Surfaces which behave like vortex lines

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- How they occur in hydrodynamics of ideal fluid
- What are vortex lines
- How they move according to Helmholtz
- How we can see it using integral invariants
- What a generalization of vortex lines (to surfaces) looks like

Introduction

Integral invariants - original sources (1)

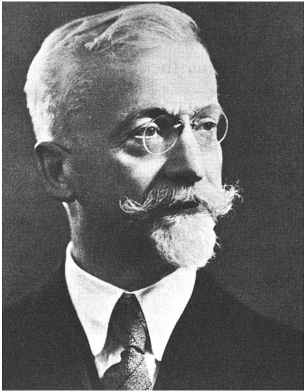
[1] H. Poincaré, "Sur le problème des trois corpses et les équations de la dynamique" *Acta Math.* , **13** (1890) pp. 1–270

[2] H. Poincaré, "Les méthodes nouvelles de la mécanique céleste" , **3**, Invatiants intégraux, Gauthier-Villars et fils (1899), Chapt. 26

[3] E. Cartan, "Leçons sur les invariants intégraux" , Hermann (1922)

Henri Poincaré and Élie Cartan

Henri Poincaré (1854 – 1912)



Élie Cartan (1869 – 1951)

Integral invariants - original sources (2)

Cartan's monograph (from 1922 :-)


E. CARTAN
PROFESSEUR A LA FACULTÉ DES SCIENCES DE PARIS

LEÇONS
SUR LES
INVARIANTS INTÉGRAUX

Integral invariant à la Poincaré (1)

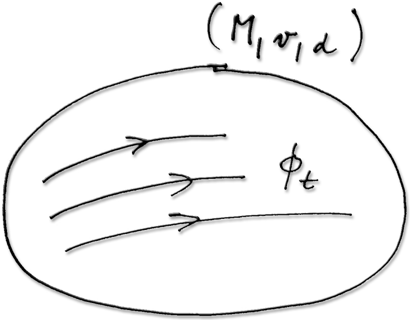
Consider a triple (M, v, α), i.e.
1. a manifold M with the flow of a vector field v

$(M, \Phi_t \leftrightarrow v)$ *phase space*

i.e. also the dynamics

$$\dot{\gamma} = v \qquad \dot{x}^i = v^i(x)$$

2. A p-form α on M.



Integral invariant à la Poincaré (2)

On (M, v, α), if

$$\boxed{\int_{\Phi_t(c)} \alpha = \int_c \alpha}$$

holds for each p-chain c, the integral

$$\int_c \alpha$$

is known as *absolute integral invariant*

If it only holds for each p-cycle c (i.e. $\partial c = 0$),
we speak of relative integral invariant.

Absolute invariant - infinitesimal condition

Since, for $t = \epsilon$, we have

$$\int_{\Phi_\epsilon(c)} \alpha = \int_c \alpha + \epsilon \int_c \mathcal{L}_v \alpha$$

($\mathcal{L}_v$ is Lie derivative) and since c is arbitrary, we get

$$(v, \alpha) \text{ gives absolute invariant} \quad \Leftrightarrow \quad \boxed{\mathcal{L}_v \alpha = 0}$$

i.e.

α is to be Lie-invariant w.r.t. v

Relative invariant - infinitesimal condition (1)

For relative invariants (i.e. when $\partial c = 0$)
Lie invariance is overly strong requirement.
What *is* needed is Lie invariance modulo exact form:

$$(v, \alpha) \text{ relative invariant} \quad \Leftrightarrow \quad \boxed{\mathcal{L}_v \alpha = d\tilde{\beta}}$$

▼ This is so due to de Rham theorem:

$$\int_c (\dots) = 0 \quad \text{for each cycle} \quad c \quad \Leftrightarrow \quad (\dots) \text{ is exact}$$

▲

Relative invariant - infinitesimal condition (2)

Now Cartan's formula

$$\mathcal{L}_v = i_v d + d i_v$$

enables one to rewrite it as

$$\mathcal{L}_v \alpha = (i_v d + d i_v)\alpha = d\tilde{\beta}$$

so that

$$i_v d\alpha = d(\tilde{\beta} - i_v \alpha) \equiv d\beta$$

Relative invariant - infinitesimal condition (3)

Therefore, an alternative (and often useful) infinitesimal condition for relative invariants reads:

$$(v, \alpha) \text{ relative invariant} \quad \Leftrightarrow \quad \boxed{i_v d\alpha = d\beta} \quad \text{(for some } \beta\text{)}$$

Well known example - Hamiltonian mechanics

On exact symplectic manifold $(M, \omega \equiv d\theta)$
Hamiltonian field is defined by

$$\boxed{i_{\zeta_H} d\theta = -dH}$$

So, it is

$$i_v d\alpha = d\beta \qquad \text{for} \qquad v \equiv \zeta_H \quad \alpha = \theta \quad \beta = -H$$

Therefore, we get relative integral invariant

$$\boxed{\int_c \theta \equiv \oint_c p_a dq^a}$$

Another example - hydrodynamics

Consider the following realization of (M, v, α, β):
- $M = E^3 = (\mathbb{R}^3, \text{"standard" } g)$
- v = velocity field of ideal fluid
- α = velocity 1-form $\tilde{v}$, i.e. $\tilde{v} \equiv g(v, \ \cdot\ ) \equiv \mathbf{v} \cdot d\mathbf{r}$
- $\beta = v^2/2 + P + \Phi \equiv \mathcal{B} \equiv$ Bernoulli function $\quad (dP = dp/\rho)$

Then

$$i_v d\alpha = d\beta \qquad \Leftrightarrow \qquad \boxed{(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P - \nabla \Phi}$$ Euler equation

Therefore, we get relative integral invariant

$$\boxed{\int_c \tilde{v} \equiv \oint_c \mathbf{v} \cdot d\mathbf{r}}$$ Kelvin's theorem

Cartan's generalization of relative invariants - step 1 (1)

Consider a relative integral invariant given by (v, α) on M.

$\boxed{\text{Cartan's 1-st idea:}}$

1. Replace M by $M \times \mathbb{R}[t]$ (extended phase space, time axis added)
2. On $M \times \mathbb{R}$, construct vector field

$$\xi := \partial_t + v$$

3. On $M \times \mathbb{R}$, construct p-form

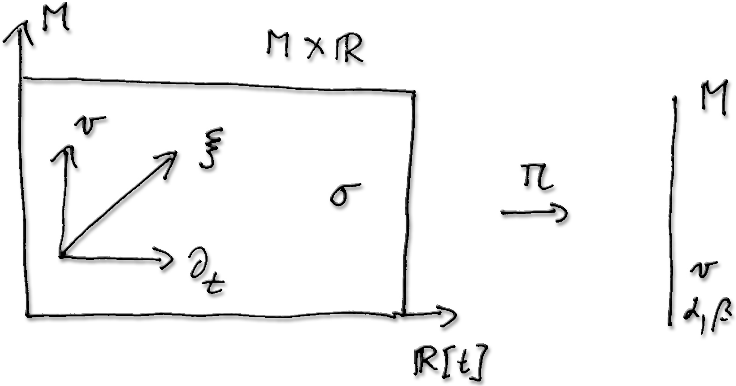
$$\sigma := \hat{\alpha} + dt \wedge \hat{\beta}$$

where

$$\hat{\alpha} \equiv \pi^* \alpha \qquad \hat{\beta} \equiv \pi^* \beta \qquad \pi : M \times \mathbb{R} \to M$$

Cartan's generalization of relative invariants - step 1 (2b)

So, v is encoded into ξ and (α, β) into σ.



Cartan's generalization of relative invariants - step 1 (2)

Cartan's observation:

$$i_v d\alpha = d\beta \qquad \Leftrightarrow \qquad \boxed{i_\xi d\sigma = 0}$$

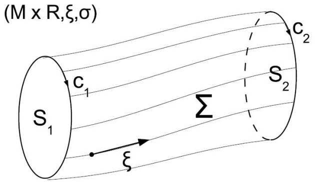
So, yet, one can express
the result of Poincaré (left, on M) in
the language of Cartan (right, on $M \times \mathbb{R}$).

Cartan's generalization of relative invariants - step 1 (3)

One can also easily prove directly:

$$i_\xi d\sigma = 0 \quad \Rightarrow \quad \int_{c_1} \sigma = \int_{c_2} \sigma$$

if c_1 and c_2 encircle the same tube of solutions.



This already generalizes Poincaré result, since here the two cycles c_1 and c_2 do not necessarily lie in fixed-time hyperplane (as is the case for Poincaré as well as at the picture on the right side).

Cartan's generalization of relative invariants - step 2 (1)

$\boxed{\text{Cartan's 2-nd idea:}}$ regard

$$\sigma = \hat{\alpha} + dt \wedge \hat{\beta}$$

as the standard (unique) decomposition
of a general p-form σ on $M \times \mathbb{R}[t]$.
Here, explicitly,

$$\hat{\beta} := i_{\partial_t}\sigma \qquad\qquad \hat{\alpha} := \sigma - dt \wedge i_{\partial_t}\sigma$$

and $\hat{\alpha}$ and $\hat{\beta}$ are just general spatial forms,

$$i_{\partial_t}\hat{\alpha} = 0 \qquad\qquad i_{\partial_t}\hat{\beta} = 0$$

Cartan's generalization of relative invariants - step 2 (2)

So, it is not necessarily true, now, that

$$\hat{\alpha} \equiv \pi^* \alpha \qquad \hat{\beta} \equiv \pi^* \beta$$

What is important, their time (Lie-) derivatives

$$\mathcal{L}_{\partial_t} \hat{\alpha} \qquad\qquad \mathcal{L}_{\partial_t} \hat{\beta}$$

do not necessarily vanish (contrary to $\pi^* \alpha$ and $\pi^* \beta$).
Forms $\hat{\alpha}$ and $\hat{\beta}$ *correspond* (in the language of Poincaré) to time-dependent forms α and β on *M*
(where time is, however, just a parameter).

Cartan's generalization of relative invariants - step 2 (3)

In this new situation, one easily checks that

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta}$$

So, there is a new term (the time derivative), in general,
in Cartan's infinitesimal condition $i_\xi d\sigma = 0$
for existence of relative integral invariant,
when expressed in Poincaré language on M:

$$\boxed{\mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta}} \qquad \Leftrightarrow \qquad \int_{c_1} \hat{\alpha} = \int_{c_2} \hat{\alpha}$$

Example - back to hydrodynamics

Consider again the realization $(M, v, \alpha, \beta) = (\mathbb{R}^3, v, \tilde{v}, \mathcal{B})$:
Then

$$\mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta} \qquad \Leftrightarrow \qquad \boxed{\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi}$$

So, we get the *complete*, time-dependent Euler equation, now.
And we see that we again get as relative integral invariant expression

$$\boxed{\int_c \tilde{v} \equiv \oint_c \mathbf{v}\cdot d\mathbf{r}} \qquad \text{Kelvin's theorem} \quad \text{(still true!)}$$

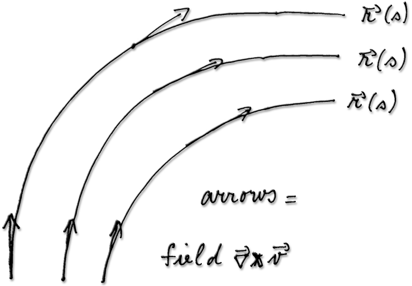
Vortex lines equations in hydrodynamics

In hydrodynamics:

$\mathbf{v}$	velocity field
$\mathrm{curl}\,\mathbf{v}$	vorticity field

Lines $\mathbf{r}(s)$, which are, at each point, tangent to vorticity vector, i.e. for which $(\mathrm{curl}\,\mathbf{v}) \parallel \mathbf{r}'$ holds, are vortex lines. So they satisfy differential equations

$$\mathbf{r}' \times \mathrm{curl}\,\mathbf{v} = \mathbf{0}$$



The same in the language of differential forms (1)

Velocity field may be encoded into 1-form

$$\tilde{v} = \mathbf{v} \cdot d\mathbf{r}$$

Its exterior derivative is 2-form

$$d\tilde{v} \equiv d(\mathbf{v} \cdot d\mathbf{r}) = (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}$$

Interior product with the tangent vector $\mathbf{r}' \equiv d\mathbf{r}/ds$ gives again 1-form

$$i_{\mathbf{r}'} d\tilde{v} = (\operatorname{curl} \mathbf{v} \times \mathbf{r}') \cdot d\mathbf{r}$$

The same in the language of differential forms (2)

This means that differential equations for finding vortex lines $\mathbf{r}(s)$

$$\mathbf{r}' \times \operatorname{curl} \mathbf{v} = \mathbf{0}$$

may also be written in the form

$$\boxed{i_{\mathbf{r}'} d\tilde{v} = 0}$$

Notice, that parametrization $\mathbf{r}(s)$ is completely irrelevant (reparametrization leads to a change which drops out).

The same in the language of differential forms (3)

For time-dependent flow (velocity field) we have
(on extended "phase space" $M \times \mathbb{R} \equiv \mathbb{R}^3 \times \mathbb{R}$)

$$\hat{v} = \mathbf{v}(\mathbf{r}, t) \cdot d\mathbf{r}$$

Its spatial exterior derivative is (spatial) 2-form

$$\hat{d}\hat{v} \equiv \hat{d}(\mathbf{v} \cdot d\mathbf{r}) = (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S} \equiv \boldsymbol{\omega}(\mathbf{r}, t) \cdot d\mathbf{S}$$

Interior product with (spatial) vector $\mathbf{r}'$ gives

$$i_{\mathbf{r}'}\hat{d}\hat{v} = (\operatorname{curl} \mathbf{v} \times \mathbf{r}') \cdot d\mathbf{r}$$

The same in the language of differential forms (4)

So differential equations

$$\mathbf{r}' \times \mathrm{curl}\, \mathbf{v} = \mathbf{0}$$

may now be written as

$$\boxed{i_{\mathbf{r}'} \hat{d} \hat{v} = 0}$$

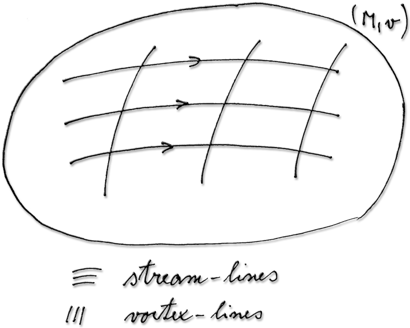
Vortex lines - how they move

Helmholtz theorem (1858): if

1. ideal and barotropic fluid
2. only conservative forces

then vortex lines

- move with the fluid
= are frozen into the fluid
= are material lines



Why it is so - distribution perspective (1)

First - time-independent case (i.e. in Poincaré approach).
On $M \equiv \mathbb{R}^3$ introduce distribution

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \; i_w d\tilde{v} = 0 \; \text{ holds}\}$$

So

$$w \in \mathcal{D} \qquad \Leftrightarrow \qquad w \text{ is tangent to vortex line}$$

Moreover,

vortex lines are exactly

integral submanifolds of $\mathcal{D}$.

Why it is so - distribution perspective (2)

▼ Check of integrability of the distribution $\mathcal{D}$:
Let $w_1, w_2 \in \mathcal{D}$, i.e.

$$i_{w_1} d\tilde{v} = 0 \qquad i_{w_2} d\tilde{v} = 0$$

Then

$$i_{[w_1,w_2]} d\tilde{v} = \mathcal{L}_{w_1} i_{w_2} d\tilde{v} - i_{w_2} \mathcal{L}_{w_1} d\tilde{v} = -i_{w_2}(i_{w_1} d + d i_{w_1}) d\tilde{v} = 0$$

So, also $[w_1, w_2] \in \mathcal{D}$.
This guarantees integrability of $\mathcal{D}$ due to Frobenius theorem. ▲

Why it is so - distribution perspective (3)

Observation: the distribution $\mathcal{D}$ is Lie-invariant w.r.t. v

▼ Indeed, application of d to Euler equation $i_v d\tilde{v} = d\mathcal{B}$ gives

$$\mathcal{L}_v(d\tilde{v}) = 0 \qquad \text{i.e.} \qquad \Phi_t^*(d\tilde{v}) = d\tilde{v}$$

so $d\tilde{v}$ *is* Lie-invariant.
But $d\tilde{v}$ carries full information about $\mathcal{D}$.
So, necessarily, $\mathcal{D}$ is Lie-invariant as well.

$$\Phi_t(\mathcal{D}) = \mathcal{D}$$

▲

Why it is so - distribution perspective (4)

Now

1. $\mathcal{D}$ is Lie-invariant w.r.t. v (i.e. w.r.t. the flow of fluid)
2. $\mathcal{D}$ is integrable (so, there exist integral submanifolds)

So,

3. integral submanifolds are invariant w.r.t. the flow

But

4. integral submanifolds coincide with vortex lines, so
5. vortex lines are invariant w.r.t. the flow (= move with the fluid)

But

6. this is exactly what Helmholtz claims

Time dependent case - distribution perspective (1)

On $M \times \mathbb{R}[t]$ (i.e. in Cartan's approach) introduce distribution

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \ i_w d\sigma = 0 \text{ and } i_w dt = 0 \ \text{ holds}\}$$

So

$$w \in \mathcal{D} \qquad \Leftrightarrow \qquad w \text{ is spatial and annihilates } d\sigma$$

Similarly as before, we easily check that $\mathcal{D}$ is

integrable and Lie-invariant w.r.t. ξ (i.e. the flow of the fluid)

So integral submanifolds of $\mathcal{D}$ move with the fluid.

Time dependent case - distribution perspective (2)

However, it is not evident, here,
that integral submanifolds of $\mathcal{D}$, which move with the fluid,
coincide with vortex lines.

Recall that
- integral submanifolds annihilate the pair $(d\sigma, dt)$,
- vortex lines annihilate the pair $(\hat{d}\hat{v}, dt)$.

Actually, the two objects do coincide.
We show it (next slide) in the general context of integral invariants.
(That is, even beyond the hydrodynamics context.)

Relative integral invariants - distinguished surfaces (1)

Consider the data needed for
general relative integral invariant in Cartan's approach,
i.e. extended phase space $M \times \mathbb{R}[t]$ endowed with
a vector field ξ and a p-form σ

$$\xi := \partial_t + v \qquad \sigma := \hat{\alpha} + dt \wedge \hat{\beta}$$

satisfying

$$\boxed{i_\xi d\sigma = 0}$$

Relative integral invariants - distinguished surfaces (2)

Introduce distribution

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \ i_w d\sigma = 0 \text{ and } i_w dt = 0 \ \text{ holds}\}$$

So

$$w \in \mathcal{D} \quad \Leftrightarrow \quad w \text{ is spatial plus annihilates } d\sigma$$

One easily checks (as it was done in hydrodynamics) that $\mathcal{D}$ is

integrable and Lie-invariant w.r.t. ξ

(i.e. invariant w.r.t. the flow Φ_t)

Relative integral invariants - distinguished surfaces (3)

So, whenever (!) we encounter relative integral invariant situation, distinguished surfaces occur,
namely integral submanifolds of the distribution $\mathcal{D}$.
They possess remarkable property that

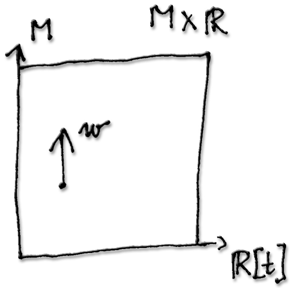
they move with (= are frozen in) the flow

This means the following:

$$\mathcal{S} \text{ is distinguished} \quad \Rightarrow \quad \Phi_t(\mathcal{S}) \text{ is distinguished, too}$$

The surfaces - regarded as living on M (1)

Since $w \in \mathcal{D} \Leftrightarrow w$ is spatial (plus ...),
the distinguished surfaces are spatial, too.
So, they may be regarded as surfaces
on M (rather than on $M \times \mathbb{R}$).
And they may also be described as surfaces
on M (in terms of $\hat{\alpha}$ rather than σ)
Namely, instead of annihilation $(dt, d\sigma)$
one can (see below) use $(dt, \hat{d}\hat{\alpha})$



The surfaces - regarded as living on M (2)

So, we can describe the same $\mathcal{D}$

either on $M \times \mathbb{R}$ as

$$\mathcal{D} \ \leftrightarrow \ \text{annih}\ (dt, d\sigma) \ \leftrightarrow \ \text{annih}\ (dt, \hat{d}\hat{\alpha})$$

or on M as

$$\mathcal{D} \ \leftrightarrow \ \text{annih}\ \hat{d}\hat{\alpha}$$

On M the distribution $\mathcal{D}$ is time-dependent
(since $\hat{\alpha}$, and consequently $\hat{d}\hat{\alpha}$, is such).

Why it is so (1)

- Recall that

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta}$$

Then

$$\begin{aligned} d\sigma &= \hat{d}\hat{\alpha} + dt \wedge (\mathcal{L}_{\partial_t}\hat{\alpha} - \hat{d}\hat{\beta}) && \text{direct computation} \\ &= \hat{d}\hat{\alpha} + dt \wedge (-i_v \hat{d}\hat{\alpha}) && \textit{on solutions} \end{aligned}$$

and, for *spatial* w,

$$i_w d\sigma = \hat{b} - dt \wedge i_v \hat{b} \qquad \hat{b} \equiv i_w \hat{d}\hat{\alpha}$$

So,

$$i_w d\sigma = 0 \quad \Leftrightarrow \quad \hat{b} = 0 \quad \Leftrightarrow \quad i_w \hat{d}\hat{\alpha} = 0$$

Dimension of the distribution $\mathcal{D}$

On M, consider the map

$$f : w \mapsto i_w d\alpha \qquad (\mathcal{D} \leftrightarrow \mathrm{Ker}\, f)$$

Then

$$\dim \mathrm{Ker}\, f =: \dim \mathcal{D} \qquad \dim \mathrm{Im}\, f =: \mathrm{rank}\, d\alpha$$

Rank-nullity theorem says

$$\dim \mathcal{D} + \mathrm{rank}\, d\alpha = \dim T_x M \equiv \dim M$$

Since α is a p-form, $\mathrm{rank}\, d\alpha \geq p+1$, finally

$$\boxed{\dim \mathcal{D} \leq \dim M - (p+1)}$$

Example 1: Hamiltonian dynamics

Here

$$\sigma = \hat{\theta} - Hdt \equiv p_a dq^a - H(q,p,t)dt$$

so $\mathcal{D}$ consists of spatial vectors

$$w = A^a(q,p,t)\partial/\partial q^a + B_a(q,p,t)\partial/\partial p_a$$

which annihilate

$$\hat{d}\hat{\theta} \equiv dp_a \wedge dq^a$$

But $\hat{d}\hat{\theta}$ has maximum rank, so only zero vector annihilates it.
The distribution is 0-dimensional, integral surfaces are just points
so the whole stuff is of no interest, here.

Example 2: ideal fluid in 3D

Here

$$\sigma = \hat{v} - \mathcal{B}dt$$

so $\mathcal{D}$ consists of spatial vectors

$$w = \mathbf{w} \cdot \boldsymbol{\nabla} \qquad \text{which annihilate} \qquad \hat{d}\hat{v} \equiv \boldsymbol{\omega}(\mathbf{r}, t) \cdot d\mathbf{S}$$

The form is annihilated iff

$$\mathbf{w} \parallel \boldsymbol{\omega}$$

so integral submanifolds of $\mathcal{D}$ are vortex lines.

Conclusions & The End

Conclusions:
In this sense, the surfaces, in the general case,
may be treated as a generalization of vortex lines.

(More details: arXiv:1603.09563 [math-ph])

Thanks for Your attention!