Galilean and Carrollian Hodge star operators

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- How to find the same operator $*$ completely differently

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- What are Galilei and Carroll spacetimes
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- Why (usual) Hodge star $*$ does not exist there
- How to construct (useful) analog of $*$ there
- How to find the same operator $*$ completely differently
- How the new $*$ works in Galilean/Carrollian electrodynamics

Contents

1. Introduction
2. Reminder 1.: Galilean and Carrollian spacetimes
3. Reminder 2.: Standard Hodge star operator
4. Analogs of the Hodge star operator
5. Application: Maxwell equations

Galilei and Carroll spacetimes

Formally - limits ($c \to \infty$ and $c \to 0$) of Minkowski spacetime.

What is common for all three spacetimes is

- (global) coordinates $(x^0, x^i) \leftrightarrow x^\mu$
- action of translations $x^\mu \mapsto x^\mu + k^\mu$ (incl. time transl.)
- action of spatial rotations $(t, \mathbf{r}) \mapsto (t, R\mathbf{r})$, $R^T R = \mathbb{I}$

What is different is action of boosts.

Galilei and Carroll spacetimes (2)

Namely, infinitesimal action of boosts is as follows:

$$
\begin{aligned}
&\text{Lorentz} & t' &= t + \epsilon \mathbf{n}\cdot\mathbf{r} & (1)\\
& & \mathbf{r}' &= \mathbf{r} + \epsilon \mathbf{n} t & (2)\\
&\text{Galilei} & t' &= t & (3)\\
& & \mathbf{r}' &= \mathbf{r} + \epsilon \mathbf{n} t & (4)\\
&\text{Carroll} & t' &= t + \epsilon \mathbf{n}\cdot\mathbf{r} & (5)\\
& & \mathbf{r}' &= \mathbf{r} & (6)
\end{aligned}
$$

One easily checks that translations, rotations and boosts close to (Poincaré, Galilei and Carroll) Lie groups.

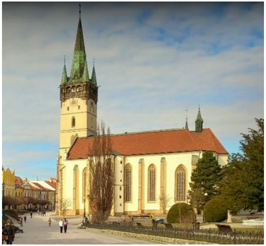
Digression - a very personal note

Btw., the concept of Carroll spacetime was introduced in

J.-M.Lévy Leblond:
Une nouvelle limite non-relativiste du groupe de Poincaré,
Annales de l' I.H.P. Phys. théor. vol. 3, no. 1, pp. 1-12, (1965)

That year, I was a small pupil
in primary school here in Prešov —>

Although I already knew
the entire alphabet
(both upper and lower case letters),
surprisingly, no teacher
turned my attention to the paper.



Galilei and Carroll spacetimes (3)

The generators of the actions read:

Common:

$$\partial_t \ , \ \partial_i \quad = \quad \text{time and space translations} \tag{7}$$

$$\epsilon_{ijk} x^j \partial_k \quad = \quad \text{spatial rotations} \tag{8}$$

Specific:

$$x^i \partial_t + t\partial_i \quad = \quad \text{Lorentz boosts} \tag{9}$$

$$t\partial_i \quad = \quad \text{Galilei boosts} \tag{10}$$

$$x^i \partial_t \quad = \quad \text{Carroll boosts} \tag{11}$$

They all close, w.r.t. commutators, to (three) Lie algebras.

Invariant metric tensor - Minkowski space

The great insight of Minkowski (1908) is that
there exists a metric tensor on spacetime
whose isometries coincide with
crucial special relativistic transformations:

$$f : x \mapsto \Lambda x + a \qquad f^* g = g \qquad g = \eta_{\mu\nu} dx^\mu \otimes dx^\nu \tag{12}$$

(So g is invariant w.r.t. physically crucial transformations.)

The natural questions then arise:
Is this also the case for Galilei (Carroll) transformations?

Invariant metric tensors - Galilei and Carroll spacetimes

Short and clear answer is: NO.

In slightly more words:

1. There is NO interesting metric tensor on Galilei spacetime, i.e. such which is invariant w.r.t. Galilei transformations.

2. There is NO interesting metric tensor on Carroll spacetime, i.e. such which is invariant w.r.t. Carroll transformations.

Neither Galilei nor Carroll transformations are isometries !!!

Invariant tensors

Instead of metric tensor, however,
there are other invariant tensors available
on both Galilei and Carroll spacetimes,
i.e. one can (easily) find several tensor fields $\mathcal{T}$ such that

$$\mathcal{L}_U \mathcal{T} = 0 \tag{13}$$

holds for $U =$ any of the

- Galilei generators (on Galilei spacetime)
- Carroll generators (on Carroll spacetime)

(All have been well known for over 50 years :-)

Invariant tensors - Galilei spacetime

1. $\binom{0}{1}$-type tensor (covector, 1-form)

$$\xi = \xi_\mu dx^\mu = dt \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

2. $\binom{2}{0}$-type symmetric (degenerate) tensor

$$h = h^{\mu\nu} \partial_\mu \otimes \partial_\nu = \delta^{ij} \partial_i \otimes \partial_j \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix} \tag{15}$$

Notice that they are related via

$$h(\xi, \ \cdot \ ) = 0 \tag{16}$$

Invariant tensors - Galilei spacetime (2)

3. Of course, any tensor products, like

$\binom{0}{2}$-type symmetric tensor

$$k \equiv \xi \otimes \xi = dt \otimes dt \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

happen to be invariant as well.

Invariant tensors - Carroll spacetime

1. $\binom{1}{0}$-type tensor (vector)

$$\tilde{\xi} = \tilde{\xi}^\mu \partial_\mu = \partial_t \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{18}$$

2. $\binom{0}{2}$-type symmetric (degenerate) tensor

$$\tilde{h} = \tilde{h}_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{ij} dx^i \otimes dx^j \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \tag{19}$$

Notice that they are related via

$$\tilde{h}(\tilde{\xi}, \ \cdot \ ) = 0 \tag{20}$$

Invariant tensors - Carroll spacetime (2)

3. Of course, any tensor products, like

$\binom{2}{0}$-type symmetric tensor

$$\tilde{k} \equiv \tilde{\xi} \otimes \tilde{\xi} = \partial_t \otimes \partial_t \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{21}$$

happen to be invariant as well.

Galilean and Carrollian spacetimes

Finally we can understand motivation for
formal definitions of spacetimes of our real interest,
Galilean and Carrollian spacetimes,
generalizations (kind of "curved versions")
of Galilei and Carroll spacetimes.

They may be regarded as manifolds, where
Galilei and Carroll (vector space) structure
is defined in each tangent space.

Similarly as standard Lorentzian spacetimes may be regarded
as manifolds, where Minkowski (vector space) structure
is defined in each tangent space.

Galilean manifold (spacetime)

It is a triple (M, ξ, h), where

- M is an d-dimensional manifold, $d = 1 + n$
- ξ is an everywhere non-zero covector (i.e. a $\binom{0}{1}$-tensor) field on M
- h is an everywhere rank-n symmetric type-$\binom{2}{0}$-tensor field on M
- such that $h(\xi, \,\cdot\,) = 0$.

We call a (local) frame field $e_a = (e_0, e_i)$
and the (dual) coframe field $e^a = (e^0, e^i)$, $i = 1, \dots, n$ on (M, ξ, h)
adapted (or distinguished) if

$$e^0 = \xi \qquad\qquad h = \delta^{ij} e_i \otimes e_j \tag{22}$$

Galilean manifold (spacetime) (2)

Then, in any adapted frame field,
the components of the two tensor fields
achieve "canonical form"

$$\xi_a \leftrightarrow \begin{pmatrix} \xi_0 \\ \xi_i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{23}$$

$$h^{ab} \leftrightarrow \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{24}$$

These frames are analogs of orthonormal frames (vielbeins)
widely used on Lorentzian spacetimes.

Galilean manifold (spacetime) (3)

The (point-dependent) change-of-basis matrix A between any pair $\hat{e}_a$, e_a of adapted frame fields, given by $\hat{e}_a = A^b_a e_b$, has the structure

$$A^b_a \leftrightarrow \begin{pmatrix} A^0_0 & A^0_i \\ A^i_0 & A^i_j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v^i & R^i_j \end{pmatrix} \quad \text{i.e.} \quad A \leftrightarrow \begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix} \qquad (25)$$

where R is an n-dimensional rotation matrix.

In each point, such matrices form a Lie group G, subgroup of $GL(d, \mathbb{R})$, the (homogeneous) Galilei group (R parametrizes rotations and v boosts, respectively).

Carrollian manifold (spacetime)

It is a triple $(M, \tilde{\xi}, \tilde{h})$, where

- M is an d-dimensional manifold, $d = 1 + n$
- $\tilde{\xi}$ is an everywhere non-zero vector (i.e. a $\binom{1}{0}$-tensor) field on M
- $\tilde{h}$ is an everywhere rank-n symmetric type-$\binom{0}{2}$-tensor field on M
- such that $\tilde{h}(\tilde{\xi}, \ \cdot\ ) = 0$.

We call a (local) frame field $e_a = (e_0, e_i)$
and the (dual) coframe field $e^a = (e^0, e^i)$, $i = 1, \dots, n$ on (M, ξ, h)
adapted (or distinguished) if

$$e_0 = \tilde{\xi} \qquad \tilde{h} = \delta_{ij} e^i \otimes e^j \tag{26}$$

Carrollian manifold (spacetime) (2)

Then, in any adapted frame field,
the components of the two tensor fields
achieve "canonical form"

$$\tilde{\xi}^a \leftrightarrow \begin{pmatrix} \tilde{\xi}^0 \\ \tilde{\xi}^i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{27}$$

$$\tilde{h}_{ab} \leftrightarrow \begin{pmatrix} \tilde{h}_{00} & \tilde{h}_{0i} \\ \tilde{h}_{i0} & \tilde{h}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{28}$$

Again, these are analogs of orthonormal frames (vielbeins)
from Lorentzian spacetimes.

Carrollian manifold (spacetime) (3)

The (point-dependent) change-of-basis matrix A between any pair $\hat{e}_a$, e_a of adapted frame fields, given by $\hat{e}_a = A^b_a e_b$, has the structure

$$A^b_a \leftrightarrow \begin{pmatrix} A^0_0 & A^0_i \\ A^i_0 & A^i_j \end{pmatrix} = \begin{pmatrix} 1 & v_i \\ 0 & R^i_j \end{pmatrix} \quad \text{i.e.} \quad A \leftrightarrow \begin{pmatrix} 1 & v^T \\ 0 & R \end{pmatrix} \tag{29}$$

where R is an n-dimensional rotation matrix.

In each point, such matrices form a Lie group G, subgroup of $GL(d,\mathbb{R})$, the (homogeneous) Carroll group (R parametrizes rotations and v boosts, respectively).

Just a note on reduction of the frame bundle LM

One can also (standardly) introduce
Galilean (or Carrollian) structure on M
via reduction of the frame bundle LM.

Namely, rather than considering all linear frames,
we restrict to Galilean (or Carrollian) adapted frames
(they become points of the total space $P \subset LM$)
and, consequently, restrict the action
of the whole linear group $GL(d, \mathbb{R})$ to the action
of the Galilei (or Carroll) subgroup G:

$$G \leftrightarrow \begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix} \qquad \text{or} \qquad G \leftrightarrow \begin{pmatrix} 1 & v^T \\ 0 & R \end{pmatrix} \tag{30}$$

Just a note on Galilean and Carrollian connections

On Galilean and Carrollian spacetimes, often adapted linear connections are studied.

In complete analogy with Levi-Civita connection on (M, g), one requires

$$\nabla h = 0 = \nabla \xi \qquad \text{Galilean connection} \tag{31}$$

$$\nabla \tilde{h} = 0 = \nabla \tilde{\xi} \qquad \text{Carrollian connection} \tag{32}$$

We do not need this concept for what interests us here (namely for Galilean/Carrollian Hodge star :-)

Standard Hodge star operator on (M, g, o)

Hodge star (duality) operator $*$ is a well-known linear map on forms

$$* : \Omega^p(M, g, o) \to \Omega^{d-p}(M, g, o) \qquad d = \dim M \tag{33}$$

given in components as follows:

$$\boxed{(*\alpha)_{a\dots b} := \frac{1}{p!}\alpha_{c\dots d}\ \omega^{c\dots d}{}_{a\dots b}} \tag{34}$$

where

$$\omega^{c\dots d}{}_{a\dots b} := g^{ce}\dots g^{df}\omega_{e\dots fa\dots b} \tag{35}$$

Standard Hodge star operator on (M, g, o) (2)

So what is used for construction of the (standard) Hodge star is

- the metric volume form $\omega \equiv \omega_{g,o} \;\leftrightarrow\; \omega_{e \dots f a \dots b}$
- the cometric $g^{-1} \;\leftrightarrow\; g^{ab}$ (for raising of indices).

In terms of these objects, we can also write the operator in component-free way

$$*_g \alpha \;\sim\; C \dots C(g^{-1} \otimes \cdots \otimes g^{-1} \otimes \omega_g \otimes \alpha) \qquad (36)$$

where C denotes *contraction*.

Hodge star on Galilean and Carrollian spacetimes ?

Direct consequence of non-existence of (interesting) g:

1. Officially, there is NO Hodge star on Galilean spacetime.
2. Officially, there is NO Hodge star on Carrollian spacetime.

Hodge star on Galilean and Carrollian spacetimes ? (2)

This is a bad news for those who plan to use differential forms on Galilean and Carrollian spacetimes,
since without Hodge star
the number of interesting operators on forms is too limited.

The good news is that
one can easily construct analogs of Hodge star operators,
which (perhaps) may become almost as useful
as the full fledged Hodge star is on (M, g, o).

Hodge star on Galilean and Carrollian spacetimes ? (3)

There are (at least) two completely different ways how the analogs may be found.

One way to achieve this is simply substituting (non-existing) metric tensor with the tensors, which are available on Galilean or Carrollian spacetimes. They are well known for a long time. Just use them! It turns out it works. See below.

Another way is to compute all intertwining operators between p-forms and q-forms. It also works! See below.

Analogs of the Hodge $*$: Use (h, ξ) and $(\tilde{h}, \tilde{\xi})$

On (M, h, ξ) and $(M, \tilde{h}, \tilde{\xi})$,
one can replace the original construction (36) of the Hodge star

$$*_g \alpha \sim C \dots C(g^{-1} \otimes \dots \otimes g^{-1} \otimes \omega_g \otimes \alpha) \qquad (37)$$

with two analogs of the Hodge stars:

$$*_{h,\xi} \alpha \sim C \dots C(h \otimes \dots \otimes h \otimes \omega_{h,\xi} \otimes \alpha) \qquad (38)$$

$$*_{\tilde{h},\tilde{\xi}} \alpha \sim C \dots C(\tilde{h} \otimes \dots \otimes \tilde{h} \otimes \tilde{\omega}_{\tilde{h},\tilde{\xi}} \otimes \alpha) \qquad (39)$$

(see ArXiv:2206.09788 [math-ph])

Analogs of the Hodge $*$: Use (h, ξ) and $(\tilde{h}, \tilde{\xi})$ (2)

Here $\omega_{h,\xi}$ and $\tilde{\omega}_{\tilde{h},\tilde{\xi}}$ stand for

- top degree form (i.e. the volume form)
- top degree polyvector

They are given as

$$\omega_{h,\xi} := e^0 \wedge e^1 \wedge \cdots \wedge e^n \qquad \tilde{\omega}_{\tilde{h},\tilde{\xi}} := e_0 \wedge e_1 \wedge \cdots \wedge e_n \tag{40}$$

w.r.t. any adapted frame/coframe.
It turns out that both of them are canonical at both spacetimes!
(Simply because $\det A = 1$ for both Galilei and Carroll groups :-)

Analogs of the Hodge $*$: Use (h, ξ) and $(\tilde{h}, \tilde{\xi})$ (3)

Another way to express the same idea is to recall the formula

$$(*\alpha)_{a\ldots b} := \frac{1}{p!}\alpha_{c\ldots d}\ \omega^{c\ldots d}{}_{a\ldots b} \qquad (41)$$

and display formulas giving the crucial mixed tensor $\omega^{c\ldots d}{}_{a\ldots b}$:

$$\begin{aligned}
\omega^{c\ldots d}{}_{a\ldots b} &= (g^{-1})^{ce}\ldots(g^{-1})^{df}(\omega_g)_{e\ldots f a\ldots b} && \text{(standard)} \qquad (42)\\
&= h^{ce}\ldots h^{df}\omega_{e\ldots f a\ldots b} && \text{(Galilean)} \qquad (43)\\
&= \tilde{h}_{ae}\ldots\tilde{h}_{bf}\tilde{\omega}^{c\ldots d e\ldots f} && \text{(Carrollian)} \qquad (44)
\end{aligned}$$

Action of the three Hodge star operators on forms

In all three spacetimes, any p-form α, $p = 0, 1, \dots, d$, may be uniquely decomposed as follows:

$$\alpha = e^0 \wedge \hat{s} + \hat{r} \tag{45}$$

where the two hatted forms $(\hat{s}, \hat{r})$ are spatial (no e^0 present). Explicit computation of the three Hodge stars leads to

$$* \, (e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \qquad \text{Lorentzian Hodge star} \tag{46}$$

$$* \, (e^0 \wedge \hat{s} + \hat{r}) = e^0 \wedge \hat{*}\hat{r} \qquad \text{Galilean Hodge star} \tag{47}$$

$$* \, (e^0 \wedge \hat{s} + \hat{r}) = \hat{*}\hat{\eta}\hat{s} \qquad \text{Carrollian Hodge star} \tag{48}$$

Here $\hat{*}$ stands for standard Euclidean Hodge star and $\hat{\eta}$ is just ± 1.

Action of the three Hodge star operators on F

Important example: On $(1+3)-$ dimensional Minkowski, Galilei and Carroll spacetimes, we get for action of $*$ on the 2-form of electromagnetic field F

$$F = dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S} \tag{49}$$

these results:

$$*_M (dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = dt \wedge (-\mathbf{B}) \cdot d\mathbf{r} - \mathbf{E} \cdot d\mathbf{S} \tag{50}$$

$$*_G (dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = dt \wedge (-\mathbf{B}) \cdot d\mathbf{r} \tag{51}$$

$$*_C (dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = -\mathbf{E} \cdot d\mathbf{S} \tag{52}$$

Action of the three Hodge star operators on $(\mathbf{E}, \mathbf{B})$

So, effectively, in terms of electric and magnetic fields $(\mathbf{E}, \mathbf{B})$, this reads

$$*_M \; : \; (\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{E}) \tag{53}$$

$$*_G \; : \; (\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{0}) \tag{54}$$

$$*_C \; : \; (\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{0}, \mathbf{E}) \tag{55}$$

Recall that $*_M$ is known to be a duality
(when applied twice, we essentially come back to the original).

It is clear from (54) and (55) that we can
no longer speak of Galilean and Carrollian Hodge duality!

2-nd approach: Intertwining operators (1)

Lorentz, Galilei and Carroll groups
(denoted here collectively as G)
act on the space $\hat{\Lambda}^p$ of components of p-forms,

Namely the components get scrambled according to

$$\alpha_{a\dots b} \ \mapsto \ (A^{-1})^c_a \dots (A^{-1})^d_b \alpha_{c\dots d} \tag{56}$$

This is a representation of G on $\hat{\Lambda}^p$

$$\rho_p : G \to \mathrm{Aut}\,\hat{\Lambda}^p \tag{57}$$

2-nd approach: Intertwining operators (2)

Now one can regard all three star operators $*$ as linear maps

$$* : \hat{\Lambda}^p \to \hat{\Lambda}^{d-p} \tag{58}$$

The fact that the stars really convert p-forms to $(d-p)$-forms says that the following commutative diagram is true

$$\begin{array}{ccc} \hat{\Lambda}^p & \xrightarrow{\ *\ } & \hat{\Lambda}^{d-p} \\ {\scriptstyle \rho_p(g)}\downarrow & & \downarrow{\scriptstyle \rho_{d-p}(g)} \\ \hat{\Lambda}^p & \xrightarrow[\ *\ ]{} & \hat{\Lambda}^{d-p} \end{array} \qquad \text{i.e.} \qquad \rho_{d-p}(g) \circ * = * \circ \rho_p(g) \tag{59}$$

(The two scramblings, via ρ_{d-p}/ρ_p and via $*$, do commute.)

Definition of intertwining operator

In representation theory parlance, intertwining operator acting between *general* representations ρ_1 and ρ_2 is defined as

$$\begin{array}{ccc} V_1 & \xrightarrow{a} & V_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow[a]{} & V_2 \end{array} \qquad \text{i.e.} \qquad \rho_2(g)\circ a = a \circ \rho_1(g) \qquad (60)$$

Then all three Hodge star operators $*$
may be regarded as an intertwining operators
between representations ρ_p and ρ_{d-p}
on component spaces of p-forms and $(d-p)$-forms, respectively.

2-nd approach: Intertwining operators (3)

So, in our case, on component spaces of differential forms on Lorentzian, Galilean and Carrollian spacetimes, we can define intertwining operators a_{qp} acting between spaces of general pair of degrees p and q, i.e. defined as

$$\begin{array}{ccc} \Omega^p & \xrightarrow{a_{qp}} & \Omega^q \\ \rho_p \downarrow & & \downarrow \rho_q \\ \Omega^p & \xrightarrow[a_{qp}]{} & \Omega^q \end{array} \qquad \text{i.e.} \qquad \rho_q \circ a_{qp} = a_{qp} \circ \rho_p \qquad (61)$$

Since ρ_p depend on the choice of spacetime (boosts act differently), also operators a_{qp} are expected to be different for different spacetimes.

2-nd approach: Intertwining operators (3)

In a sense, we try to find all algebraic operators on forms sharing local G-invariance property with the Hodge star (all (Hodge star)-like operators, including the Hodge star itself).

Surprisingly (at least for me),
all this can be explicitly computed for
- 1+3 case (yet) and for
- all three spacetimes and
- all pairs p and q

(see SIGMA 19 (2023), 024, 24 pages).

2-nd approach: Intertwining operators (4)

The results may be summarized as follows:
The operators of interest are just

- Lorentzian Hodge star and nothing more
- Galilean Hodge plus one more operator ($\xi\wedge$)
- Carrollian Hodge plus one more operator ($i_{\tilde{\xi}}$)

The first result again confirms
the unique value of the (standard Lorentzian) Hodge.

The remaining two results similarly confirm
the value of the two new Hodge stars.

(Free) Maxwell equations - $(\mathbf{E}, \mathbf{B})$ language

Recall that standard (free) electromagnetism in vacuum is governed by (sourceless) Maxwell equations

$$\operatorname{div} \mathbf{E} = 0 \tag{62}$$

$$\operatorname{curl} \mathbf{B} - \partial_t \mathbf{E} = 0 \tag{63}$$

$$\operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{64}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{65}$$

(Free) Maxwell equations - F language

In terms of the 2-form on Minkowski spacetime

$$F = dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S} \tag{66}$$

Maxwell equations become a system of just two equations

$$\boxed{d *_\eta F = 0 \qquad dF = 0} \quad \text{Maxwell equations on } (R^4, \eta, o) \tag{67}$$

(Free) Maxwell equations on (M, g, o)

This is naturally generalized for any Lorentzian spacetime as

$$\boxed{d *_\eta F = 0 \qquad dF = 0} \qquad \text{Maxwell equations on } (R^4, \eta, o) \tag{68}$$

$$\boxed{d *_g F = 0 \qquad dF = 0} \qquad \text{Maxwell equations on } (M, g, o) \tag{69}$$

So one only modifies the Hodge star operator, $*_\eta \mapsto *_g$.

Galilei and Carroll electrodynamics

Starting from papers

M.Le Bellac, J.M.Lévy Leblond (1973)

Ch.Duval, G.W.Gibbons, P.A.Horvathy, P.M.Zhang (2014)

Galilei and Carroll electrodynamics (in $(\mathbf{E}, \mathbf{B})$ language)
are studied intensively and systematically
(they are invariant w.r.t. Galilei/Carroll group, respectively,
rather than Lorentz group).

Galilei and Carroll electrodynamics - $(\mathbf{E}, \mathbf{B})$ language

The corresponding field ("Maxwell") equations read

$$
\begin{array}{rrrr}
\text{Minkowski} & \text{Galilei} & \text{Carroll} & (70) \\
\operatorname{div} \mathbf{E} = 0 & \operatorname{div} \mathbf{E} = 0 & \operatorname{div} \mathbf{E} = 0 & (71) \\
\operatorname{curl} \mathbf{B} - \partial_t \mathbf{E} = 0 & \operatorname{curl} \mathbf{B} \qquad = 0 & \partial_t \mathbf{E} = 0 & (72) \\
\operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0 & \operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0 & \operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0 & (73) \\
\operatorname{div} \mathbf{B} = 0 & \operatorname{div} \mathbf{B} = 0 & \operatorname{div} \mathbf{B} = 0 & (74)
\end{array}
$$

Galilei and Carroll electrodynamics - spacetime language

What about their (Galilei/Carroll-) spacetime formulation?

It is possible (and standardly done).
Namely, with the help of Galilei/Carroll linear connection:

$$F^{\mu\nu}{}_{;\nu} = 0 \qquad F_{[\mu\nu;\rho]} = 0 \tag{75}$$

(as is a possibility in Lorentzian case as well).

Notice that the fact, that F is a differential form
(rather than a general tensor field)
plays virtually no role, here.

Galilei and Carroll electrodynamics - just new Hodge stars?

Another idea (for which it is crucial that F is a form):
In general, perhaps (= hypothesis) just replace

$$*_\eta \mapsto *_g \qquad \text{for Lorentzian electrodynamics} \tag{76}$$

$$*_\eta \mapsto *_{h,\xi} \qquad \text{for Galilean electrodynamics} \tag{77}$$

$$*_\eta \mapsto *_{\tilde{h},\tilde{\xi}} \qquad \text{for Carrollian electrodynamics} \tag{78}$$

in standard Minkowski version of electrodynamics

$$d *_\eta F = 0 \qquad dF = 0 \tag{79}$$

Just new Hodge stars?

That is, consider

$$\boxed{d *_{\text{any}} F = 0 \qquad dF = 0} \tag{80}$$

as electrodynamics equations "in general"
and get concrete versions
by specifying the meaning of the word "any"
on the Hodge star operator..

Various versions of (free) Maxwell equations

In particular, for

$$*_{\text{any}} = *_{\eta} \qquad *_{\text{any}} = *_{h,\xi} \qquad *_{\text{any}} = *_{\tilde{h},\tilde{\xi}} \tag{81}$$

we get

Minkowski	Galilei	Carroll	(82)
$\operatorname{div}\mathbf{E} = 0$		$\operatorname{div}\mathbf{E} = 0$	(83)
$\operatorname{curl}\mathbf{B} - \partial_t\mathbf{E} = 0$	$\operatorname{curl}\mathbf{B} = 0$	$\partial_t\mathbf{E} = 0$	(84)
$\operatorname{curl}\mathbf{E} + \partial_t\mathbf{B} = 0$	$\operatorname{curl}\mathbf{E} + \partial_t\mathbf{B} = 0$	$\operatorname{curl}\mathbf{E} + \partial_t\mathbf{B} = 0$	(85)
$\operatorname{div}\mathbf{B} = 0$	$\operatorname{div}\mathbf{B} = 0$	$\operatorname{div}\mathbf{B} = 0$	(86)

Various versions of (free) Maxwell equations

This basically matches what we saw before
except for missing of Gauss's law in Galilei case
(not clear for me why, yet ...)

So we always get correct equations :-)
but in Galilean case we do not get the whole story :-(

Why we always get correct equations

It is clear, that the equations

$$d * F = 0 \qquad dF = 0 \tag{87}$$

represent, just because of basic properties of d and $*$,

- a system of 1-st order linear partial differential equations
- for the fields **E** and **B**
- depending on what $*$ is actually there.

But there is actually more, there.
Namely, depending on the concrete choice of $*$,
- they are Lorentz, Galilei or Carroll invariant, respectively.

Why?

Why we always get correct equations (2)

Right from the construction of our $*_{\text{any}}$
we can see that it behaves nicely w.r.t. diffeomorphisms, i.e.

$$\boxed{f^*(*_{\text{any}}\alpha) = *_{f^*\text{any}}(f^*\alpha)} \tag{88}$$

And from this directly follows a useful observation:

$$\boxed{f^*\text{any} = \text{any} \qquad \Rightarrow \qquad f^* \, *_{\text{any}} = *_{\text{any}} \, f^*} \tag{89}$$

Why we always get correct equations (3)

In words:
Each particular version of Hodge star operator is invariant w.r.t. corresponding structure preserving diffeomorphisms (i.e. diffeomorphisms which preserve concrete choice of "any").

$$f = \text{Poincaré transformation} \quad \Rightarrow \quad f^* *_\eta = *_\eta f^* \tag{90}$$

$$f = \text{Galilei transformation} \quad \Rightarrow \quad f^* *_{h,\xi} = *_{h,\xi} f^* \tag{91}$$

$$f = \text{Carroll transformation} \quad \Rightarrow \quad f^* *_{\tilde{h},\tilde{\xi}} = *_{\tilde{h},\tilde{\xi}} f^* \tag{92}$$

Why we always get correct equations (4)

Therefore the system

$$d * F = 0 \qquad dF = 0 \tag{93}$$

is, for each particular version of $*$, invariant w.r.t. corresponding structure preserving diffeomorphisms.

And this is then clearly true
when the equations are rewritten in terms of fields $(\mathbf{E}, \mathbf{B})$.

So, by this method we necessarily get system of equations in terms of fields $(\mathbf{E}, \mathbf{B})$, possessing additional property:
it is Poincaré/Galilei/Carroll - invariant.

Summing up (1)

Let us recapitulate:

Galilean and Carrollian spacetimes:

- Galilean spacetime is a triple (M, h, ξ)
- Carrollian spacetime is a triple $(M, \tilde{h}, \tilde{\xi})$
- Galilei/Carroll spacetimes are "just" special cases
- (Lorentzian spacetime is a pair (M, g))

Summing up (2)

The Hodge star operator:

- Hodge star operator $*_g$ needs (M, g)
- So there is no Hodge star on Galilean/Carrollian spacetimes
- That's bad news for using differenial forms there
- Good news: There are (useful) analogs $*_{h,\xi}$ and $*_{\tilde{h},\tilde{\xi}}$
- One can perhaps use them in physics there

For Further Reading (1)

J.-M. Lévy Leblond
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