100 years (plus epsilon) of Noether theorem

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Topics to be discussed:

- Some historical context

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- Its original look (using coordinate language)

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- Some historical context
- Its original look (using coordinate language)
- Its modern look 1 (using jet spaces language)
- Its modern look 2 (due to P.Ševera - boldly simple language)

Contents

(Amalie) Emmy Noether [1, 2]

1882: born in Erlangen (Germany).
1900: teacher of French and English
1900: study at University of Erlangen
(2 women out of 986 students)
1907: dissertation in Erlangen
1915: invited to Göttingen
1918: our beloved theorem
1919: habilitation in Göttingen
1933: expulsion from Göttingen
1935: died in Bryn Mawr, Pennsylvania

Noether theorem praised in a popular book [3]

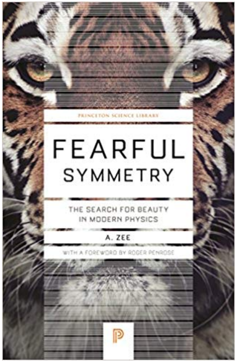
Chapter 8: The Lady and the Tiger.

"The conservation laws of physics say that you get out what you put in, and no more. Nature says that there is no free lunch."

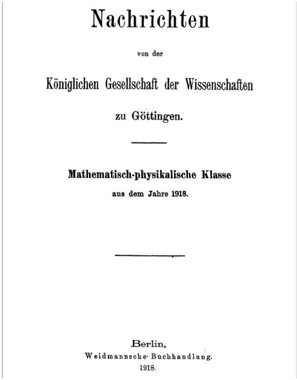
...

"Before Noether came along physicists resorted to trial and error, juggling the given equations until they found a combination that did not change in time."

...



Her paper (1)



Nachrichten

von der

Königlichen Gesellschaft der Wissenschaften

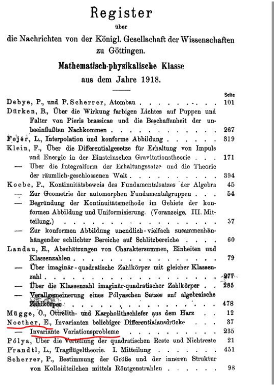
zu Göttingen.

Mathematisch-physikalische Klasse

aus dem Jahre 1918.

Berlin,
Weidmannsche Buchhandlung.
1918.

The journal



Register
über
die Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen.

Mathematisch-physikalische Klasse
aus dem Jahre 1918.

Table of Contents

Her paper (2)

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17

The paper (first page)

Fall des einfachen Integrals die Identität (3) identisch mit der von Heun sogenannten „Lagrangeschen Zentralgleichung“:

$$(4)\qquad \sum \psi_i \delta u_i = \delta f - \frac{d}{dx}\left(\sum \frac{\partial f}{\partial u_i'}\delta u_i\right), \qquad \left(u_i' = \frac{du_i}{dx}\right),$$

während für das n-fache Integral (3) übergeht in:

$$(5)\quad \sum \psi_i \delta u_i = \delta f - \frac{\partial}{\partial x_1}\left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_1}}\delta u_i\right) - \cdots - \frac{\partial}{\partial x_n}\left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_n}}\delta u_i\right).$$

Für das einfache Integral und $\varkappa$ Ableitungen der u ist (3) gegeben durch:

$$(6)\qquad \sum \psi_i \delta u_i = \delta f -$$
$$- \frac{d}{dx}\left\{\sum\left(\binom{1}{1}\frac{\partial f}{\partial u_i^{(1)}}\delta u_i + \binom{2}{1}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i^{(1)} + \cdots + \binom{\varkappa}{1}\frac{\partial f}{\partial u_i^{(\varkappa)}}\delta u_i^{(\varkappa-1)}\right)\right\} +$$
$$+ \frac{d^2}{dx^2}\left\{\sum\left(\binom{2}{2}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i + \binom{3}{2}\frac{\partial f}{\partial u_i^{(3)}}\delta u_i^{(1)} + \cdots + \binom{\varkappa}{2}\frac{\partial f}{\partial u_i^{(\varkappa)}}\delta u_i^{(\varkappa-2)}\right)\right\} + \cdots$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$+ (-1)^\varkappa \frac{d^\varkappa}{dx^\varkappa}\left\{\sum \binom{\varkappa}{\varkappa}\frac{\partial f}{\partial u_i^{(\varkappa)}}\delta u_i\right\}$$

und eine entsprechende Identität gilt beim n-fachen Integral; A enthält insbesondere δu bis zur $(\varkappa-1)$ten Ableitung. Daß durch (4), (5), (6) tatsächlich die Lagrangeschen Ausdrücke ψ_i definiert sind, folgt daraus, daß durch die Kombinationen der rechten Seiten alle höheren Ableitungen der δu eliminiert sind, während andererseits die Relation (2) erfüllt ist, zu der die partielle Integration eindeutig führt.

Es handelt sich nun im folgenden um die beiden Sätze:

I. Ist das Integral I invariant gegenüber einer $\mathfrak{G}_\varrho$, so werden ϱ linear-unabhängige Verbindungen der Lagrangeschen Ausdrücke zu Divergenzen — um-

Invariante Variationsprobleme. 239

gekehrt folgt daraus die Invarianz von I gegenüber einer $\mathfrak{G}_\varrho$. Der Satz gilt auch noch im Grenzfall von unendlich vielen Parametern.

The theorem

Publication in Transport Theory and Statistical Physics (1971)

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Invariant Variation Problems

Emmy Noether, M. A. Tavel

(Submitted on 8 Mar 2005 (v1), last revised 30 May 2018 (this version, v3))

The problems in variation here concerned are such as to admit a continuous group (in Lie's sense); the conclusions that emerge from the corresponding differential equations find their most general expression in the theorems formulated in Section 1 and proved in following sections. Concerning these differential equations that arise from problems of variation, far more precise statements can be made than about arbitrary differential equations admitting of a group, which are the subject of Lie's researches. What is to follow, therefore, represents a combination of the methods of the formal calculus of variations with those of Lie's group theory. For special groups and problems in variation, this combination of methods is not new; I may cite Hamel and Herglotz for special finite groups, Lorentz and his pupils (for instance Fokker), Weyl and Klein for special infinite groups. Especially Klein's second Note and the present developments have been mutually influenced by each other, in which regard I may refer to the concluding remarks of Klein's Note.

Comments:	M. A. Tavel's English translation of Noether's Theorems (1918), reproduced by Frank Y. Wang. Thanks to Lloyd Kannenberg for corrigenda
Subjects:	**History and Philosophy of Physics (physics.hist-ph)**
Journal reference:	Gott.Nachr.1918:235-257,1918; Transp.Theory Statist.Phys.1:186-207,1971
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	(or **arXiv:physics/0503066v3 [physics.hist-ph]** for this version)

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P.Ševera - 1-st Letter to Weinstein (1998)

Letter 1

How Courant algebroids appear in 2-dim variational problems (or maybe in string theory)

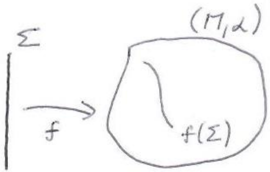
Suppose that we look for extremal surfaces in a manifold M. The thing we want to be extremal (stationary, more precisely) is the integral of a 2-form α living on M (this is for simplicity, but after all it can always be achieved by passing to a jet space). According to Noether, if v is a vector field preserving α (I shall use $v(\alpha)$ for the Lie derivative, i.e. $v(\alpha) = 0$) then $v \lrcorner \alpha$ is closed on extremal surfaces. Here it is just a consequence of Cartan's $v(\alpha) = d(v \lrcorner \alpha) + v \lrcorner d\alpha$. But conservation laws also come from symmetries up to total divergences. If v is a vector field and θ a 1-form on M and $v(\alpha) + d\theta = 0$ then $v \lrcorner \alpha + \theta$ is conserved

P.Ševera - missing details: action integral

Data: dim Σ = p = deg α, α on M

Action integral

$$\boxed{S[f] = \int_{f(\Sigma)} \alpha \; \equiv \int_{\Sigma} f^* \alpha} \qquad (1)$$



as a functional of a mapping

$$f : \Sigma \to (M, \alpha) \qquad (2)$$

We want to find its extremals.

P.Ševera - missing details: variation of f

Variation of f: via (infinitesimal) flow:

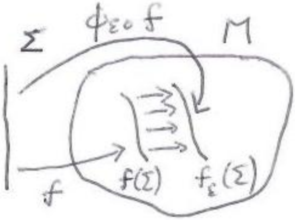
$$f \mapsto f_\epsilon \equiv \Phi_\epsilon \circ f \qquad (3)$$

where

$$\Phi_t : M \to M \qquad \Phi_t \leftrightarrow W \qquad (4)$$

is a flow corresponding to (any, fixed,"variational") vector field W. So

$$\Sigma \xrightarrow{f} M \xrightarrow{\Phi_\epsilon} M$$



P.Ševera - missing details: variation of S (1)

Variation of S: then

$$
\begin{aligned}
S[f_\epsilon] \equiv S[\Phi_\epsilon \circ f] &= \int_{\Phi_\epsilon(f(\Sigma))} \alpha \\
&= \int_{f(\Sigma)} \Phi_\epsilon^* \alpha \\
&= \int_{f(\Sigma)} (\hat{1} + \epsilon \mathcal{L}_W + \dots)\alpha \\
&= S[f] + \epsilon \int_{f(\Sigma)} \mathcal{L}_W \alpha + \dots
\end{aligned}
$$

or (up to first order)

$$
\begin{aligned}
\delta S \equiv S[f_\epsilon] - S[f] &= \epsilon \int_{f(\Sigma)} \mathcal{L}_W \alpha \\
&= \epsilon \int_{f(\Sigma)} (i_W d + d i_W)\alpha \\
&= \epsilon \int_\Sigma f^* i_W d\alpha + \epsilon \int_{\partial f(\Sigma)} i_W \alpha
\end{aligned}
$$

P.Ševera - missing details: variation of S (2)

So, the general variation formula naturally contains two terms:

$$\delta S = \epsilon \int_{\Sigma} f^* i_W d\alpha + \epsilon \int_{\partial f(\Sigma)} i_W \alpha \tag{5}$$

P.Ševera - missing details: equations of motion (1)

In the variation formula

$$\delta S = \epsilon \int_{\Sigma} f^* i_W d\alpha + \epsilon \int_{\partial f(\Sigma)} i_W \alpha$$

consider no variations at the boundary:

$$W = 0 \qquad \text{at} \quad \partial f(\Sigma)$$

and W arbitrary variations (i.e. W) inside $f(\Sigma)$. Then we get

$$\delta S = 0 \qquad \Leftrightarrow \qquad f^* i_W d\alpha = 0 \tag{6}$$

P.Ševera - missing details: equations of motion (2)

So the equation of motion (EOM) reads

$$\boxed{f^* i_W d\alpha = 0} \tag{7}$$

by the principle of extremal action.
Here, W is arbitrary on $f(\Sigma)$ (but s.t. $W|_{\partial f(\Sigma)} = 0$).
(The unknown in the equation is the mapping f.)

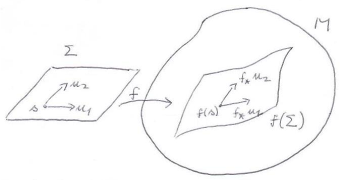
P.Ševera - missing details: equations of motion (3)

Let $(u_1, \dots, u_p)$ be a basis at $T_s\Sigma$.
On extremal $f(\Sigma)$ we have

$$\begin{aligned} 0 &= (f^* i_W d\alpha)(u_1, \dots, u_p) \\ &= (i_W d\alpha)(f_* u_1, \dots, f_* u_p) \end{aligned}$$

But $(f_* u_1, \dots, f_* u_p)$ is a basis at $T_{f(s)} f(\Sigma)$. Therefore

$$f \text{ extremal } \Leftrightarrow i_W d\alpha = 0 \text{ on } f(\Sigma)$$



P.Ševera - missing details: equations of motion (4)

Recall: f is extremal means "f is solution of EOM".
So we have an alternative form of EOM:

$$f \text{ is "solution of EOM"} \quad \Leftrightarrow \quad \boxed{i_W d\alpha = 0} \text{ on } f(\Sigma) \tag{8}$$

(*This particular* form of EOM turns out to be suitable for Noether theorem.)

P.Ševera - missing details: equations of motion (5)

So, we have two differently (and both strange :-) looking expressions of

f "is a solution of equations of motion" :

1. $$\boxed{f^* i_W d\alpha = 0} \qquad (9)$$

2. $$\boxed{i_W d\alpha = 0 \quad \text{on } f(\Sigma)} \qquad (10)$$

(Both of them for *arbitrary* vector field W.)

Where is physics?

Ok, it is perhaps nice mathematics connected with the action

$$S[f] = \int_{\Sigma} f^* \alpha$$

But is it also some interesting physics encoded in it?

We do not seem to meet this particular action integral
(neither the strange looking equations of motion)
while doing theoretical physics.

How various concrete actions look like

Physicists know (and mostly like) actions of the form

$$
\begin{aligned}
S[q] &= \int_{t_1}^{t_2} L(q,\dot q,t)dt && \text{Lagrangean mechanics}\\
S[q,p] &= \int_{t_1}^{t_2} (p_a\dot q^a - H(q,p,t))dt && \text{Hamiltonian mechanics}\\
S[\phi] &= \int d^4x\mathcal{L}(x,\phi(x),\partial_\mu\phi(x)) && \text{scalar field}\\
S[y] &= \int d^nx\partial_i y^a\partial_j y^b h_{ab}(y(x))g^{ij}(x) && \text{non-lin. sigma model}\\
&\text{etc.}\\
&\text{etc.}
\end{aligned}
$$

Various concrete forms α and mappings f

It turns out that our general action indeed becomes one of the above mentioned for

$$\alpha = L(q,v,t)dt + \frac{\partial L}{\partial v^a}(dq^a - v^a dt) \qquad \text{Lagrangean mechanics}$$
$$\alpha = p_a dq^a - H(q,p,t)dt \qquad \text{Hamiltonian mechanics}$$
$$\alpha = \mathcal{L}d^4x + \frac{\partial \mathcal{L}}{\partial y^a_i}(dy^a - y^a_j dx^j) \wedge dS_i \qquad \text{1-st order field theory}$$

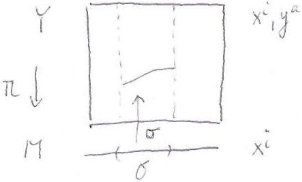
if

$$f: \ t \mapsto (t, q^a(t), \dot{q}^a(t)) \qquad \text{Lagrangean mechanics}$$
$$f: \ t \mapsto (t, q^a(t), p_a(t)) \qquad \text{Hamiltonian mechanics}$$
$$f: \ x^i \mapsto (x^i, y^a(x), \partial_i y^a(x)) \qquad \text{1-st order field theory}$$

Where (t, q^a, v^a) or (x^i, y^a, y^a_i) serve as coordinates?

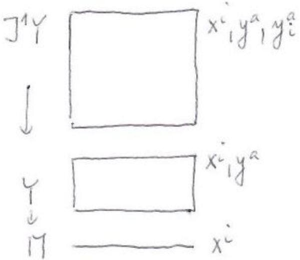
Natural geometric setting for Lagrangean theory provide jet bundles. Fields in physics are described as sections σ of a fiber bundle. In local coordinates, if x^i are on M and (x^i, y^a) on Y, then

$$\begin{aligned} \pi &: (x^i, y^a) \mapsto x^i \\ \sigma &: x^i \mapsto (x^i, y^a(x)) \end{aligned}$$



First jet prolongation $J^1 Y$ of a fibre bundle

Coordinates y^a on Y correspond to values of fields $y^a(x)$.
But we also need derivatives of fields, $\partial_i y^a(x)$.
So we construct a manifold, which "has enough room" to encompass all that data. It is $J^1 Y$.
(If we needed second derivatives, $\partial_i \partial_j y^a(x)$, we proceed to $J^2 Y$ etc.)



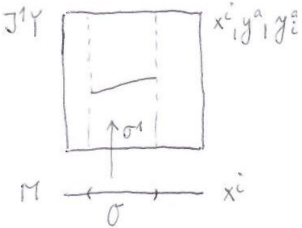
First jet prolongation σ^1 of a section σ

Consider

$$\begin{aligned} \pi : Y &\to M \qquad (x^i, y^a) \mapsto x^i \\ \pi^1 : J^1 Y &\to M \qquad (x^i, y^a, y^a_i) \mapsto x^i \end{aligned}$$

A section σ may be prolonged to σ^1

$$\begin{aligned} \sigma &: \quad x^i \mapsto (x^i, y^a = y^a(x)) \\ \sigma^1 &: \quad x^i \mapsto (x^i, y^a = y^a(x), y^a_i = \partial_i y^a(x)) \end{aligned}$$



How it works in (1-st order) Lagrangean field theory

1. Take a function $\mathcal{L}(x^\mu, y^a, y^a_\mu)$ on $J^1 Y$ (Lagrangian density)
2. Construct a 4-form on $J^1 Y$ (Lepage equivalent) as follows

$$\alpha = \mathcal{L} d^4 x + \frac{\partial \mathcal{L}}{\partial y^a_\mu}(dy^a - y^a_\nu dx^\nu) \wedge dS_\mu$$

3. Take a section σ of $\pi : Y \to M$ (a field) and its prolongation σ^1
4. Let Σ be a domain in the Minkowski space $\mathbb{R}^4$.
4. Then $\sigma^1(\Sigma)$ is a surface in $J^1 Y$
5. Construct the functional (the action)

$$S[\sigma] = \int_{\sigma^1(\Sigma)} \alpha = \cdots = \int_\Sigma \mathcal{L}(x^\mu, y^a(x), \partial_\mu y^a(x)) d^4 x$$

Long history of the jet approach to (k-th order) variational calculus

There is a long list of researchers who helped, step by step, to established today's jet bundles understanding of the subject (see, e.g., Krupková [1] and Kosmann-Schwarzbach [1]).
Just some of them:
E.T.Whittaker (1917), E.Cartan (1922), Th.De Donder (1930), Th.Lepage (1936), P.Dedecker (1957), A.Trautman (1967), J.Sniaticky (1970), *D.Krupka* (1973), I.M.Gelfand and L.A.Dikii (1975), S.Sternberg (1978), L. Mangiarotti and M. Modugno (1982), M.Marvan (1983), C.M.Marle (1983), I.Kolář (1984), J.Musilová (1984), D.E.Betounes (1984), O.Krupková (1986), P.J.Olver (1993)
and many others.

What is a symmetry of the action

Natural definition: v is a(n infinitesimal) symmetry iff its flow $\Phi_\epsilon \leftrightarrow v$ preserves the action on any argument f:

$$\text{symmetry}: \quad \boxed{S[\Phi_\epsilon \circ f] = S[f]} \quad \text{for any } f \tag{11}$$

Because of the formula derived previously

$$S[\Phi_\epsilon \circ f] = S[f] + \epsilon \int_{f(\Sigma)} \mathcal{L}_v \alpha$$

this also means

$$\text{symmetry}: \quad \boxed{\mathcal{L}_v \alpha = 0} \tag{12}$$

Derivation of the Noether theorem

Let v be a symmetry od $S[f]$. Then $\mathcal{L}_v \alpha = 0$

$$\begin{aligned} &\Rightarrow \quad i_v d\alpha + d i_v \alpha = 0 \\ &\Rightarrow \quad (i_v d\alpha)|_{f(\Sigma)} + (d i_v \alpha)|_{f(\Sigma)} = 0 \\ &\Rightarrow \quad d(i_v \alpha)|_{f(\Sigma)} = 0 \end{aligned}$$

So we have (after just 3 short lines :-)

the Noether theorem : $\boxed{\text{on extremal surface } f(\Sigma), \quad i_v\alpha \quad \text{is closed}}$

Closed form - who ordered that? (Rabi might ask.)

In business parlance,
Noether theorem says:

Give me a symmetry and I give You a closed form.

Is it a good deal for us to give her a symmetry?
Who needs closed forms?
What she promised (see folklore), is a conserved quantity.
Give me my conserved quantity! (Or my symmetry back.)
I am probably a victim of a fraud!

Where are conserved quantities? (1)

Example: Hamiltonian mechanics.

Here α is a 1-form, $i_v\alpha$ is a 0-form, i.e. a function.
Here, extremal surface = a curve $(t, q^a(t), p_a(t))$,
which is solution of Hamilton equation.
And closed 0-form is constant function.
So Noether theorem says, here, that

on solutions, $i_v\alpha$ is a constant function

Or: Give me a symmetry and I give You a function, which is constant on any solution.
Fair trade. Standard conservation law.

Where are conserved quantities? (2)

Ex.: Field theory in Minkowski space.
Here, $i_v\alpha$ is a closed 3-form. So $d(i_v\alpha) = 0$ on extremal $f(\Sigma)$.
Then also $d(f^* i_v\alpha) = 0$ for extremal f. Therefore

$$\int_{\mathcal{D}_4} d(f^* i_v\alpha) = 0$$

where $\mathcal{D}_4$ is any 4-dim domain in Minkowski space and f is extremal (solution of field equations).
So, Noether theorem (effectively) says, here, that

$$\boxed{\text{on solutions } f\ , \quad \int_{\partial\mathcal{D}_4} f^* i_v\alpha = 0}$$

Where are conserved quantities? (3)

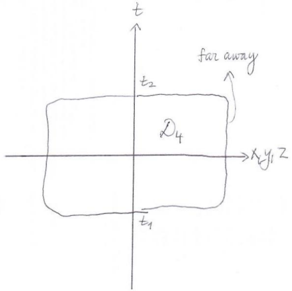
Now choose $\mathcal{D}_4 = \mathcal{D}_3 \times \langle t_1, t_2 \rangle$:
Then the boudary $\partial\mathcal{D}_4$ has 3 pieces:

1. the whole 3-space $\mathcal{D}_3$ at t_1
2. the whole 3-space $\mathcal{D}_3$ at t_2
3. $\partial\mathcal{D}_3$ "at infinity" at all $t \in \langle t_1, t_2 \rangle$

Since fields at spatial infinity vanish,

$$\int_{t=t_1,\mathcal{D}_3} f^* i_v \alpha - \int_{t=t_2,\mathcal{D}_3} f^* i_v \alpha = 0$$

So, a conservation law :-)



Symmetry up to (total) divergence

Definition:

$$\mathcal{L}_v \alpha = d\theta \qquad \text{(rather than } \mathcal{L}_v \alpha = 0\text{)}$$

for some θ. Why also interesting?

$$\begin{aligned} &\Rightarrow \quad i_v d\alpha + d i_v \alpha = d\theta \\ &\Rightarrow \quad (i_v d\alpha)|_{f(\Sigma)} + (d(i_v\alpha - \theta))|_{f(\Sigma)} = 0 \\ &\Rightarrow \quad d(i_v\alpha - \theta)|_{f(\Sigma)} = 0 \end{aligned}$$

So we have (again after just 3 short lines :-)

the Noether theorem : | on extremal surface $f(\Sigma)$, $i_v\alpha - \theta$ is closed |

The same profit as from a "standard" symmetry.

Symmetry up to (total) divergence (2)

This possibility (generalization of the original Noether theorem) was observed (and published in *Matematische Annalen*) by Erich Bessel-Hagen in 1921.
(In connection with conformal invariance of Maxwell equations.)
He writes that he will formulate his theorems slightly more generally than they were formulated in Noether's article, but that he "owe[s] these to an oral communication by Miss Emmy Noether herself."

Still (slightly) another approach (1)

momentum from flowing off the ends of the string. The general solution of the wave equation with these boundary conditions is given by

$$X^\mu(\sigma,\tau) = x^\mu + l^2 p^\mu \tau + il \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma. \qquad (2.1.56)$$

The open-string boundary conditions cause the left- and right-moving components to combine into standing waves. In particular,

$$2\partial_\pm X^\mu = \dot{X}^\mu \pm X'^\mu = l \sum_{-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)}, \qquad (2.1.57)$$

where we have set $\alpha_0^\mu = lp^\mu$.

Analogous formulas for closed strings are

$$\partial_- X_R^\mu = \dot{X}_R^\mu = l \sum_{-\infty}^{\infty} \alpha_n^\mu e^{-2in(\tau-\sigma)} \qquad (2.1.58)$$

$$\partial_+ X_L^\mu = \dot{X}_L^\mu = l \sum_{-\infty}^{\infty} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}. \qquad (2.1.59)$$

The important difference is that the left- and right-moving modes are independent in this case. Also $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2} lp^\mu$ in the closed-string case, and there is an extra factor of two in the exponents.

We turn next to a description of the D-dimensional Poincaré invariance. Since the Poincaré transformations $\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu$ are simply global symmetries from the point of view of the two-dimensional theory, they are associated with conserved 'Noether currents'. There is a standard procedure in field theory, known as the 'Noether method', for constructing the conserved current J_α associated with the global symmetry transformation $\phi(\sigma) \to \phi(\sigma) + \epsilon\delta\phi(\sigma)$, where $\phi(\sigma)$ is any field in the theory and ϵ is a constant infinitesimal parameter. One considers the transformation

$$\phi(\sigma) \to \phi(\sigma) + \epsilon(\sigma)\delta\phi(\sigma) \qquad (2.1.60)$$

where ϵ is an infinitesimal parameter that is *not* constant on the world sheet. The action is not invariant under such transf nations for general ϵ, since the symmetry we are considering is on'y a global symmetry. Since the action would be invariant for constant ϵ, its variation is proportional to the derivative of ϵ and so is of the general form

$$\delta S = \int d^2\sigma J_\alpha \partial^\alpha \epsilon, \qquad (2.1.61)$$

for some current J_α. The current defined in this way is always conserved if the equations of motion are obeyed. Indeed, when the equations are obeyed, the action is stationary under any variation and in particular under a variation of the form (2.1.60). Thus, when the equations of motion are obeyed, δS in (2.1.61) is zero for any ϵ. This is possible only if $\partial_\alpha J^\alpha = 0$.

One can readily apply this method to derive the conserved currents associated with the Poincaré transformations of X^μ:

$$P_\alpha^\mu = T\partial_\alpha X^\mu \qquad (2.1.62)$$

$$J_\alpha^{\mu\nu} = T(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu). \qquad (2.1.63)$$

Here P_α is the current associated with translation invariance while $J_\alpha^{\mu\nu}$ is the current associated with Lorentz invariance. Current conservation is the statement that

$$\partial_\alpha P^{\alpha\mu} = \partial_\alpha J^{\alpha\mu\nu} = 0. \qquad (2.1.64)$$

These currents describe the density of D-dimensional momentum and angular momentum on the two-dimensional world sheet. The amount of momentum flowing across an arbitrary line segment in the world sheet, $(d\sigma, d\tau)$ is given by

$$dP^\mu = P_\tau^\mu d\sigma + P_\sigma^\mu d\tau, \qquad (2.1.65)$$

so that the boundary conditions at the ends of an open string, (2.1.32), indeed imply that there is no momentum flowing out of the ends of the string. Similar statements apply to the current of angular momentum, $J_\alpha^{\mu\nu}$.

The total conserved momentum and angular momentum of a string are found by integrating the currents of (2.1.62) and (2.1.63) over σ at $\tau = 0$. For example, the total momentum of a closed string is given by

$$P^\mu = T \int_0^\pi d\sigma \frac{dX^\mu(\sigma)}{d\tau} = \pi T(l\alpha_0^\mu + l\tilde{\alpha}_0^\mu) = p^\mu, \qquad (2.1.66)$$

so that the t [illegible] m of the string is the same as the 'momentum' p^μ of the zero mo [illegible] also holds for open strings. The total angular

Green, Schwarz, Witten: Superstring Theory

Still (slightly) another approach (2)

to fiddle about with the individual equations of motion. We begin by illustrating her

1.3 Lagrangian mechanics 15

technique in the case of angular momentum, whose conservation is a consequence of the rotational symmetry of the central force problem. The action integral for the central force problem is

$$S = \int_0^T \left\{ \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \right\} dt. \qquad (1.58)$$

Noether observes that the integrand is left unchanged if we make the variation

$$\theta(t) \to \theta(t) + \varepsilon\alpha \qquad (1.59)$$

where α is a fixed angle and ε is a small, time-independent, parameter. This invariance is the symmetry we shall exploit. It is a mathematical identity: it does not require that r

M.Stone, P.Goldbart: Mathematics for Physics

Still (slightly) another approach (3)

where α is a fixed angle and ε is a small, time-independent, parameter. This invariance is the symmetry we shall exploit. It is a mathematical identity: it does not require that r and θ obey the equations of motion. She next observes that since the *equations of motion* are equivalent to the statement that S is left stationary under *any* infinitesimal variations in r and θ, they necessarily imply that S is stationary under the specific variation

$$\theta(t) \to \theta(t) + \varepsilon(t)\alpha \qquad (1.60)$$

where now ε is allowed to be time-dependent. This stationarity of the action is no longer a mathematical identity, but, because it requires r, θ, to obey the equations of motion, has physical content. Inserting $\delta\theta = \varepsilon(t)\alpha$ into our expression for S gives

$$\delta S = \alpha \int_0^T \left\{ mr^2\dot{\theta} \right\} \dot{\varepsilon}\, dt. \qquad (1.61)$$

Note that this variation depends only on the time derivative of ε, and not ε itself. This is because of the invariance of S under time-independent rotations. We now assume that $\varepsilon(t) = 0$ at $t = 0$ and $t = T$, and integrate by parts to take the time derivative off ε and put it on the rest of the integrand:

$$\delta S = -\alpha \int \left\{ \frac{d}{dt}(mr^2\dot{\theta}) \right\} \varepsilon(t)\, dt. \qquad (1.62)$$

M.Stone, P.Goldbart: Mathematics for Physics

Still (slightly) another approach (4)

Consider an action integral $S[\psi]$ for a field ψ in the domain $\mathcal{U}$ of the space-time M (there are possibly several fields, ψ denotes all of them). Assume there is a "global" action $\psi(x) \mapsto \rho(g)\psi(x)$ of the group G on the fields ψ (so that g does not depend on x; at each point x the same g acts), such that the action integral is invariant with respect to the "global" action of the group

$$S[\rho(g)\psi] = S[\psi]$$

In particular, for an infinitesimal global transformation we have

$$S[\rho(e^{\epsilon X})\psi] = S[\psi] \qquad X \in \mathcal{G}$$

Consider now an infinitesimal "*local*" transformation

$$\psi(x) \mapsto \rho(e^{\epsilon s(x)})\psi(x)$$

given in terms of the function $s : \mathcal{U} \to \mathcal{G}, x \mapsto s(x) \equiv s^i(x)E_i \in \mathcal{G}$. The action integral is no longer invariant in general.

M.Fecko: Differential Geometry and Lie Groups for Physicists

Still (slightly) another approach (5)

21.6.1 Since for a *constant* function the action integral is supposed to remain unchanged, the variation is now expected to be proportional to the 1-form $\epsilon\, ds = \epsilon\, ds^i\, E_i$ (it is a 1-form on $\mathcal{U}$ with values in the Lie algebra $\mathcal{G}$)

$$S[\rho(e^{\epsilon s(x)})\psi] = S[\psi] + \epsilon \int_{\mathcal{U}} ds^i \wedge J_i(\psi) + o(\epsilon)$$

where $(n-1)$-forms $J_i(\psi)$, constructed from the fields ψ, depend on the detailed structure of the action (n is the dimension of the space-time M). Check that *Noether's theorem* holds, i.e. that

(i) the forms $J_i(\psi)$, when evaluated *on the solutions of the equations of motion*, are *closed*

$$\psi \text{ is a solution of the equations of motion} \quad \Rightarrow \quad dJ_i(\psi) = 0$$

[456] It is named after the distinguished German mathematician ("whose innovations in higher algebra," according to Encyclopaedia Britannica, "gained her recognition as the most creative abstract algebraist of modern times") Emmy Noether; she published the celebrated result in 1918.

M.Fecko: Differential Geometry and Lie Groups for Physicists

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THE END

Than You for Your attention!