends, we indeed get solutions of Hamilton equations

$$i_{\dot\gamma} d\sigma = 0$$

as extremals.

Action integral for Nambu equations (1)

Finding action integral for Nambu equations is a delicate matter.
See (already mentioned) papers:

On a variational principle for the Nambu dynamics

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On Foundation of the Generalized Nambu Mechanics

Leon Takhtajan
Department of Mathematics, State University of New York at Stony Brook, Stony Brook, NY 11794-3651, USA

Action integral for Nambu equations (2)

Where is the issue?
Although both Nambu and Hamilton equations formally look

$$i_{\dot{\gamma}} d\sigma = 0$$

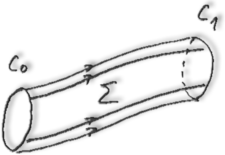
for Nambu equations σ is 2-form.
So it cannot be integrated along a curve, but rather over a surface!
Although we search for exceptional curves,
we are forced to use surfaces in the theory.
(This is written in both papers.)

Action integral for Nambu equations (3)

The idea of Takhtajan (1994) is more interesting:

1. At time t_1 take arbitrary loop c_0.
2. Let it evolve via Nambu equations up to t_2.
3. You get a surface Σ.
4. Integrate the 2-form σ over this surface.
5. Call the resulting number the

action of the family of curves ($=$ surface Σ)



Action integral for Nambu equations (4)

Computation shows that it really works!
Namely the action is stationary for surfaces,
which are composed of solutions of Nambu equations.

(One can perform variation via a vector field W
similarly, as we showed in the Hamiltonian case.)

What is infinitesimal symmetry of Hamiltonian system

It is a small change

$$\gamma \mapsto \gamma_\epsilon$$

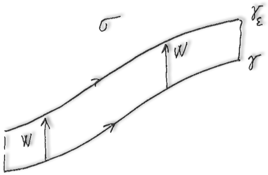
(generated by a flow of a vector field W), which does not change the value of the action

$$S[\gamma_\epsilon] = S[\gamma]$$

i.e. for which

$$\delta S = 0$$

If we find such (very special) field W, the reward is a conserved quantity.



Conserved quantity for infinitesimal symmetry W (1)

For a general field W and a general curve γ one gets the expression

$$\delta S = \epsilon \int_{t_1}^{t_2} \langle -\, i_{\dot\gamma} d\sigma, W \rangle dt + \epsilon \langle \sigma, W \rangle_{\gamma(t_1)}^{\gamma(t_2)}$$

But our W is not general, since it leads to

$$\delta S = 0$$

Consider also special curves, namely solutions of Hamilton equations

$$i_{\dot\gamma} d\sigma = 0$$

Conserved quantity for infinitesimal symmetry W (2)

What remains in this particular situation is:

$$\langle \sigma, W \rangle_{\gamma(t_1)}^{\gamma(t_2)} = 0$$

The expression

$$f := \langle \sigma, W \rangle \equiv i_W \sigma$$

is a 0-form, i.e. a function. So we get the statement that

$$f(\gamma(t_2)) = f(\gamma(t_1))$$

This is the promised conserved quantity.

Example: Conservation of energy

Take W to be ∂_t (i.e. we examine time translation).
One easily shows that it is a symmetry
iff H does not depend on time explicitly.
Then the following expression is conserved

$$f := \langle \sigma, W \rangle \equiv \langle p_a dq^a - H dt, \partial_t \rangle = -H$$

So the function H is conserved.

What is infinitesimal symmetry of Nambu system

It is a small change

$$\Sigma \mapsto \Sigma_\epsilon$$

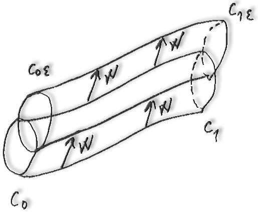
(generated by the flow of a vector field W), which does not change the value of the action

$$S[\Sigma_\epsilon] = S[\Sigma]$$

i.e. for which

$$\delta S = 0$$

If we find such (very special) field W, the reward is also here a conserved quantity.



Conserved quantity for infinitesimal symmetry W (1)

In Nambu case, for a general field W and a general surface Σ,

$$\delta S = \epsilon \int_\Sigma i_W d\sigma + \epsilon \left(\oint_{c_2} - \oint_{c_1} \right) i_W \sigma$$

But our W is not general, since it leads to

$$\delta S = 0$$

Consider also special surfaces, namely those composed of solutions

$$i_{\dot\gamma} d\sigma = 0$$

Then (one easily shows that) the first integral vanishes.

Conserved quantity for infinitesimal symmetry W (2)

What remains in this particular situation is:

$$\oint_{c_1} i_W \sigma = \oint_{c_2} i_W \sigma$$

The expression (function)

$$f(t) := \oint_{c_t} i_W \sigma$$

is the promised conserved quantity:

$$f(t_1) = f(t_2)$$

Where is the essential difference?

In Hamiltonian case, directly $i_W\sigma$ is conserved:

$$f := i_W\sigma$$

In Nambu case, only the integral of $i_W\sigma$

$$f := \oint_c i_W\sigma$$

over (an arbitrary) loop c is conserved.
(Expression $i_W\sigma$ is a 1-form, now, and we get a number only upon integration.)

Another formulation of the result

In Hamiltonian case, a function is the reward for a symmetry. (Energy, component of momentum, component of angular momentum etc.)

In Nambu case, a relative integral invariant is the reward, i.e. only the integral of a 1-form over a loop is conserved.

One can read more details here

On symmetries and conserved quantities in Nambu mechanics

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(Received 8 August 2013; accepted 23 September 2013; published online 15 October 2013)

In Hamiltonian mechanics, a (continuous) symmetry leads to conserved quantity, which is a *function* on (extended) phase space. In Nambu mechanics, a straightforward consequence of symmetry is just a *relative integral invariant*, a differential form which only upon integration over a cycle provides a conserved real number. The

...

102901-2 M. Fecko J. Math. Phys. **54**, 102901 (2013)

Standard integral invariants in Hamiltonian mechanics (1)

There are also some well-known integral invariants
in Hamiltonian mechanics,
but they are not related to symmetries
(they hold for any Hamiltonian).

Standard integral invariants in Hamiltonian mechanics (2)

First, Poincaré discovered, that the integral

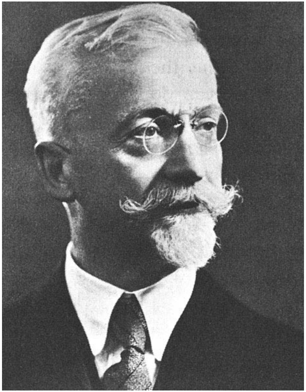
$$\oint_c p_a dq^a$$

is invariant (c is a loop in a fixed-time hyper-plane).
Later, Cartan generalized it to loops not necessarily lying in a fixed-time hyper-plane.
But then a more general 1-form is to be integrated, namely

$$\oint_c (p_a dq^a - H dt)$$

Henri Poincaré and Élie Cartan

Henri Poincaré (1854 – 1912)



Élie Cartan (1869 – 1951)

Poincaré integral invariant

A nice drawing from V.I.Arnold

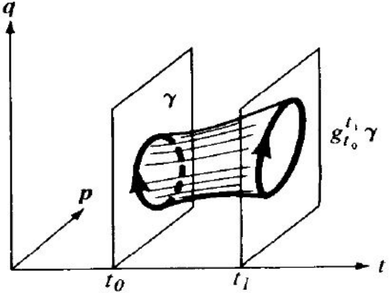


Figure 183 Poincaré's integral invariant

Poincaré-Cartan integral invariant (1)

Also this V.I.Arnol'd can draw nicer than me :-(

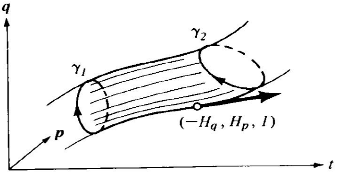


Figure 182 Hamiltonian field and vortex lines of the form $\mathbf{p}\, d\mathbf{q} - H\, dt$.

Poincaré-Cartan integral invariant (2)

A famous book by Cartan (from 1922 :-)

The End

Thanks for Your attention!