# Differential Geometry in Physics

An introductory exposition for **true non-experts**


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Syllabus of lectures held at University Regensburg, July 2007

Modern differential geometry provides a remarkably useful and, at the same time, simple language in which numerous ideas and concepts, crucial for theoretical physics, may be clearly stated and effectively treated. The lectures are aimed as a short introduction into this beautiful subject and focus primarily to essential *underlying ideas* (with many important topics completely omitted). Without attending the lectures themselves, these notes might provide insufficient information for adequate grasping of what is written here (at least since no pictures, which were drawn on the blackboard, are reproduced here). On the other hand, I hope the notes can serve as a helpful review of what was discussed for those who came and followed the lectures in person. I did my best in order to make them as useful as possible.

## 1. The playground - a smooth manifold

- Manifolds - why do we actually need them?
- Vector fields on manifolds
- Linear algebra of tensors and, in particular, of forms
- Tensor fields and forms on manifolds, pull-back, Lie derivative
- Differentiation and integration of forms, exterior derivative

**Manifolds - why do we actually need them?** In physics, we need to differentiate and integrate. Although we know quite well how to do it in $\mathbb{R}^n$, in real life often more general objects are encountered, which only *locally* look like $\mathbb{R}^n$ (they are endowed with just *local* coordinates $x^1, \dots, x^n$). They are called (smooth) manifolds (and denoted $M, N, \dots$).

Example 1.1. Configuration spaces of mechanical systems. Consider a pendulum ($M = S^2$), in particular the plane pendulum ($M = S^1 =$ the circle) or the double pendulum ($M = S^1 \times S^1 \equiv T^2 =$ torus). Configuration spaces are often given by a system of *constraints* imposed upon the system of point masses.

Example 1.2. Local coordinates on spheres may be given either by various angles or by stereographical projection.

Example 1.3. Local coordinates on real *projective space* $\mathbb{R}P^n$ (where points are given as lines in $\mathbb{R}^{n+1}$, which pass through the origin): start with any point $(x^1, x^2, \dots, x^{n+1})$ on the line. Then, if $x^1 \neq 0$, there is a unique point on the same line with coordinates $(1, x^2/x^1, \dots, x^{n+1}/x^1)$. Take $(u^1, \dots, u^n) \equiv (x^2/x^1, \dots, x^{n+1}/x^1)$ as local coordinates. If $x^1 = 0$, play the same game instead with any nonzero $x^k \neq 0$ (for each $k = 1, \dots, n+1$ we get such local coordinates, at least one of them suits for any point in $\mathbb{R}P^n$).

Example 1.4. Similar story holds for *complex* projective space $\mathbb{C}P^n$, (points being *complex* lines in $\mathbb{C}^{n+1}$, which pass through the origin). Real dimension is $2n$, here.
Example 1.5. The set of all (pure) states of a quantum system with $n$ levels is the manifold $\mathbb{C}P^{n-1}$ ("rays" in the Hilbert space $\mathbb{C}^n$). In particular for a qubit ($n = 2$, say, spin 1/2) we get $\mathbb{C}P^1$. This manifold happens to coincide with the sphere $S^2$ (usually called the *Bloch sphere* in this context; so a state of spin 1/2 may be characterized by a *unit* vector $\mathbf{n}$ - "spin directed along $\mathbf{n}$").
Example 1.6. Lie groups are smooth (even "analytical") manifolds. Sometimes quite simple (like $U(1) = S^1$, $SU(2) = S^3$), sometimes more complicated (already $SO(3) = \mathbb{R}P^3$, see below) and sometimes even more complicated.
Example 1.7. Parametrize rotations in $E^3$ by a pair $(\mathbf{n}, \alpha)$. Combine into a single $\mathbf{a} \equiv \alpha\mathbf{n}$. All rotations fill the ball of radius $\pi$ in the **a**-space. However, $(\mathbf{n}, \pi)$ is the same rotation as $(-\mathbf{n}, \pi)$. That's why the antipodal points on the *boundary* of the ball should be *identified* (glued together). We thus come to $\mathbb{R}P^3$. (This is more easily seen in two dimensions. Consider a *disc* with the antipodal points on the boundary - the circle - identified. Deform the disc into the upper hemisphere. Then there is a bijection between the points of the hemisphere and lines through the origin.)
Example 1.8. The equation of state $pV - RT = 0$ may be regarded as a constraint in $\mathbb{R}^3$, leaving a 2-dimensional manifold (surface) of allowed states.

**Vector fields on manifolds**. A vector $v$ in a point $x$ of a manifold $M$ may be identified with a *directional derivative* operator at that point or, in local coordinates $(x^1, \dots, x^n)$, as a first order differential operator $v = v^i \partial_i|_x$. A vector field is a collection of vectors on $M$ (one vector in each point of $M$). In coordinates we get $V = V^i(x)\partial_i$, so that it is given by a general first order differential operator. Then solutions of the equations $\dot{x}^i = V^i(x)$ correspond to integral curves of the field $V$. If $\gamma(t) \leftrightarrow x^i(t)$ is such a solution, then the map

$$\Phi_t : M \to M \qquad x \mapsto \gamma(t) \qquad \gamma(0) = x$$

is called the flow associated with the vector field $V$.
Example 1.9. The electric field of a point charge situated at the origin of Cartesian coodinate system reads (in associated spherical polar coordinates) $E \sim (1/r^2)\partial_r$
Example 1.10. The vector field $\partial_\varphi$ on the ordinary Euclidean plane $E^2$ generates the flow

$$(r, \varphi) \mapsto \Phi_t(r, \varphi) = (r, \varphi + t)$$

since the equations for integral curves are $\dot{r} = 0, \dot{\varphi} = 1$. We see that the flow consists of uniform rotations of the plane around the origin. In Cartesian coordinates the same result looks more elaborate:

$$(x, y) \mapsto \Phi_t(x, y) = (x \cos t - y \sin t, x \sin t + y \cos t)$$

Example 1.11. Equally looking vector field $\partial_\varphi$ in the ordinary Euclidean *space* $E^3$ generates the flow

$$(r, \vartheta, \varphi) \mapsto \Phi_t(r, \vartheta, \varphi) = (r, \vartheta, \varphi + t)$$

(since the equations for integral curves are $\dot{r} = 0, \dot{\vartheta} = 0, \dot{\varphi} = 1$, now). The flow consists of uniform rotations around the $z$-axis. In Cartesian coordinates this gives

$$(x, y, z) \mapsto \Phi_t(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z)$$

Example 1.12. Quantum mechanical operators $L_x, L_y, L_z$ of the *orbital angular momentum* of a single particle *are* (modulo a constant) *vector fields* in $E^3$. (From the preceding example we can see "what they do" - how their integral curves and flows look like. They simply rotate the space $E^3$ around the $x$, $y$ and $z$ axes respectively.)

**Linear algebra of tensors and, in particular, of forms**. For an arbitrary real $n$-dimensional vector space $L$, first consider its dual space $L^*$. Its elements, called covectors, are linear functions on $L$, i.e.

$$\alpha : L \to \mathbb{R} \qquad \text{such that} \qquad \alpha(v + \lambda w) = \alpha(v) + \lambda\alpha(w)$$

It is a vector space in its own right where the linear combination is defined by

$$(\alpha + \lambda\beta)(v) := \alpha(v) + \lambda\beta(v)$$

If the standard notation

$$\alpha(v) \equiv \langle \alpha, v \rangle$$

is adopted, then the map $\langle\ (\ .\ )\ ,\ (\ .\ )\ \rangle$ is linear in both slots. If $e_a$, $a = 1, \dots, n$ is a basis in $L$, then a general vector may be written as $v = v^a e_a$. The dual basis in $L^*$ is defined by

$$e^a\ ,\ a = 1, \dots, n \qquad \langle e^a, e_b \rangle := \delta^a_b \qquad \text{i.e.} \qquad \langle e^a, v \rangle := v^a$$

A general covector may be then written as

$$\alpha = \alpha_a e^a \qquad \text{where} \qquad \alpha_a = \langle \alpha, e_a \rangle$$

and for the pairing of a covector and a vector we get

$$\langle \alpha, v \rangle = \alpha_a v^a$$

Example 1.13. Consider a menu card in a mensa. It offers just three items, a soup (1 Euro), a Schnitzel (2 Euro) and a small bier (1/2 Euro). Each menu card may be regarded as a covector $\alpha$ (!). You take 1 soup, 2 portions of the Schnitzel (you did hard work $\Delta W$ in your research and you spent (a lot of) energy $\Delta E$) and a bier (= 2 small biers). This is a vector $v$. The total payment is given by the pairing

$$\langle \alpha, v \rangle = 1 \times 1 + 2 \times 2 + (1/2) \times 2 = 6 \text{ Euro}$$

Homework: Find natural mutually dual bases in $L$ and $L^*$. (Note: In a respectable mensa, like here in Regensburg, the dimension of $L^*$ is much higher than three and a typical $v$ necessarily contains many zeros in the natural basis.)

A map $t$, which assigns to $p$ vectors a real number

$$t : L \times \cdots \times L \to \mathbb{R} \qquad v, \dots, w \mapsto t(v, \dots, w) \in \mathbb{R}$$

and is *polylinear*, or equivalently *multilinear* (i.e. it depends linearly on each argument), is called a *tensor* of type $(0, p)$. The special cases are given by covectors ($p = 1$) and bilinear forms ($p = 2$). For $p = 0$ one identifies tensors of type $(0, 0)$ with real numbers (no vector available results in a number $\Rightarrow$ we necessarily know the number at the outset). The space $T^0_p(L)$ of tensors of type $(0, p)$ is a linear space in its own right where the linear combination is defined by

$$(t + \lambda s)(v, \dots, w) := t(v, \dots, w) + \lambda s(v, \dots, w)$$

A general tensor of type $(0, p)$ may be expressed in terms of components and basis tensors as follows:

$$t = t_{a\dots b} e^a \otimes \cdots \otimes e^b$$

where

$$t_{a\dots b} := t(e_a, \dots, e_b) \qquad \text{and} \qquad (e^a \otimes \cdots \otimes e^b)(v, \dots, w) := v^a \dots w^b$$

Then

$$t(v, \dots, w) = t_{a\dots b} v^a \dots w^b$$

The dimension of the space $T^0_p(L)$ is $n^p$.

*Forms* in $L$ enter the play naturally when (oriented) *volumes* of *parallelepipeds* are computed. Consider $n$ vectors $v, \dots, w$. The volume of the parallelepiped spanned on these vectors is clearly a real number, so that in order to compute the volume one needs to know a *map*

$$\alpha : L \times \cdots \times L \to \mathbb{R} \qquad v, \dots, w \mapsto \alpha(v, \dots, w) \in \mathbb{R}$$

From appropriate pictures in two and three-dimensional space one can easily see that the map is linear in each argument. This means that there is a *tensor* of type $(0, n)$ behind the computation of the volume. The tensor is, however, rather specific. In particular, the volume of *any degenerate* parallelepiped (such that the vectors $v, \dots, w$ fail to be linearly independent) should *vanish.* From this it follows that the tensor (map) $\alpha$ is to be *completely antisymmetric* (= *skew* symmetric; use linearity in $0 = \alpha(\dots, v + u, \dots, v + u, \dots)$). The space of completely antisymmetric tensors of type $(0, p)$ in $n$-dimensional space - *p-forms* in $L$ - is denoted by $\Lambda^p L^*$ and its dimension is $\binom{n}{p}$. The tensor product $\alpha \otimes \beta$ of two forms is not a form (just a tensor). If one, however, projects out the antisymmetric part of the result, one obtains a form. This specific product of forms is called the *wedge product* $\alpha \wedge \beta$. It is antisymmetric on basis 1-forms (i.e. $e^a \wedge e^b = -e^b \wedge e^a$) and any $p$-form may be expressed in terms of the basis built solely by wedge products:

$$\alpha = \frac{1}{p!} \, \alpha_{a\dots b} \, e^a \wedge \cdots \wedge e^b$$

Example 1.14. From this formula (and the property $e^a \wedge e^b = -e^b \wedge e^a$) one can see easily that nontrivial (= nonzero) $p$-forms in $n$-dimensional space can only exist for $p = 0, 1, \dots, n$ (otherwise there always exist at least two equal basis 1-forms in the monomial $e^a \wedge \cdots \wedge e^b$ (like $e^1 \wedge e^2 \wedge e^1 = -e^1 \wedge e^1 \wedge e^2 = 0$).

Example 1.15. If the formula is worked out in a 3-dimensional space $L$ with a basis $e_1, e_2, e_3$ (and the dual basis $e^1, e^2, e^3$ in $L^*$), the following most general $p$-forms arise:

$$\begin{array}{ll} p = 0 & \alpha = k_1 \\ p = 1 & \alpha = k_1 e^1 + k_2 e^2 + k_3 e^3 \\ p = 2 & \alpha = k_1 e^1 \wedge e^2 + k_2 e^2 \wedge e^3 + k_3 e^1 \wedge e^3 \\ p = 3 & \alpha = k_1 e^1 \wedge e^2 \wedge e^3 \end{array}$$

$k_i$ being arbitrary real numbers.

The presentation of forms in terms of sums of wedge products of basis 1-forms is by far the most suitable way of performing practical manipulations. The algorithm for, say, the wedge product of two forms reduces to the following steps:

- juxtapose the two forms
- multiply out all terms
- reshufle all constants to the left
- *delete* those of the resulting terms which contain some basis covector more than once (remember $e^a \wedge e^b = -e^b \wedge e^a \;\; \Rightarrow e^1 \wedge e^1 = e^2 \wedge e^2 = \cdots = 0$).

Example 1.16. Consider dim $L = 3$, a basis $e^1, e^2, e^3$ in $L^*$ and let

$$\alpha = 2e^1 + e^3 \qquad \beta = -3e^1 \wedge e^3 + 4e^2 \wedge e^3$$

Then

$$\begin{aligned}\alpha \wedge \beta &= (2e^1 + e^3) \wedge (-3e^1 \wedge e^3 + 4e^2 \wedge e^3) \\ &= -6 \underbrace{e^1 \wedge e^1}_{0} \wedge e^3 + 8e^1 \wedge e^2 \wedge e^3 - 3 \underbrace{e^3 \wedge e^1}_{-e^1 \wedge e^3} \wedge e^3 + 4 \underbrace{e^3 \wedge e^2}_{-e^2 \wedge e^3} \wedge e^3 = \\ &= 8e^1 \wedge e^2 \wedge e^3 + 3e^1 \wedge \underbrace{e^3 \wedge e^3}_{0} - 4e^2 \wedge \underbrace{e^3 \wedge e^3}_{0} = \\ &= 8e^1 \wedge e^2 \wedge e^3\end{aligned}$$

Note that the components of $p$-forms carry $p$ indices. In particular components of 1-forms are one-index objects and components of 2-forms are *two-index* objects, both indices run the same number of values (in $n$-dimensional space from 1 to $n$). This means that components of two-forms may be represented graphically as *antisymmetric square matrices.*

Example 1.17. Consider *four*-dimensional space $L$. Then in particular

$$\alpha = e^1 \wedge e^2 + e^3 \wedge e^4 \qquad \leftrightarrow \qquad \alpha_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

There is a simple albeit important operation on forms called the *interior product* of a form and a vector. For a given vector $v \in L$ it is the map $\alpha \mapsto i_v \alpha \equiv v \lrcorner\, \alpha$, consisting in plugging $v$ as the first argument into the $p$-form $\alpha$, i.e.

$$\begin{aligned}(i_v \alpha)(u, \dots, w) &:= \alpha(v, u, \dots, w) & \alpha \in \Lambda^p L^*,\, p \geq 1 \\ i_v \alpha &:= 0 & p = 0\end{aligned}$$

In spite of the simple definition it enjoys a lot of remarkable properties (useful for practical manipulations). For example:

$$\begin{aligned}(i_v \alpha)_{a \dots b} &= v^c \alpha_{ca \dots b} && \text{component formula} \\ i_v i_w &= -i_w i_v && \text{skew symmetry} \\ i_{v + \lambda w} &= i_v + \lambda i_w && \text{linearity} \\ i_v(\alpha + \lambda \beta) &= i_v \alpha + \lambda\, i_v \beta && \text{linearity (another one)} \\ i_v(\alpha \wedge \beta) &= (i_v \alpha) \wedge \beta + (\hat{\eta} \alpha) \wedge (i_v \beta) && \text{graded Leibniz rule}\end{aligned}$$

Example 1.18. Let $\alpha = -3e^1 \wedge e^3 + 4e^2 \wedge e^3$ and $v = 2e_1 - e_3$. Then

$$i_v\alpha = (2i_{e_1} - i_{e_3})(-3e^1 \wedge e^3 + 4e^2 \wedge e^3) = \dots = -6e^3 - 3e^1 + 4e^2$$

Still another operation on forms is the *Hodge star* operator (or *duality* operator) $* : \Lambda^p L^* \to \Lambda^{n-p} L^*$. One needs first a *metric tensor* and an *orientation* in $L$. Then the *metric volume form* $\omega \equiv \omega_{g,o}$ is constructed and finally the dual form is given by

$$(*\alpha)_{a\dots b} := \frac{1}{p!}\, \alpha^{c\dots d}\, \omega_{c\dots da\dots b} \qquad\qquad \alpha^{c\dots d} \equiv g^{cr} \dots g^{ds}\, \alpha_{r\dots s}$$

The duality means that if applied twice it gives (plus or minus) the identity operator, i.e. $** \alpha = \pm\alpha$. (The idea behind is the existence of the unique orthogonal complement to each subspace of $L$. The duality property results from the elementary fact that the complement to the complement is the original subspace itself.)
Example 1.19. Consider the mundane three dimensional Euclidean space $E^3$ and an orthonormal $e^1, e^2, e^3$. Then

$$*1 = e^1 \wedge e^2 \wedge e^3 \qquad * (e^1 \wedge e^2 \wedge e^3) = 1$$

$$*e^1 = e^2 \wedge e^3 \ \text{ etc.} \qquad * (e^1 \wedge e^2) = e^3 \ \text{ etc.}$$

**Tensor fields and forms on manifolds, Lie derivative**. One should simply apply linear algebra of tensors (in particular forms) to the special case $L =$ the *tangent* space (i.e. vectors are $v = v^i\partial_i|_x$ now and vector fields exist, $V = V^i(x)\partial_i$). What we need first is how *covectors* look like. The dual coordinate basis is

$$\langle dx^i, \partial_j\rangle = \delta^i_j \quad \text{ where } \quad \langle df, V\rangle := Vf \equiv V^i(x)(\partial_i f)$$

so that the *gradient* $df$ of a function $f$ (which is a *covector* field) is given by the formula

$$df = (\partial_i f)dx^i \qquad \text{just like the "total differential"}$$

Example 1.20. For polar coordinates in a plane this just says that

$$\langle dr, \partial_r\rangle = 1 \quad \langle dr, \partial_\varphi\rangle = 0 \quad \langle d\varphi, \partial_r\rangle = 0 \quad \langle d\varphi, \partial_\varphi\rangle = 1$$

$$df = (\partial_r f)dr + (\partial_\varphi f)d\varphi \qquad \langle df, V\rangle = V^r(r,\varphi)(\partial_r f)(r,\varphi) + V^\varphi(r,\varphi)(\partial_\varphi f)(r,\varphi)$$

Then a general tensor field of type $(0,p)$ on a manifold and a general $p$-form respectively may be locally (in a given coordinate patch) expressed as

$$A = A_{i\dots j}(x)dx^i \otimes \dots \otimes dx^j \qquad\qquad \alpha = (1/p!)\alpha_{i\dots j}(x)dx^i \wedge \dots \wedge dx^j$$

Forms on manifolds are sometimes called *differential forms* (so as to distinguish them from just forms in a fixed linear space $L$).
Example 1.21. On the standard two-dimensional unit sphere the following expressions correspond to the usual ("round" = rotationally invariant) *metric* tensor and the *volume* 2-form

$$g = d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi \qquad\qquad \omega = \sin\vartheta d\vartheta \wedge d\varphi$$

Tensor fields of type $(0,p)$ (in particular $p$-forms) may be *pulled-back* with respect to a map $f : M \to N$. First, individual vectors (not always vector fields) may be *pushed-forward* from $x \in M$ to $f(x) \in N$ (via mapping of curves associated with vectors). This is denoted as $v \mapsto f_* v$. Then, if a tensor $A$ of type $(0,p)$ is available at $f(x) \in N$, we can construct pulled-back tensor $f^* A$ of the same type at $x \in M$ as follows

$$(f^* A)(v, \dots, w) := A(f_* v, \dots, f_* w)$$

Let the map $f$ be in coordinates $y^a(x^i)$ and let $A$ be $A_{a\dots b}(y) dy^a \otimes \dots \otimes dy^b$. Then this reduces technically to the following algorithm: compute $dy^a(x) = (\partial y^a / \partial x^i) dx^i$ and replace $y$ by $y(x)$ in components and all $dy^a$ by $dy^a(x)$. That's all.

It turns out that pull-back exists for all smooth maps (contrary to push-forward of fields) and it plays eminent role in the whole stuff. For example, metric tensor can be pulled-back onto a manifold from a larger manifold (then $f$ is an embedding of the smaller manifold into the larger one; this was the case for the two-dimensional sphere mentioned in Example 1.21). Or, it is used to define the *Lie derivative* of a tensor field (in particular of a form). The idea goes as follows. Consider a flow $\Phi_t : M \to M$ generated by a vector field $V$ on $M$. Then, for any tensor field $A$ on $M$, we can compute its pull-back $\Phi_t^* A$. It depends, in general, on $t$. The derivative with respect to $t$ at $t = 0$ (so the *rate of change* at $t = 0$ of $A$ along integral curves of $V$) is called the Lie derivative of the tensor field $A$ with respect to the vector field $V$

$$\mathcal{L}_V A := (d/dt)_{t=0} \Phi_t^* A$$

From this definition one can readily derive the following useful expansion of the operator $\Phi_t^*$ itself:

$$\Phi_t^* = \hat{1} + t\mathcal{L}_V + \frac{t^2}{2!}\mathcal{L}_V\mathcal{L}_V + \frac{t^3}{3!}\mathcal{L}_V\mathcal{L}_V\mathcal{L}_V + \cdots \equiv e^{t\mathcal{L}_V}$$

It is then clear that

$$\mathcal{L}_V A = 0 \qquad \Leftrightarrow \qquad \Phi_t^* A = 0$$

so that vanishing of the Lie derivative of a tensor already guarantees invariance of the tensor with respect to the flow.

Example 1.22. We learned in Ex.1.11 that the vector field $\partial_\varphi$ generates rotations around the $z$ axis (in the sense that the flow of the field consists of uniform rotation of the space around the $z$ axis). This means that the condition

$$\mathcal{L}_{\partial_\varphi} A = 0$$

expresses the fact that the tensor field $A$ is *invariant* with respect to *rotations* around the $z$ axis.

There are simple rules (based on simple properties of the operator $\mathcal{L}_V$) for computing it in components. [1] Just to see an example, the system of first order partial differential equations

$$(\mathcal{L}_V g)_{ij} = V^k g_{ij,k} + V^k_{\ ,i} g_{kj} + V^k_{\ ,j} g_{ik} = 0 \qquad \textit{Killing equations}$$

[1] In my book there is a table for that in the form of a cook-recipe: "For preparation of $\mathcal{L}_V A$, first put on a bottom of a pan ... plus for each ... add ... plus for each ... add ...

is the component version of the abstract equation $\mathcal{L}_V g = 0$, saying that the Lie derivative of a metric tensor $g$ along $V$ *vanishes*, or, put it differently, that the metric tensor $g$ is *invariant* with respect to the flow $\Phi_t$ generated by $V$. (In these equations the unknown functions are $V^k(x)$, the components of the *Killing vector* $V$.) Looking for Killing vectors is thus looking for *symmetries* of a metric tensor (= of the space endowed with the metric tensor).

**Differentiation and integration of forms, exterior derivative**. First, differentiation. Differential forms, being just special tensor fields, can be Lie-differentiated along a vector field $V$. However, there exists also another way of differentiation of forms which is fully specific for forms. It is called the *exterior derivative.* If applied on a $p$-form $\alpha$, it produces $(p+1)$-form $d\alpha$. The most important properties of $d$ read

$$\begin{aligned} dd\alpha &= 0 && \text{for any } \alpha \\ d(\alpha \wedge \beta) &= (d\alpha) \wedge \beta + (-1)^p \alpha) \wedge (d\beta) && \text{for any } p\text{-form } \alpha \end{aligned}$$

plus the fact that on 0-forms (functions $f$) $df$ coincides with the *gradient* $df$ introduced before. Therefore $ddx^i = 0$ and one can show then that

$$d(f dx^i \wedge \cdots \wedge dx^j) = df \wedge dx^i \wedge \cdots \wedge dx^j = (\partial_k f) dx^k \wedge dx^i \wedge \cdots \wedge dx^j$$

This makes practical computation of $d\alpha$ *extremely simple.*
Example 1.23. For the 1-form $\alpha = ydx - xdy$ in the plane $\mathbb{R}^2$ we get the 2-form

$$\begin{aligned} d\alpha &= d(ydx - xdy) = dy \wedge dx - dx \wedge dy = -dx \wedge dy - dx \wedge dy \\ &= -2dx \wedge dy \end{aligned}$$

Example 1.24. For the 2-form $\beta = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ in $\mathbb{R}^3$ we get the 3-form

$$\begin{aligned} d\beta &= d(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\ &= dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy \\ &= 3dx \wedge dy \wedge dz \end{aligned}$$

It turns out that the exterior derivative commutes with any pull-back

$$d(f^* \alpha) = f^*(d\alpha)$$

and that *on forms* the Lie derivative can be expressed in terms of the exterior derivative and the interior product (on general tensors this makes no sense since the last two operations are not defined!). The explicit formula (known as the *Cartan's formula*) reads

$$\mathcal{L}_V = i_V d + d i_V$$

Differential forms can also be *integrated.* Actually (and perhaps surprisingly), all integrals you ever did in your life were (in a sense) integrals of forms! So you are already quite renowned experts in integral calculus of forms. Though, a few words about.

The idea of integration clearly shows that it is very *natural* to integrate *forms.* Why? Performing an integration, we first cut the domain of integration into (very very) small pieces. So small that they have the shape of a *parallelepiped* spanned on $p$ edge vectors in a point $x$ (if $p$ is the dimension of the domain). Now we are to assign a number to the vectors. This assignment is, however, exactly the *full-time job* of *$p$-forms* at $x$.

Official theory is a bit more involved but from a *completely pragmatic* point of view, we first should know that *always* $p$-forms are integrated over $p$-dimensional "domains" on manifolds (this is clear from the general considerations mentioned in the preceding paragraph). In order to integrate a $p$-form $\alpha$ over a $p$-dimensional domain $\mathcal{D}$, the following steps are needed. First, parametrize the domain $\mathcal{D}$ in terms of some $u^1, \ldots, u^p$, i.e. express all the coordinates $x^1, \ldots, x^n$ on the manifold in terms of $u^1, \ldots, u^p$. Then compute $dx^i = (\partial x^i / \partial u^a) du^a$. If $\alpha = f(x) dx^i \wedge \cdots \wedge dx^j$, replace $x$ by $x(u)$ and all $dx^i$ by $(\partial x^i / \partial u^a) du^a$. You obtain something like

$$F(u) du^a \wedge \cdots \wedge du^b$$

(Note that this is already a $p$-form in the "parametric space" and it is actually the *pull-back* of the original form $\alpha$.) Now *forget* about all wedge signs $\wedge$ and perform the usual multiple *Riemann* integral over relevant values of $u^1, \ldots, u^p$

$$\int F(u) du^a \ldots du^b$$

Example 1.25. We want to compute the integral of the 1-form $\alpha = y dx - x dy$ (the hero of Ex.1.23) over the circle $\bigcirc$ with radius $R$ centered at the origin of the plane $\mathbb{R}^2$. Parametrize the circle by $x = R \cos u, \ y = R \sin u$. Then

$$\begin{aligned} \alpha = y dx - x dy &\mapsto R \sin u d(R \cos u) - R \cos u d(R \sin u) \\ &= -R^2 (\sin^2 u + \cos^2 u) du = -R^2 du \end{aligned}$$

so that

$$\int_{\bigcirc} \alpha = -R^2 \int_0^{2\pi} du = -2\pi R^2$$

Example 1.26. We want to compute the integral of the 2-form $\beta = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ (the hero of Ex.1.24) over the upper hemisphere $S$ with radius $R$ centered at the origin of the space $\mathbb{R}^3$. Parametrize the hemisphere by $x = R \sin u \cos v, \ y = R \sin u \sin v, z = R \cos u$. Then $dx = R(\cos u \cos v du - \sin u \sin v dv)$ etc. and, after some struggle we get the simple result

$$\beta = R^3 \sin u du \wedge dv$$

Then

$$\int_S \beta = R^3 \int_0^{\pi/2} \int_0^{2\pi} \sin u du dv = R^3 \int_0^{\pi/2} \sin u du \int_0^{2\pi} dv = 2\pi R^3$$

A truly important and useful result in the integral calculus of forms is the *Stokes theorem*. It asserts that

$$\int_{\mathcal{D}} d\alpha = \int_{\partial \mathcal{D}} \alpha$$

Example 1.27. We want to verify the validity of the Stokes theorem for the 1-form $\alpha = y dx - x dy$ (the hero of Ex.1.23) and $\mathcal{D}$ equal to the 2-dimensional disc such that its boundary $\partial\mathcal{D}$ just coincides with the circle from Ex.1.23. Then, according to the result of Ex.1.23, the right-hand side of the Stokes formula gives $-2\pi R^2$. For the left-hand side we need the integral of $d\alpha \equiv -2dx \wedge dy$ over the disc which is (check! - parametrize etc.) just $-2\times$ the area, of the disc, being exactly $-2\pi R^2$.

Example 1.28. Try to formulate and solve similar exercise based on Example 1.24 instead of 1.23!

Another simple but, nevertheless, very useful result of integral calculus of differential forms concerns behavior of integrals with respect to *mappings* of manifolds. It says that (with some assumptions being fulfilled), given $f: M \to N$, $\mathcal{D}$ a domain in $M$ and $\alpha$ a form on $N$, it holds

$$\int_{f(\mathcal{D})} \alpha = \int_{\mathcal{D}} f^*\alpha$$

Instead of performing integration over the *image* of the domain, we can integrate over the domain itself, but we are to integrate the *pull-back* of the original form rather than the form itself.

## 2. Classical Hamiltonian mechanics

- Hamilton equations - how symplectic geometry emerges
- Liouville theorem and more general integral invariants
- Time dependent hamiltonian, action integral

**Hamilton equations - how symplectic geometry emerges.** Consider *Hamilton canonical equations*

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a} \qquad a = 1, \dots n$$

Restrict first to the case when the hamiltonian does not depend on time. It is a system of first order ordinary differential equations. It may therefore be regarded as a system for finding integral curves of an appropriate vector field in $\mathbb{R}^{2n}[q^a, p_a]$. We easily read off the field to be

$$\zeta_H = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a}$$

Introduce new coordinates by

$$z^i \equiv (z^1, \dots, z^n, z^{n+1}, \dots, z^{2n}) := (q^1, \dots, q^n, p_1, \dots, p_n) \equiv (q^a, p_a)$$
$$i = 1, \dots, 2n; \;\; a = 1, \dots, n$$

(i.e. forget about „unnatural" division of coordinates into two parts labeled with different letters and use a single letter $z$ instead with indices running from

1 to the dimension of the manifold, as is usual). In the new coordinates we can see the structure of the field more clearly:

$$\zeta_H = (dH)_j \mathcal{P}^{ji} \partial_i$$

where

$$\mathcal{P}^{ij}(z) = \begin{pmatrix} 0_n & -\mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix} = -\mathcal{P}^{ji}(z)$$

so that the Hamilton equations are

$$\dot{z}^i = \zeta^i_H(z) =: (\partial_j H)\mathcal{P}^{ji} \equiv (dH)_j \mathcal{P}^{ji} \qquad\qquad i = 1, \dots, 2n$$

The last equation resembles "raising of an index" procedure on $dH$, known from relativity course ($(dH)_j \mapsto (dH)_j \mathcal{P}^{ji}$). The only difference is that the matrix $\mathcal{P}^{ji}$ is *antisymmetric* (contrary to relativity, where metric tensor was *symmetric*). It turns out, however, that this difference is not essential (in the sense that *it is* possible to raise indices with an antisymmetric tensor, too). What really matters is the matrix were *regular* (but $\mathcal{P}^{ji}$ *is* regular, its determinant being equal to 1 - check!).

Define now the inverse matrix (except for the sign) $\omega_{kj}$

$$\mathcal{P}^{ik}\omega_{kj} = -\delta^i_j$$

and apply this matrix to the last equation:

$$\zeta^i_H = (dH)_j \mathcal{P}^{ji} \qquad\quad | \times \omega_{ik}$$

We get

$$\zeta^i_H \omega_{ik} = - \ (dH)_k$$

Now if we introduce the 2-form

$$\omega = \frac{1}{2}\omega_{ij} dz^i \wedge dz^j = dp_a \wedge dq^a$$

the last equation can be written (at last!) in a coordinate-free ("geometrical") way as

$$i_{\zeta_H}\omega \equiv \zeta_H \lrcorner\, \omega = -dH$$

In general, the vector field $\zeta_f$ defined by the equation

$$i_{\zeta_f}\omega = -df \qquad\quad \text{(i.e. in components as } \zeta^i_f \omega_{ik} = - \ (df)_k)$$

is called the *hamiltonian field* generated by the function $f$. We see that, from the geometrical point of view, Hamilton equations are just equations for integral curves of hamiltonian vector field generated by the function $H$ (the Hamiltonian of the system):

$$\dot{\gamma} = \zeta_H \qquad\quad i_{\zeta_H}\omega = -dH$$

Note also that the 2-form $\omega$ enables one to express the *Poisson bracket* of two functions $f, g$ in a coordinate-free form as

$$\{f, g\} = \omega(\zeta_f, \zeta_g) = \zeta_f g = -\zeta_g f$$

All the well-known properties of Poisson bracket are immediately seen from this formulas *except for* the *Jacobi identity.* Here the check is more technical (well, just a line provided one knows so called Cartan formulas for computation of $d$; it turns out that the identity is just reexpressing of $d\omega = 0$).

The differential 2-form $\omega$ introduced above has the following two properties:

- it is *closed* (meaning that $d\omega = 0$)
- it is *non-degenerate* (meaning that the matrix of its components is regular)

*Any* closed and non-degenerate 2-form on a manifold $M$ is called *symplectic form* and a pair $(M, \omega)$ is a *symplectic manifold.* (According to the theorem of *Darboux*, one can introduce on *any* symplectic manifold *local* coordinates $(q^a, p_a)$ such that $\omega$ takes the *canonical* form $dp_a \wedge dq^a$. A change of coordinates which *preserves* this canonical form (so that $dp_a \wedge dq^a \mapsto d\hat{p}_a \wedge d\hat{q}^a$) is called *canonical transformation.*)

We see that any phase space is a symplectic manifold. On the contrary, there are symplectic manifolds which do not have the structure of a *usual* phase space. By "usual" phase spaces I mean those coming from *configuration* spaces by "adding generalized momenta", as it is standard in analytical mechanics. (There is a geometrical construction producing such a phase space from a configuration space. If $M$ is the configuration space then the phase space is $T^*M$, called the *cotangent bundle* of $M$. We will not need it, however.)

Example 2.1. Consider ordinary 2-sphere $S^2$ endowed with the usual *volume form* $\omega$ (integral of $\omega$ over a domain on the sphere gives the volume (= area) of the domain). Now $\omega$ is a *closed* 2-form (trivial: $d\omega$ is a 3-form and there are no nonzero 3-forms on a 2-dimensional manifold). It is also nondegenerate (check its matrix). This means that $(S^2, \omega)$ *is* a symplectic manifold. However, no (1-dimensional) configuration space exists such that its usual phase space is $S^2$. (A simple argument to see this is that $S^2$ is *compact* whereas any $T^*M$, the usual phase space, is non-compact (because of $p$'s).)

Example 2.2. We want to write down explicitly Hamilton equations on the sphere from Ex.2.1. In coordinates $\vartheta, \varphi$ we have $\omega = \sin\vartheta d\vartheta \wedge d\varphi$. Let $\zeta_H = A\partial_\vartheta + B\partial_\varphi$ (with functions $A, B$ to be determined). Then $i_{\zeta_H}\omega = \sin\vartheta(Ad\varphi - Bd\vartheta)$. This should be equal to $-dH = -((\partial_\vartheta H)d\vartheta + (\partial_\varphi H)d\varphi)$. Equating the expressions standing by coordinate basis $d\vartheta, d\varphi$ we get $A\sin\vartheta = -\partial_\varphi H, B\sin\vartheta = \partial_\vartheta H$ so that $\zeta_H = (-\partial_\varphi H/\sin\vartheta)\partial_\vartheta + (\partial_\vartheta H/\sin\vartheta)\partial_\varphi$. Then the Hamilton equations read

$$\dot{\vartheta} = -\frac{1}{\sin\vartheta}\frac{\partial H}{\partial\varphi} \qquad \dot{\varphi} = \frac{1}{\sin\vartheta}\frac{\partial H}{\partial\vartheta}$$

We can see from the result of Ex.2.2 that Hamilton equations came out in a somewhat unusual form (but in spite of it they *are* correct!). The reason is that the coordinates $\vartheta, \varphi$ are not "canonical" (a coordinate and its conjugate momentum). Technically speaking we have $\omega = \sin\vartheta d\vartheta \wedge d\varphi$ whereas the general "canonical" expression $\omega = dp_a \wedge dq^a$ gives simply $\omega = dp \wedge dq$ on a 2-dimensional phase space. By inspection we see that if we introduce $q := \varphi, p := -\cos\vartheta$, then already $\omega = \sin\vartheta d\vartheta \wedge d\varphi = dp \wedge dq$ and the reader can check that the equations of motion from Ex.2.2 indeed acquire their well known structure

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

**Liouville theorem and more general integral invariants**. Use the Cartan's formula $\mathcal{L}_V = i_V d + d i_V$ for computation of the Lie derivative of a *sym-*

*plectic* form $\omega$ with respect to any *hamiltonian* field $\zeta_f$. We get

$$\mathcal{L}_{\zeta_f}\omega = i_{\zeta_f}d\omega + di_{\zeta_f}\omega = 0 + d(-df) = -ddf = 0$$

This means that the symplectic form remains unchanged under action of the *flow* $\Phi_t$ of a hamiltonian field (called, perhaps not too much surprisingly, a *hamiltonian flow*)

$$\Phi_t^*\omega = \omega$$

Moreover, the behavior of pull-back on wedge product of forms (i.e. $f^*(\alpha\wedge\beta) = f^*\alpha \wedge f^*\beta$) yields

$$\Phi_t^*(\omega\wedge\cdots\wedge\omega) = \omega\wedge\cdots\wedge\omega$$

so that any "exterior power" $\omega^k$ of the symplectic form is invariant with respect to the flow of any hamiltonian field. This statement has important consequences for certain integrals. Since $\omega^k$ is a $2k$-form (on a $2n$-dimensional phase space $(M,\omega)$), it can be integrated over a $2k$-dimensional domain. Let the integral be $\int_{\mathcal{D}}\omega^k$. Now imagine our domain starts to flow along the integral curves of the hamiltonian field. We obtain a time-dependent domain $\mathcal{D}(t) \equiv \Phi_t(\mathcal{D})$, the $\Phi_t$-image of the original domain $\mathcal{D} \equiv \mathcal{D}(0)$ at time 0. In principle the integral $\int_{\mathcal{D}(t)}\omega^k$ should be time-dependent, since the domain of integration depends on time. However, it turns out *it is not*. Indeed, recall the simple rule for computing integrals over *image* of a domain, namely $\int_{f(\mathcal{D})}\alpha = \int_{\mathcal{D}} f^*\alpha$. This gives here

$$\int_{\Phi_t(\mathcal{D})}\omega\wedge\cdots\wedge\omega = \int_{\mathcal{D}}\Phi_t^*(\omega\wedge\cdots\wedge\omega) = \int_{\mathcal{D}}\omega\wedge\cdots\wedge\omega$$

And that's all! (Perhaps it's time to pay a tribute to the astonishing effectiveness of differential forms in hamiltonian mechanics, now.)

What we've learned is that there exists a sequence $I_2, I_4, \ldots, I_{2n}$ of integrals

$$I_2 := \int_{\mathcal{D}}\omega \qquad I_4 := \int_{\mathcal{D}}\omega^2 \qquad \ldots \qquad I_{2n} := \int_{\mathcal{D}}\omega^n$$

(the subscript $2k$ on $I_{2k}$ meaning that the integral is performed over $2k$-dimensional domain) with a remarkable property that the value of any of them does dot change when the domain evolves in time under the influence of a hamiltonian flow. Note, that all this holds for *any* domain (just with right dimension) as well as for *any* generator $f$ (the "hamiltonian" behind the field $\zeta_f$ corresponding to the flow $\Phi_t$. The integrals are called *Poincaré-Cartan integral invariants*. The far best known is the last one. The property of invariance of this particular integral is known as the *Liouville theorem* - it states that the *phase volume* is invariant under the flow of any hamiltonian field. (Actually the proof shows that all the integrals are invariant with respect to any *symplectomorphism* - a map $f : M \to M$ such that it preserves $\omega$ in the sense $f^*\omega = \omega$. $\Phi_t$ represents a one-parameter family (group) of symplectomorphisms.)

It often happens that the symplectic form $\omega$ is *exact*, meaning that $\omega = d\theta$ for some 1-form $\theta$.
(Actually this is, according to the *Poincaré lemma*, always the case *locally*, but it sometimes also happens globally, with a single $\theta$ for the whole $M$. It *is so*, for example, on any "ordinary" phase space, where $M = T^*N$ with $N$ being the configuration space. But it *is not* so, say, on the sphere $S^2$.)

Then consider the 3-form $\theta \wedge \omega$. If we proceed exactly as we did before with the exception that now the (3-dimensional) domain $\mathcal{D}$ represents the *boundary* of some (4-dimensional) $\mathcal{U}$ (i.e. $\mathcal{D} = \partial\mathcal{U}$), we get, using the Stokes theorem, commutation of $d$ with pull-back and $d(\theta \wedge \omega) = \omega \wedge \omega$

$$\begin{aligned}\int_{\Phi_t(\partial\mathcal{U})} \theta \wedge \omega &= \int_{\partial\mathcal{U}} \Phi_t^*(\theta \wedge \omega) = \int_{\mathcal{U}} d\Phi_t^*(\theta \wedge \omega) = \int_{\mathcal{U}} \Phi_t^*(\omega \wedge \omega) \\ &= \int_{\mathcal{U}} \omega \wedge \omega = \int_{\mathcal{U}} d(\theta \wedge \omega) = \int_{\partial\mathcal{U}} \theta \wedge \omega \end{aligned} \tag{1}$$

This means that there is also another sequence $I_1, I_3, \ldots, I_{2n-1}$ of integrals

$$I_1 := \int_{\partial\mathcal{U}} \theta \qquad I_3 := \int_{\partial\mathcal{U}} \theta \wedge \omega \qquad \ldots \qquad I_{2n-1} := \int_{\partial\mathcal{U}} \theta \wedge \omega^{n-1}$$

which serve as integral invariants, but in this case only for domains which are *boundaries.* They are called *relative* Poincaré-Cartan integral invariants.

**Time dependent hamiltonian, action integral**. So far we assumed that our Hamiltonian does not depend on time. When this is *not* the case, the geometrical picture underlying Hamilton equations should be modified, since $H(q, p, t)$ says we are no longer on a $2n$-dimensional manifold. A possible extension of the manifold is simply the *product* of a symplectic manifold $(M, \omega)$ with the time axis $\mathbb{R}$. (Natural coordinates on $M \times \mathbb{R}$) are just $(q^1, \ldots, q^n, p_1, \ldots, p_n, t)$, as needed.)

Rewrite the Hamilton equations as

$$dq^a - \frac{\partial H}{\partial p_a}\, dt = 0 \qquad dp_a + \frac{\partial H}{\partial q^a}\, dt = 0 \qquad a = 1, \ldots n$$

What precisely we mean, however, by these formulas? Consider the simpler case, the differential equation $\dot{x} = 1$. Its solutions, $x(t) = t +$ const., represent slant lines in the plane $\mathbb{R}^2[x, t]$. Now rewrite the equation as $dx - dt = 0$. Is it possible to regard the left-hand side as a *differential form* in $\mathbb{R}^2[x, t]$? Yes, it is. Plug the tangent vector $\dot{\gamma}$ of a general curve $\gamma(\tau) \leftrightarrow (x(\tau), t(\tau))$ into the 1-form $\alpha \equiv dx - dt$. Get $\langle \alpha, \dot{\gamma} \rangle = \dot{x} - \dot{t}$. (Recall that $\dot{\gamma}$ is just a symbol for the tangent vector whereas $\dot{x}$ and $\dot{t}$ represent the ordinary derivatives of the functions $x(\tau)$ and $t(\tau)$). Now demand this to *vanish.* Get $\dot{x} = \dot{t}$, or $\dot{\gamma} = \dot{x}(\partial_x + \partial_t)$. Notice, that the *direction* of $\dot{\gamma}$ coincides in each point of the plane (irrespective of particular value of $\dot{x}$, i.e. of the parametrization of the curve) with the direction given by the *solutions* of of the equation $\dot{x} = 1$. This shows that the *solutions* of the differential equation are encoded into the *null-space* (the subspace of all vectors annihilated by the form) of the corresponding 1-form $\alpha = dx - dt$.

Now return to Hamilton equations. Introduce a system of $2n$ differential 1-forms defined on the *extended phase space* $M \times \mathbb{R}$

$$\alpha^a := dq^a - \frac{\partial H}{\partial p_a}\, dt \qquad \beta_a := dp_a + \frac{\partial H}{\partial q^a}\, dt$$

In terms of these 1-forms the Hamilton equations read

$$\alpha^a = 0 \qquad \beta_a = 0$$

This should, of course, *not* to be understood as that the forms *identically* vanish on $M \times \mathbb{R}$. They clearly don't. Rather, in the spirit of the simple example above, we should first consider all curves $\gamma(\tau) \leftrightarrow ((q^a(\tau), p_a(\tau), t(\tau))$ on $M \times \mathbb{R}$ and the above equations are then to be regarded in the sense of vanishing (all of them at once!) *on the tangent vector* to the *solution curve*

$$\langle \alpha^a, \dot{\gamma} \rangle = 0 \qquad \langle \beta_a, \dot{\gamma} \rangle = 0$$

(When working out these equations in detail we actually get, by the way

$$\dot{q}^a = \dot{t}\,\frac{\partial H}{\partial p_a} \qquad \dot{p}_a = -\dot{t}\,\frac{\partial H}{\partial q^a} \qquad a = 1, \ldots n$$

where the dot means the derivative with respect to an arbitrary parameter $\tau$. Choosing $\tau$, as is usual, to be just $t$ (which is one of the coordinates here!), we get $\dot{t} = 1$ and finally the standard form of Hamilton equations.)

The point of all this procedure is to *distinguish the solutions* of Hamilton equations by specifying the 1-dimensional subspace in the $(2n+1)$-dimensional tangent space in each point. We already succeeded in this effort in the sense that the subspace is characterized as a *common null-space* of the 1-forms $\alpha^a$ and $\beta_a$. This can be, however, done in even more economical way. Define the 2-form $\beta_a \wedge \alpha^a$. Then

$$i_{\dot{\gamma}}(\beta_a \wedge \alpha^a) = (i_{\dot{\gamma}}\beta_a) \wedge \alpha^a - \beta_a \wedge (i_{\dot{\gamma}}\alpha^a) \equiv \langle \beta^a, \dot{\gamma} \rangle \; \alpha^a - \langle \alpha^a, \dot{\gamma} \rangle \; \beta_a$$

Since (as one can show) the 1-forms $\alpha^a$ and $\beta_a$ are linearly independent in each point of $M \times \mathbb{R}$, this means that it's the same thing to simultaneously annihilate all of the 1-forms $\alpha^a$ and $\beta_a$ and to annihilate *a single* 2-form $\beta_a \wedge \alpha^a$

$$i_{\dot{\gamma}}(\beta_a \wedge \alpha^a) = 0 \qquad \Leftrightarrow \qquad \{ \; \langle \alpha^a, \dot{\gamma} \rangle = 0 = \langle \beta_a, \dot{\gamma} \rangle \; , \; a = 1, \ldots, n \; \}$$

This computation shows that there is a simple way to single out the key subspace of interest in each point of $M \times \mathbb{R}$ (the subspace of the solution curve passing through the point). It is given as the null-subspace of the 2-form $\beta_a \wedge \alpha^a$. This actually means that the Hamilton equations may also be written as $i_{\dot{\gamma}}(\beta_a \wedge \alpha^a) = 0$. Although this *is* the case, one usually rewrites it in a different form since an important point of the stuff is still hidden yet. Compute explicitly $\beta_a \wedge \alpha^a$:

$$\begin{aligned} \beta_a \wedge \alpha^a &= (dp_a + \frac{\partial H}{\partial q^a}\; dt) \wedge (dq^a - \frac{\partial H}{\partial p_a}\; dt) = dp_a \wedge dq^a - dH \wedge dt \\ &= d(p_a dq^a - H dt) \end{aligned}$$

The important point here is that the 2-form $\beta_a \wedge \alpha^a$ is *exact*, i.e. that there exists a 1-form $\Theta := p_a dq^a - H dt$ on $M \times \mathbb{R}$ such that $\beta_a \wedge \alpha^a = d\Theta$. (This may be true just locally - we did coordinate computations. In general the 2-form is only guaranteed to be *closed*, i.e. $d(\ldots) = 0$.) So the final version of the Hamilton equations reads

$$i_{\dot{\gamma}} d\Theta = 0 \qquad \Theta := p_a dq^a - H dt$$

If the hamiltonian actually *does not* depend on time, one can easily "project" the good old symplectic picture on $M$ out of the just obtained more general equation on $M \times \mathbb{R}$ (not done here).

In order to find an appropriate *variational principle* leading to Hamilton equations, realize that, because an extremal *curve* is to be determined from it, the action integral necessarily has to be an integral of a 1-*form*. Now, there is only a single 1-form available, $\Theta \equiv p_a dq^a - H dt$. So try

$$S[\gamma] := \int_\gamma \Theta \equiv \int_{t_1}^{t_2} (p_a(t)\dot{q}^a(t) - H(q(t), p(t), t))dt$$

The check that this action integral indeed leads to Hamilton equations is straightforward (it is done in each undergraduate textbook on analytical mechanics). It is more instructive, however, to see this in a more geometrical way. We want to understand why a curve which obeys $i_{\dot\gamma} d\Theta = 0$ necessarily extremizes at the same time the integral $S[\gamma] := \int_\gamma \Theta$.

Let $\gamma(t)$ be a solution at the interval between $t_1$ and $t_2$. Produce a small variation, $\gamma(t) \mapsto \gamma_\epsilon(t)$ making use of the flow $\Phi_t$ of a vector field $W$ (so far let the endpoints held fixed). What we get is a 2-dimensional narrow strip $\Sigma$ bounded by the two curves: $\partial\Sigma = \gamma - \gamma_\epsilon$ . Consider integral of the 2-form $d\Theta$ over the strip, $\int_\Sigma d\Theta$. The integral, according to the Stokes' theorem, just equals (minus of) the variation of the action

$$\int_\Sigma d\Theta = \int_{\partial\Sigma} \Theta = S[\gamma] - S[\gamma_\epsilon] \equiv -\delta S$$

We want to understand why this integral *vanishes*. Well, imagine the actual summation of infinitesimal contributions. Because of the shape of the domain of integration (= the narrow strip), a typical contribution is proportional to $(d\Theta)(\dot\gamma, W)$. But this may be written as $(i_{\dot\gamma} d\Theta)(W)$ and this expression *vanishes* simply because of $i_{\dot\gamma} d\Theta = 0$. We are done. Finally, note that the idea in fact does not need the assumption of fixed endpoints, provided that variations at the endpoints do not contribute to the integral $\int_{\partial\Sigma} \Theta$. This is the case if, as is standardly assumed, only variations "along the $p$-direction" are allowed (i.e. $\delta p \neq 0$ are allowed at the endpoints).

## 3. Classical electrodynamics

- Usual vector analysis in terms of forms - oh, it becomes easier!
- Forms in Minkowski space - 4 dimensions versus 3+1
- Maxwell equations via forms - the most succinct formulation of the EM laws
- Potentials, gauge transformations, action integral and all that

**Vector analysis in $E^3$ in terms of forms**. In vector analysis we encounter integrals of the form $\int \mathbf{A}.d\mathbf{r}$, $\int \mathbf{B}.d\mathbf{S}$ and $\int f dV$ (line, surface and volume integrals). Remember, however, that lines, surfaces and volumes are 1, 2 and

3-dimensional domains respectively and so, treating the integrals as integrals of differential forms, we have to have necessarily a 1-form, a 2-form and a 3-form respectively under the integral sign in the three cases under consideration.

The Euclidean space $E^3$, being a 3-dimensional manifold, admits the existence of (nonzero) differential forms of degrees 0, 1, 2 and 3. The most general expressions for those forms are as follows

$$f \qquad \mathbf{A}.d\mathbf{r} \equiv A_i dx^i = A^j g_{ji} dx^i \qquad \mathbf{B}.d\mathbf{S} \equiv B^i dS_i \qquad hdV$$

Put another way, one can take

$$1 \qquad dx^i \qquad dS_i \qquad dV$$

as *a basis* for differential forms of degrees 0,1,2 and 3 on $E^3$ respectively. Now, what exactly we mean by $dS_i$ and $dV$? The most useful answer may be given in terms of the following results:

$$*f = fdV \qquad *(hdV) = h$$
$$*(\mathbf{B}.d\mathbf{S}) = \mathbf{B}.d\mathbf{r} \qquad *(\mathbf{A}.d\mathbf{r}) = \mathbf{A}.d\mathbf{S}$$

This may serve as a *definition* of $dS_i$ and $dV$, provided that we know what 1 and $dx^i$ means (we do). The general formulas (valid in arbitrary coordinates in $E^3$) are

$$dS_i = \frac{1}{2}\sqrt{|g|}\epsilon_{ijk} dx^j \wedge dx^k \qquad dV = \sqrt{|g|} dx^1 \wedge dx^2 \wedge dx^3$$

(where $g$ is the determinant of the matrix $g_{ij}$). They might look a bit too complicated but in concrete coordinates everything gets simple.

Example 3.1. In cartesian coordinates ($g_{ij} = \delta_{ij}$) we obtain the well-known expressions

$$\mathbf{B}.d\mathbf{S} \equiv B^i dS_i = B^x dy \wedge dz + B^y dz \wedge dx + B^z dx \wedge dy$$

$$fdV = fdx \wedge dy \wedge dz$$

Example 3.2. In spherical polar coordinates ($g_{rr} = 1$, $g_{\vartheta\vartheta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2 \vartheta$) we get

$$\mathbf{B}.d\mathbf{S} \equiv B^i dS_i = r^2 \sin\vartheta (B^r d\vartheta \wedge d\varphi + B^\vartheta d\varphi \wedge dr + B^\varphi dr \wedge d\vartheta$$

$$fdV = fr^2 \sin\vartheta dr \wedge d\vartheta \wedge d\varphi$$

(Warning: The components displayed here are "coordinate" components of the quantities under consideration. More often "orthonormal" components are used. The two possibilities differ if in given coordinates $g_{ij} \neq \delta_{ij}$)

Now recall that the operator $d$ acts on forms (exterior derivative) raising the degree of forms in one unit. At the same time, we see that all forms are "parametrized" by either *scalar* fields (0- and 3-forms) or by *vector* fields (1- and 2-forms). That means that the operator $d$, when acting on forms with degrees 0,1 and 2, manifests itself in three differently looking ways, as an operator of type scalar $\mapsto$ vector, vector $\mapsto$ vector and vector $\mapsto$ scalar respectively. These three manifestations of $d$ are nothing but good old grad , rot and div well-known from the vector analysis. Explicitly it holds

$$df = (\mathrm{grad}\, f).d\mathbf{r} \equiv (\nabla f).d\mathbf{r} \qquad d(\mathbf{A}.d\mathbf{r}) = (\mathrm{rot}\,\mathbf{A}).d\mathbf{S}$$
$$d(\mathbf{B}.d\mathbf{S}) = (\mathrm{div}\,\mathbf{B})dV \qquad d(hdV) = 0$$

This gives immediately three (differently looking) particular cases of (the same general) *Stokes theorem*:

$$\int_C (\nabla f).d\mathbf{r} = f(B) - f(A)$$
$$\oint_{\partial S} \mathbf{A}.d\mathbf{r} = \int_S (\mathrm{rot}\,\mathbf{A}).d\mathbf{S}$$
$$\oint_{\partial D} \mathbf{A}.d\mathbf{S} = \int_D (\mathrm{div}\,\mathbf{A})dV$$

**Forms in Minkowski space - 4 dimensions versus 3+1**. Now add the *fourth* coordinate (dimension) $t = x^0$ to $E^3$ and end with $E^{1,3}$ - the *Minkowski space* ($(1,3)$ means that the metric tensor has the form $\eta = dt \otimes dt - (dx \otimes dx + dy \otimes dy + dz \otimes dz)$ with 1 plus and 3 minuses). What do general differential forms look like now? Consider a $p$-form $\alpha$. As is always the case, it is a sum of terms such that each of them contains the wedge product of exactly $p$ differentials. Each particular differential $dx^\mu$, $\mu = 0,1,2,3$, is present *at least once* (since the wedge product vanishes for equal 1-forms). This holds, in particular, for the differential $dt$. This simple reasoning shows that *any p-form* in Minkowski space may be uniquely decomposed as follows

$$\alpha = dt \wedge \hat{s} + \hat{r}$$

where (that's the point) neither the $(p-1)$-form $\hat{s}$ nor the $p$-form $\hat{r}$ contains $dt$. Such forms are called *spatial* since they only use spatial (from "space" as opposed to "time") differentials $dx, dy, dz$.

Note that this decomposition is *reference frame dependent.* Another observer uses $(t', x', y', z')$ instead of $(t, x, y, z)$ and his (sometimes even her) decomposition of the same $\alpha$ then necessarily differs from the first one

$$\alpha = dt' \wedge \hat{s}' + \hat{r}'$$

Example 3.3. Perform the decomposition for a general 2-form. We get

$$\begin{aligned}
\alpha &= \frac{1}{2}\alpha_{\mu\nu} dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}\alpha_{00} dx^0 \wedge dx^0 + \frac{1}{2}\alpha_{0i} dx^0 \wedge dx^i + \frac{1}{2}\alpha_{i0} dx^i \wedge dx^0 + \frac{1}{2}\alpha_{ij} dx^i \wedge dx^j \\
&= dt \wedge (\alpha_{0i} dx^i) + \frac{1}{2}\alpha_{ij} dx^i \wedge dx^j \\
&= dt \wedge (\hat{s}_i dx^i) + \frac{1}{2}\hat{r}_{ij} dx^i \wedge dx^j \equiv dt \wedge \hat{s} + \hat{r}
\end{aligned}$$

Since spatial forms only use differentials $dx, dy, dz$, they may be parametrized in exactly the same way as was the case for general forms in vector analysis (i.e.

for forms in $E^3$). So, in particular, a general 2-form in Minkowski space may be written as

$$\alpha = dt \wedge \mathbf{a}.d\mathbf{r} + \mathbf{b}.d\mathbf{S}$$

i.e. parametrized in terms of *two vector fields in* $E^3$, $\mathbf{a}$ and $\mathbf{b}$. Note, however, that these fields also may depend on the coordinate $t$. This results in the following expression for the exterior derivative of $\alpha$

$$d(dt \wedge \mathbf{a}.d\mathbf{r} + \mathbf{b}.d\mathbf{S}) = dt \wedge (\partial_t \mathbf{b} - \operatorname{rot} \mathbf{a}).d\mathbf{S} + (\operatorname{div} \mathbf{b})dV$$

**Maxwell equations via forms - all EM laws in just a line**. The last 3-form *vanishes* exactly when

$$\operatorname{rot} \mathbf{a} - \partial_t \mathbf{b} = 0 \qquad \operatorname{div} \mathbf{b} = 0$$

Now compare this result with the second series of Maxwell equations in vacuum (i.e. the homogeneous half of the equations),

$$\operatorname{rot} \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0} \qquad \operatorname{div} \mathbf{B} = 0$$

Everybody sees that if we define the 2-*form of the electromagnetic field*

$$F := dt \wedge \mathbf{E}.d\mathbf{r} - \mathbf{B}.d\mathbf{S}$$

then

$$dF = 0 \qquad \Leftrightarrow \qquad \text{homogeneous Maxwell equations hold}$$

Example 3.4. If we write the 2-form $F$ in standard coordinate form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

we get for the component matrix the standard well-known expression

$$F_{0i} = E_i$$

$$F_{ij} = -\epsilon_{ijk} B_k \qquad \text{i.e.} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Example 3.5. Transformation of fields $\mathbf{E}, \mathbf{B}$. Consider in particular the fields $\mathbf{E} = (0, E, 0), \mathbf{B} = (0, 0, B)$ in Cartesian coordinates $(t, x, y, z)$. Let the new frame be related to the old one by

$$\begin{aligned} t &= t' \cosh \alpha + x' \sinh \alpha \\ x &= t' \sinh \alpha + x' \cosh \alpha \\ y &= y' \\ z &= z' \end{aligned} \tag{2}$$

(i.e. the primed one moves uniformly in the $x$ direction). In general, *the same* $F$ is represented in both coordinate systems as

$$F = dt \wedge \mathbf{E}.d\mathbf{r} - \mathbf{B}.d\mathbf{S} = dt' \wedge \mathbf{E}'.d\mathbf{r}' - \mathbf{B}'.d\mathbf{S}'$$

Then, in our particular case, we get

$$
\begin{aligned}
F &= dt \wedge E dy - B dx \wedge dy \\
&= (dt' \cosh\alpha + dx' \sinh\alpha) \wedge E dy' - B(dt' \sinh\alpha + dx' \cosh\alpha) \wedge dy' \\
&= dt' \wedge E' dy' - B' dx' \wedge dy'
\end{aligned}
$$

for

$$
\begin{aligned}
E' &= E \cosh\alpha - B \sinh\alpha \\
B' &= B \cosh\alpha - E \sinh\alpha
\end{aligned}
$$

This shows that no components appear in "new" dimensions (we still have $\mathbf{E}' = (0, E', 0), \mathbf{B}' = (0, 0, B')$), but the two magnitudes of the fields do scramble in a usual lorentzian way. Note, however, that for *pure electric* field in the unprimed frame ($B = 0$) we get *both* electric *and magnetic* fields in the primed frame. (And the same phenomenon occurs for pure *magnetic* field in unprimed frame.)
Example 3.6. Uniformly rotating frame. We have (in *cylindrical* coordinates) $(t, r, \varphi, z) = (t', r', \varphi' - \omega t', z')$, so that

$$(dt, dr, d\varphi, dz) = (dt', dr', d\varphi' - \omega dt', dz')$$

The fact that $d\varphi = d\varphi' - \omega dt'$ results in "generation of" an *electric* field out of magnetic one. For example, if $F = B dr \wedge d\varphi$ (pure magnetic field, oriented along $z$-axis), then

$$F = B dr \wedge d\varphi = B dr' \wedge (d\varphi' - \omega dt') = B dr' \wedge d\varphi' + dt' \wedge (\omega B) dr'$$

so that in primed reference frame we see the same magnetic field *plus* in addition *electric* field oriented along $r$ (i.e. radial in the sense of cylindrical coordinates), proportional to the angular velocity $\omega$ and the magnitude of the original magnetic field.

It turns out that the first (= inhomogeneous) series of Maxwell equations leads to

$$d * F = -J \equiv - * j$$

or, equivalently,

$$\delta F = -j$$

where the three-dimensional quantities $\rho$ (electric charge density) and $\mathbf{j}$ (electric current density) are built into a single object living in Minkowski space, the 1-form of current or, alternatively, its dual 3-form of current

$$
\begin{aligned}
j &= \rho dt - \mathbf{j}.d\mathbf{r} \equiv j_\mu dx^\mu \\
J &= dt \wedge (-\mathbf{j}.d\mathbf{S}) + \rho dV \equiv j^\mu d\Sigma_\mu \equiv *j
\end{aligned}
$$

(The operator $*d* = \delta$ is called the *codifferential*; note, that $\delta\delta = 0$ because of $dd = 0$ and $** = \pm\hat{1}$.) So the *complete set* of Maxwell equations in vacuum turns out to be as short as

$$dF = 0 \qquad \delta F = -j$$

Applying $\delta$ to both sides of the second equation we get $\delta j = 0$. This is nothing but the *continuity equation* $\partial_t \rho + \operatorname{div} \mathbf{j} = 0$. This means that the *charge conservation* is automatically inherent in the equations (the consistency of Maxwell equations needs "conserved current" $j$).

**Potentials, gauge transformations, action integral and all that**. There is, in general, a highly nontrivial relation between *closed* forms (such that $d\alpha = 0$) and *exact* forms (such that $\alpha = d\beta$ for some $\beta$). From $dd = 0$ it is clear that each exact form is necessarily also closed. The answer to the *opposite* question may be, however, negative sometimes. Now *Poincaré lemma* asserts, that *locally* (in a sufficiently small neighborhood of any point) the answer is *positive*. That is to say, a *potential* (such $\beta$ that $\alpha = d\beta$) always exists at least locally for a given closed form.

Now if we apply the lemma to $F$, the Maxwell equation $dF = 0$ (simply saying that $F$ is closed) implies the existence of a potential 1-form:

$$F = dA \qquad \text{(at least locally)}$$

for some

$$A = A_\mu dx^\mu = A_0 dt + A_i dx^i = \Phi dt - \mathbf{A}.d\mathbf{r}$$

In coordinate components

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

since

$$dA = d(A_\nu dx^\nu) = dA_\nu \wedge dx^\nu = (\partial_\mu A_\nu) dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu$$

Potential $A$ is not unique. Indeed, the replacement

$$A \mapsto A + d\chi \qquad \chi = \text{ any function } (\equiv 0 - \text{form})$$

(called *gauge transformation* of the potential $A$) does not influence $F$ at all:

$$F \equiv dA \mapsto d(A + d\chi) = dA + dd\chi = dA \equiv F$$

In order to find appropriate *action integral* for deriving Maxwell equations, one should realize that it is to be a 4-dimensional volume integral, $\int dt \int d^3 r$ of something. This something is then necessarily a 4-form. So we are to construct a natural 4-form out from the material characterizing the field and the source. So from $F$ and $j$. There are not so many possibilities and it turns out that the right choice is

$$S[A] = \int_\Omega (k_1 F \wedge *F + k_2 A \wedge *j)$$

where appropriate constants $k_1, k_2$ are identified aposteriori from the resulting equations. Note that the action is to be treated as a functional of the *potential* $A$ rather than of the field $F$ (i.e. put $F := dA$ into the action and do variations w.r.t. $A$). There is a simple general technique for *actually performing* the variation $A \mapsto A + \epsilon a$ and it indeed results in Maxwell equations.

## Thank you for your attention!

**Further reading**

[A] M.Fecko: Differential geometry and Lie groups for physicists, Cambridge University Press, 2006, 714pp.
Some information concerning the book (including sample chapters) is available at my web page,
http://sophia.dtp.fmph.uniba.sk/~fecko
(choose "English version" and then "about my book")

[B] Numerous interesting books listed in Bibliography to [1];
For the convenience of the reader (of this text) I reproduce the bibliography here:


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