Selected topological concepts used in physics

An introductory exposition

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1 Introduction

In modern theoretical physics, including many-particle systems, topological reasoning sometimes proves to be remarkably effective. As an example, it enables one to conclude that some integral, relevant for physics, is insensitive to smooth changes of the domain of integration. Or, that it has always integer values. Or, that an integral, which contains some gauge fields under the sign of integration, actually does not depend at all on particular choice of the gauge fields.

These lectures are intended as a simple and short introduction into this beautiful subject (with many important topics, however, completely omitted). They primarily try to focus on essential *underlying ideas* rather than on systematic and detailed treatment of the topics.

The notes [3] from my 2007 lectures might be useful to readers who do not feel certain in basics of modern differential geometry. At the end of this text some further reading is mentioned. Of course, there is a vast amount of texts of all kinds available on the subject.

2 Closed and exact forms

2.1 Homotopic invariance of integrals of closed forms

Differential forms may be integrated over *chains*.

From the point of view of this section there are three types of differential forms:

$$\begin{array}{rcl} \text{general form} & : & \alpha \\ \textit{closed}\text{ form} & : & \alpha \text{ such that } \quad d\alpha = 0 \\ \textit{exact}\text{ form} & : & \alpha \text{ such that } \quad \alpha = d\beta \end{array}$$

And there are three types of chains

$$\begin{array}{rcl} \text{general chain} & : & c \\ \textit{cycle} & : & c \text{ such that } \quad \partial c = 0 \\ \textit{boundary} & : & c \text{ such that } \quad c = \partial S \end{array}$$

Let α be a *closed* k-form on a manifold M and let c be a k-cycle (*closed* k-dimensional surface) on M. Then the integral

$$\int_c \alpha \tag{1}$$

has the following remarkable property. Replace

$$\alpha \quad \mapsto \quad \alpha' = \alpha + d\beta \tag{2}$$

$$c \quad \mapsto \quad c' = c + \partial S \tag{3}$$

where β is a $(k-1)$-form and S is a $(k+1)$-chain ($(k+1)$-dimensional domain). Then it turns out that

$$\boxed{\int_{c'} \alpha' = \int_c \alpha} \tag{4}$$

so that the integral (1) is completely *insensitive* to both replacements (2) and (3).

▼ The standard nomenclature is

$$\alpha \ \text{ and } \ \alpha' = \alpha + d\beta \quad \text{are called} \quad \textit{cohomological} \tag{5}$$

$$c \ \text{ and } \ c' = c + \partial S \quad \text{are called} \quad \textit{homological} \tag{6}$$

Then, in words, we can replace the form to be integrated by a cohomological one and the chain by a homological one.

Indeed, the general

$$\textit{Stokes theorem} \qquad \int_s d\sigma = \int_{\partial s} \sigma \tag{7}$$

gives

$$\int_{c+\partial S} (\alpha + d\beta) = \int_c \alpha + \int_c d\beta + \int_{\partial S} \alpha + \int_{\partial S} d\beta = \int_c \alpha$$

since $d\alpha = 0$, $\partial c = 0$, $\partial\partial = 0$ and $dd = 0$. ▲

Let us divide the combined statement (4) into two parts. The first important observation is that the integral of an *exact* form over a *closed* surface always vanishes

$$\int_c d\beta = 0 \qquad \text{if} \qquad \partial c = 0 \tag{8}$$

The second important observation is that the integral of a *closed* form over a *boundary* always vanishes so that *addition of a boundary* to a closed surface c *does not change* the value of the integral, i.e.

$$\int_{c+\partial S} \alpha = \int_c \alpha \tag{9}$$

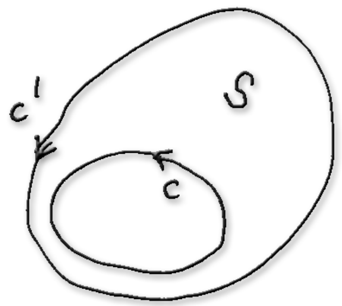

Figure 1: Here c' equals c plus a boundary ∂S, so integrals over c and c' are the same. (c' and c are k-dimensional closed surfaces, S is a $(k+1)$-dimensional domain.)

Example 2.1.1: Perhaps the best known example provides elementary complex analysis. It is well known that the value of the integral

$$\oint_c f(z)dz \tag{10}$$

remains unchanged if we *deform* the closed curve c ($k = 1$; the deformation must not cross poles, if there are any, of $f(z)$). The reason is that the *1-form* $f(z)dz$ is *closed*

$$d(f(z)dz) = f'(z)dz \wedge dz = 0 \tag{11}$$

Example 2.1.2: In the *punctured* plane (the plane with the origin removed, $M = \mathbb{R}^2 \backslash \{0, 0\}$), consider *polar* coordinates (r, φ) and the 1-form (again, $k = 1$)

$$\alpha = d\varphi \tag{12}$$

When expressed in the *cartesian* coordinates (x, y), we get [1]

$$\alpha = \frac{xdy - ydx}{x^2 + y^2} \tag{13}$$

(One easily verifies by (not so) brute force that it is indeed closed.) Let $c \leftrightarrow (x(t), y(t))$ be a *closed* curve (a loop) in $\mathbb{R}^2 \backslash \{0, 0\}$. Then no smooth deformation of the loop (never crossing the origin) changes the value of the integral (1).

For example, take the circle

$$x(t) = R\cos t \qquad y(t) = R\sin t \qquad t \in \langle 0, 2\pi\rangle \tag{14}$$

(with R being arbitrary). Then we get

$$\int_c \alpha = \int_0^{2\pi} \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} dt = \cdots = 2\pi \tag{15}$$

Notice that it is independent of R. So the smooth deformation of the circle consisting in increasing of the radius has no influence on the integral.

We checked explicitly the invariance of the integral on a very simple example. The invariance holds, however, for a much wider class of deformations - all smooth deformations for which the final loop c' together with the inversely oriented original loop c realize the boundary of some area S (i.e. the boundary of S consists of two parts, the final c' plus inversely oriented original c, $\partial S = c' - c$) should work. Indeed, as we already saw,

$$\int_{c'} \alpha = \int_{c+\partial S} \alpha = \int_c \alpha + \int_{\partial S} \alpha = \int_c \alpha + \int_S d\alpha = \int_c \alpha + \int_S 0 = \int_c \alpha \tag{16}$$

So, the essential point here is the *closedness* of α.

Of course we could evaluate the integral much more easily in *polar* coordinates. The expression of the circle is

$$r(t) = R \qquad \varphi(t) = t \qquad t \in \langle 0, 2\pi\rangle \tag{17}$$

and we get

$$\int_c \alpha = \int_0^{2\pi} \dot{\varphi} dt = [t]_0^{2\pi} = 2\pi \tag{18}$$

From the expression of α in polar coordinates (1) the additional remarkable property of the integral is clear - whatever the closed curve is, the value of the integral is always an *integer* multiple of 2π. Indeed, the integral just expresses, when divided by 2π, how many times the curve encircles the origin, i.e. it represents the

[1] The 1-form $d\varphi$ is actually ill-defined at the x-axis (where $\varphi = 0$), since the coordinate φ itself is ill-defined there. However, $d\varphi$ *may* be uniquely defined also in the missing points so that it is smooth everywhere in the punctured plane (i.e. except for the origin; the resulting 1-form being, when expressed in cartesian coordinates, just (13).)

winding number of the particular loop. (Notice that in the case of curves which are not convex, the angle contribution may be locally fairly complicated, both positive (when the curve rotates counter-clockwise) and negative (clockwise). The resulting *net* angle is, however, always an *integer* multiple of 2π.)

Example 2.1.3: This is a generalization of the preceding example to n dimensions. If one looks for a *rotation-invariant* and at the same time *closed* $(n-1)$-form in the *punctured* $\mathbb{R}^n$, one finds (see Appendix A.2) that it is a constant multiple of the form

$$\alpha = \frac{\mathbf{n}\cdot d\mathbf{S}}{r^{n-1}} \equiv \frac{\mathbf{r}\cdot d\mathbf{S}}{r^n} \equiv \frac{x^i dS_i}{r^n} \tag{19}$$

For the case $n=2$ we get

$$\frac{x^i dS_i}{r^2} = \frac{x^i \epsilon_{ij} dx^j}{r^2} = \frac{x^1 dx^2 - x^2 dx^1}{r^2} \equiv \frac{xdy - ydx}{x^2+y^2} \tag{20}$$

which is exactly (13) or, in polar coordinates, (12). Similarly, for the case $n=3$ we get

$$\frac{x^i dS_i}{r^3} = \frac{x^i \epsilon_{ijk} dx^j \wedge dx^k}{r^3} = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \tag{21}$$

This gives, when expressed in spherical polar coordinates (r,ϑ,φ), simply

$$\alpha = \sin\vartheta d\vartheta \wedge d\varphi \tag{22}$$

so that it measures, when integrated over a surface, the net *solid angle* subtended by the surface.

▼ For general n, by usual physical/dimensional language/reasoning

$$d\mathbf{S}(r) = r^{n-1} d\mathbf{S}(1)$$

so that

$$\frac{\mathbf{n}\cdot d\mathbf{S}}{r^{n-1}} \equiv \frac{\mathbf{n}\cdot d\mathbf{S}(r)}{r^{n-1}} = \mathbf{n}\cdot d\mathbf{S}(1) = dS(1)$$

So when this form is integrated over some $(n-1)$-dimensional (hyper)surface, it gives projected *area* on the *unit* sphere (by projection $\mathbf{r}\mapsto\mathbf{n}\equiv\mathbf{r}/r$), which is - by definition - the corresponding *solid angle*. ▲

For any n, whenever $\partial D = S' - S$, i.e. whenever an n-dimensional volume D (not containing the origin) is enclosed by two $(n-1)$-dimensional (hyper)surfaces S' and S, we get

$$\int_{S'}\alpha = \int_{S+\partial D}\alpha = \int_S \alpha + \int_{\partial D}\alpha = \int_S \alpha \tag{23}$$

So one can deform the hyper-surface with no effect on the value of the integral.

Notice also that, for $n=3$, (19) is just $\mathbf{E}\cdot d\mathbf{S}$ for the *electric* field of static point charge sitting in the origin. The integral (23) just represents, due to *Gauss* law from electrostatics, the total amount of *charge* enclosed by S.

2.2 De Rham cohomologies

It turns out that the relation between closed and exact forms has a potential to tell us a lot about topology of the underlying manifold. In general terms it means that *geometry* of manifolds (represented here by a delicate difference between two types of *geometric* objects living on manifolds) is deeply interrelated with their *topology.*

Recall (see section (2.1)) that $dd = 0$ implies that exact forms are automatically closed. Further, both closed and exact forms are (infinite dimensional) *subspaces* of the (infinite dimensional) vector space of all forms. Denote the three relevant spaces as follows:

$$\begin{aligned} \text{general } p\text{-forms on } M &: \quad \Omega^p(M) \\ \text{closed } p\text{-forms on } M &: \quad Z^p(M) \\ \text{exact } p\text{-forms on } M &: \quad B^p(M) \end{aligned}$$

Then the subspace structure is

$$B^p(M) \subset Z^p(M) \subset \Omega^p(M)$$

For us the first part is important, the statement that the (infinite dimensional) vector space of *exact* p-forms on M is a(n infinite dimensional) *subspace* of the (infinite dimensional) vector space of *closed* p-forms on M

$$B^p(M) \subset Z^p(M)$$

Highly valuable topological information about the manifold M lies in "precise relation" between these vector spaces. All this may be used in two directions. If we are able to compute the structure of the forms, we can learn from it something about the topology of M. If we, on the contrary, need the know the relation between closed and exact forms on M, we are to get knowledge (by some independent means) of appropriate topological property of M.

What is meant by "precise relation" between these vector spaces? The just measure of the "difference" of the two vector spaces is their *factor-space*

$$H^p(M) := Z^p(M)/B^p(M) \tag{24}$$

This is a vector space in its own right.

▼ Let $W \subset V$. Declare v_1 to be equivalent to v_2 if $v_2 = v_1 + w$ for some $w \in W$. Then equivalence *classes* (sets, denoted as $[v]$, of all mutually equivalent vectors) have a natural vector space structure. Indeed, we can define the linear combination of classes in terms of representatives (i.e. $[v_1] + \lambda[v_2] := [v_1 + \lambda v_2]$) and check that it actually does not depend on representatives. ▲

What is much more surprising is that it is *finite-dimensional.* It is known as the p-th *cohomology* space of M.

▼ $H^p(M)$ is more frequently known as the p-th cohomology *group* of M (although it is a *vector space*, here). First, a vector space *is* a group (w.r.t. addition) and second, our $H^p(M)$ is, in more accurate notation, $H^p(M;\mathbb{R})$, the *real* cohomology. There is also a more general "G-valued" case, denoted $H^p(M;G)$ where the word group is more deserved. ▲

If $H^p(M)$ comes out to be 0-dimensional (one writes $H^p = 0$), it means there is *no* difference between close and exact p-forms on M. Or, that each closed form is necessarily exact. This is often a very valuable information. If $H^p(M)$ is 1-dimensional (one often writes $H^p = \mathbb{R}^1$, where $=$ stands for linear isomorphism), it means there is exactly one "type" of closed forms which fail to be exact. (So closed forms are either exact or not exact and there is just "one way" to be not exact.) If $H^p(M)$ is 2-dimensional ($H^p = \mathbb{R}^2$), it means there are two "independent types" of closed forms which fail to be exact. (So closed forms are either exact or not exact and there are "two different ways" to be not exact.) And so on.

The case $p = 0$ is a special one. You can check right from the definition (it is quite easy) that dim $H^0(M) = k$ just means that the manifold M has k connected components. (So, for example, the sphere and the torus, both being connected, have $H^0(S^n) = H^0(T^n) = 1$.)

For real computation of cohomology spaces of concrete manifolds, there is a number of methods, from fairly easy algorithms up to ingenious tricks needed for complicated cases. Using these methods a lot of concrete results were computed.

For example, for the spheres the following statement holds:

$$H^0(S^n) = H^n(S^n) = 1 \qquad\qquad \text{all other cases vanish} \tag{25}$$

See Appendix A.2 for explicit computation of the simplest case, S^1.

There is a lemma, due to Poincaré, stating that if a manifold is *contractible* (to a point), all $H^p(M)$ (except for $p = 0$) *vanish*. So, on such manifolds, there is no difference between closed and exact forms (see Section 9.2 in [2]). Any coordinate patch *is* contractible, so closed $=$ exact on any such patch. (E.g. the form $d\varphi$ *is* exact in the domain of the coordinate φ, but it is *not* so on the whole circle, as we can see from (18).)

2.3 Betti numbers

Betti numbers b^p (of a manifold M) are just *dimensions* of the cohomology spaces discussed in the last section

$$b^p \equiv b^p(M) := \dim H^p(M) \tag{26}$$

They are important topological invariants of a manifold. So, instead of writing

$$H^p(T^n) = \mathbb{R}^{\binom{n}{p}} \tag{27}$$

for cohomological groups of n-dimensional torus one simply writes

$$b^p(T^n) = \binom{n}{p} \qquad p = 0, 1, \ldots, n \tag{28}$$

Alternating sum of Betti numbers is of special interest:

$$\chi(M) := b^0 - b^1 + b^2 - \ldots \qquad \textit{Euler characteristic of } M \tag{29}$$

For example, we see from (25), (28) and Poincaré lemma, that the Euler characteristics of two-dimensional sphere, two-dimensional torus and two-dimensional disk

$$\begin{aligned} \chi(S^2) &:= b^0 - b^1 + b^2 = 1 - 0 + 1 = 2 && (30) \\ \chi(T^2) &:= b^0 - b^1 + b^2 = 1 - 2 + 1 = 0 && (31) \\ \chi(D^2) &:= b^0 - b^1 + b^2 = 1 - 0 + 0 = 1 && (32) \end{aligned}$$

We will encounter *alternative* expressions of Euler characteristic later on (see (58), (63) and (64)). Clearly, they are expected to give the same number for the same manifold (which *is* the case; it is a highly non-trivial fact).

2.4 Hopf invariant

Consider a map of two spheres

$$f : S^{2n-1} \to S^n \qquad n = 2, 3, \ldots \tag{33}$$

Take a normalized volume form ω on the target S^n

$$\int_{S^n} \omega = 1 \tag{34}$$

It is trivially closed (because of its highest possible degree on S^n), but not exact (otherwise the l.h.s. of (34) was zero by Stokes theorem). However, its pull-back $f^*\omega$ to S^{2n-1} is not only closed (this is guaranteed by $f^*d = df^*$) but also *exact*

$$f^*\omega = d\alpha \qquad \text{for some } (n-1)\text{-form } \alpha \tag{35}$$

This is a consequence of the results on cohomology groups of spheres (25) and the clever observation, that n is neither equal to 0 nor $2n-1$:-). So, *any* closed n-form on S^{2n-1} is necessarily exact. Then, $\alpha \wedge d\alpha$ is a $(2n-1)$-form on S^{2n-1}. So, it may be integrated over S^{2n-1}. We get

$$\textit{Hopf invariant} \qquad H(f) := \int_{S^{2n-1}} \alpha \wedge d\alpha \tag{36}$$

It is a well-defined number associated with any map f of type (33).

▼ Let $\alpha \mapsto \alpha + d\beta$ (freedom in potential (35)). Then, for $\alpha \wedge d\alpha \equiv \sigma$,

$$\sigma \mapsto \sigma + d\tau \qquad \tau := \beta \wedge d\alpha$$

and since

$$\int_{S^{2n-1}} (\sigma + d\tau) = \int_{S^{2n-1}} \sigma$$

by Stokes theorem, $H(f)$ is not sensitive to the choice of potential. ▲

What is interesting about the number $H(f)$ is that it is, for any f, an *integer* (the proof is far from being easy). Knowing this, however, we can immediately conclude, that it is a *topological invariant* of f.

▼ Standard argument: Small changes of f should produce small changes of $\alpha \wedge d\alpha$ and, finally, small changes of $H(f)$. If $H(f)$ is an integer, no *small* changes are possible. ▲

If f *is* a *constant* map (each point of S^{2n-1} is mapped to a *single* point of S^n), the pull-back $f^*\omega$ vanishes and $H(f)$ vanishes too. If f can be smoothly deformed to a constant map, $H(f)$ is zero as well (it *is* zero at the *end* of a *smooth* procedure, being an *integer* all the time). So a *non-zero* value of $H(f)$ indicates that f *cannot* be smoothly deformed to a constant map (it is essential).

In particular, the *Hopf map*

$$f : S^3 \to S^2 \qquad \chi \mapsto \mathbf{n} \equiv \chi^+ \boldsymbol{\sigma} \chi \qquad \chi \in \mathbb{C}^2 \qquad \chi^+\chi = 1 \tag{37}$$

has $H(f) = 1$.

▼ This shows (*very* loosely speaking) that it is not always possible "to shrink to a point a 3-dimensional sphere's image on a 2-dimensional sphere" and, consequently, that the "3-rd homotopic group of S^2 is non-trivial". ▲

3 Gauss-Bonnet theorem

Gauss-Bonnet theorem [2] states that the sum of the integral of the Gaussian curvature K over a two-dimensional surface S and the integral of geodesic curvature k along the boundary ∂S of the surface is an integer multiple of 2π, the integer being the Euler characteristics $\chi(S)$ of the surface:

$$\boxed{\int_S K dA + \oint_{\partial S} k ds = 2\pi\chi(S)} \tag{38}$$

(The two terms on the left are also known as the *total Gaussian* and the *total geodesic* curvature respectively.) In the simpler situation, when the surface is closed ($\partial S = 0$) the boundary integral is vacuous and we get

$$\int_S K dA = 2\pi\chi(S) \qquad\qquad \text{if } \partial S = 0 \tag{39}$$

The statement (38) is known for more than one and half century and it is highly remarkable. First, the Gaussian curvature is a function on the surface and the

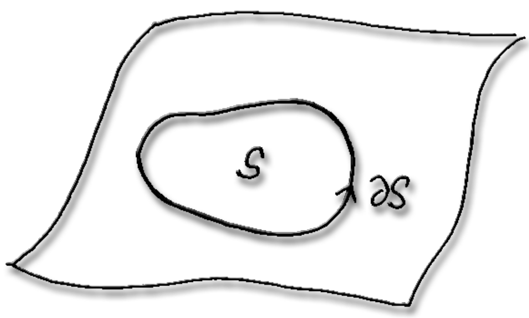

Figure 2: A surface S and its boundary ∂S

geodesic curvature is a function on its boundary. (Both depend on the metric g on the surface: $K \equiv K_g$ and $k \equiv k_g$.) It is indeed a remarkable restriction on the two functions that the sum of their integrals (when divided by 2π) *always* gives an *integer*.

Then, the Euler charakteristics $\chi(S)$ is a topological invariant of the surface. That's why a change of the metric g on the surface (caused, for example, by a change of the embedding of the surface into the Euclidean space E^3) by no means influences the right-hand side of the equation (its topology remains the same). Then it may not influence the left-hand side as well. However, the metric *is* formally present there. This is highly remarkable.

[2] Wikipedia teaches us that it is named after Carl Friedrich Gauss (1777-1855) "who was aware of a version of the theorem but never published it", and Pierre Ossian Bonnet (1819-1892) "who published a special case in 1848."

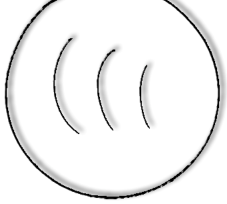

Figure 3: A sphere

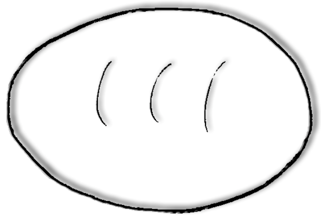

Figure 4: An egg

Imagine, to be more concrete, a sphere and (the surface of) an egg. The egg is nothing but a smoothly deformed sphere, so that both surfaces share the same topology. They share, consequently, the same value of the Euler characteristics (being 2, as we know from (30) and we will confirm independently later). The induced metrics from their embedding into E^3, however, differ. So their Gaussian curvatures differ, too. (And it can be a fairly complicated function for a "general egg".) In spite of this, the *integral* of the new (egg) curvature over the new surface leads to the same number as the integral of the old (sphere) curvature over the old surface. (Irrespective, of course, of any details of the egg shape or size.)

Gauss-Bonnet theorem is a classical prototype of a number of modern (20-th century) deep statements, which generalize its content in various directions. We touch some of them in next chapters. In this chapter, however, we concentrate on the good old original version.

3.1 Geodesic and Gaussian curvature - intuitive picture

Let us begin with a curvature of a curve in the *Euclidean plane*. Imagine we travel along a road in a car and we want to introduce a numerical measure of an "intensity of a curve" (big number = severe bending, small number = moderate bending). A good idea is to use the *centrifugal force*, which causes the push against the door. (Severe push apparently means tight turn.) The formula for the *circular* motion is, as is well known,

$$F = \frac{mv^2}{r} \tag{40}$$

However, in this formula there are still elements which characterize *ourselves* rather than the curve under consideration. Say, (our) mass m is not a property of the curve. Deleting the mass we get centrifugal *acceleration*

$$a = \frac{v^2}{r} \tag{41}$$

But, neither the velocity we pass the curve is a property of the latter, it is still our characteristic. So take some fixed velocity. The simplest choice being the *unit* velocity. Now, the magnitude of the centrifugal acceleration is exclusively a property of the curve. Huge acceleration - tight turn, negligible acceleration - negligible turn (and, in particular, zero acceleration - straight road).

$$a_{(v=1)} = \frac{1}{r} \tag{42}$$

We will call this the *curvature* of the circle. It behaves reasonably - small radius produces big curvature and vice versa. Further, let add to the definition that it is positive for a turn to the left and negative for a turn to the right:

$$k = \pm\,\frac{1}{r} \qquad \pm \;=\; \text{to the left/to the right} \tag{43}$$

So, within a single number, we pack information about both intensity of the curve as well as whether it is to the left or to the right .

And what about roads which are *not* circular? Take *very small* piece of the road and measure the acceleration. Define this acceleration to be the curvature within this small piece. So, curvature becomes, in general, *a function* on the curve. Notice that this also means that in a small vicinity of each point of an *arbitrary* (smooth) curve the curve behaves like a *circle*. (This is exactly like when we say, that *any* motion becomes *uniform* when restricted to short enough time intervals.)

Now, on a general two-dimensional surface, the same idea applies. We just need first think of what a *uniform and straight line* motion is. This is studied in differential geometry. One comes to the concept of *geodesic* curves as a natural generalization of uniform straight motion.

Consider a curve γ on a surface. Parametrize it "naturally", i.e. as $\gamma(s)$ such that $||\dot{\gamma}|| = 1$ (unit velocity). Then the *acceleration* vector $a = \nabla_{\dot{\gamma}}\dot{\gamma}$ is necessarily *perpendicular* to the velocity vector $\dot{\gamma}$

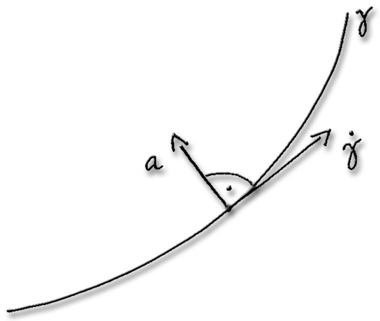

Figure 5: When traveling with unit speed, the acceleration is othogonal to the velocity

$$g(a, \dot{\gamma}) = 0 \tag{44}$$

▼

Since $0 = \nabla_{\dot{\gamma}} 1 = \nabla_{\dot{\gamma}} g(\dot{\gamma}, \dot{\gamma}) = 2g(a, \dot{\gamma})$. ▲

There are just two discrete possibilities for the direction of a. Perpendicular "left" or "right". Denote w the *unit* ($||w|| = 1$) vector perpendicular to $\dot{\gamma}$, to the left. (The frame $(\dot{\gamma}, w)$ is to be right-handed.)

Then, the definition of the *geodesic curvature* k is as follows:

$$a =: kw \tag{45}$$

So, it is plus/minus the *magnitude* of the vector a, plus when a sticks to the left

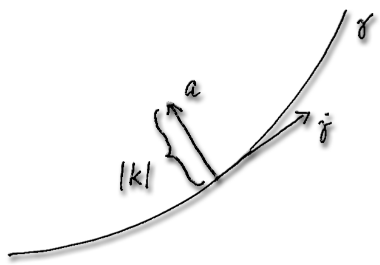

Figure 6: Geodesic curvature k is plus/minus the length of the vector a

and minus to the right w.r.t. $\dot{\gamma}$, respectively.
(On surfaces in E^3 it is also possible to express it in terms of the outer normal $\mathbf{n}$. One constructs the vector product $\mathbf{n} \times \dot{\mathbf{r}}$ (which is indeed a *unit* vector directed to the *left* with respect to $\dot{\mathbf{r}}$) and then makes the dot product of the acceleration $\mathbf{a}$ and the vector. This gives exactly plus/minus of the magnitude of the vector $\mathbf{a}$.)

Now we would like to introduce a measure of the bending of a *surface*. Consider a 2-dimensional surface. Fix a point on it and erect a vector normal to the surface in the point. Imagine two planes, mutually perpendicular and containing the vector. (There is infinite number of such pairs. Fix any of them.) The planes cut two curves out of the surface in the vicinity of the point. (Two mutually perpendicular meridians, if we think of the North pole on the Earth surface). Each of them is a planar curve, so it possesses the curvature introduced above. Let us call them k_1 and k_2.

[Here, the sign convention is as follows: if k_1 and k_2 have equal sign, it means, that the surface is, in the vicinity of the point, bent towards *the same* side with respect to the tangent plane in the point (and, in particular, positive k_1 and k_2 indicate that the surface lies on the side of the normal vector). If the signs of k_1 and k_2 differ, the surface behaves like a *saddle* - it is bent towards one side along the first curve and towards the other side along the second curve.]

It turns out it exists an optimal choice of the planes. They are known as *principal directions*.

[For this case, the values of k_1 and k_2 are extremal. They are given as eigenvectors of the Hessian matrix of the height function at the point.]

So in each point we have two mutually perpendicular directions and corresponding

$$\textit{principal curvatures}: \qquad k_1 = \pm\frac{1}{r_1} \qquad k_2 = \pm\frac{1}{r_2} \tag{46}$$

Example 3.1.1.: Surface = a *plane*. Let fix the plane xy. Then the orthogonal vector in each point has z-direction and both planes are *vertical*. The xy-plane is cut by them in two *lines*. Lines may be regarded as circles with ∞ radius, so both principal curvatures *vanish*. Thus the principal curvatures of a plane read

$$\text{plane}: \qquad k_1 = 0 \quad k_2 = 0 \tag{47}$$

Example 3.1.2.: Surface = a *sphere* of radius R. Take the North Pole. The normal vector is directed along z-axis and the planes cut two (mutually orthogonal) *meridians*. They are circles with radius R, consequently curvature $k = 1/R$. Thus the principal curvatures of the sphere read

$$\text{sphere}: \qquad k_1 = 1/R \quad k_2 = 1/R \tag{48}$$

(Equal signs - there is *no* saddle at the North Pole.)

Example 3.1.3.: Surface = a *cylinder* (say, infinite) of radius R. The principal directions are (at each point) along the axis and around the cylinder. The two lines cut by the planes are a line (along the axis), and the circle of radius R. So, the principal curvatures of the cylinder read

$$\text{cylinder}: \qquad k_1 = 0 \quad k_2 = 1/R \tag{49}$$

There are two combinations of principal curvatures k_1 and k_2 which deserve special attention. Namely, their mean value and their product. In this way, two new kinds of curvature of surfaces appear:

$$\textit{mean curvature}: \qquad H := \frac{1}{2}(k_1 + k_2) \tag{50}$$

and

$$\textit{Gaussian curvature}: \qquad K := k_1 k_2 \tag{51}$$

Mean curvature is an important measure of how a surface is embedded into the ambient space. It plays crucial role, for example, in how soap bubbles (minimal surfaces) are created. For us, however, the Gaussian curvature is of interest. Its most astonishing property is [3] that it is an entirely *intrinsic* property of a surface. It means, it can be computed solely by means of the geometry of the surface

[3] It was shown in 1828 by Gauss in his *Theorema Egregium* ("Remarkable Theorem")

itself, without referring to its embedding into an ambient space. Recall that the two factors themselves, k_1 and k_2, *do* make use of the embedding - we need the third dimension in order to introduce curvatures of the relevant planar curves. In contrary, their product k_1k_2 turns out to be, as Gauss has discovered, an entirely intrinsic quantity.

[For readers from University of Regensburg, two glass showcases, entitled *Krümmung* and *Minimalflächen* respectively, situated near lecture room H32 in Mathematics part, are highly recommended for inspection. There display various 3D models of surfaces and one can see the difference between the Gaussian and the mean curvature. "Sie haben in jeden punkt die mittlere Krümmung Null!" says the text in the "Minimalflächen" showcase.]

3.2 The net angle of rotation for a loop γ

In this section the following problem will be discussed. Consider a 2-dimensional surface (manifold) endowed with a Riemmannian metric and orientation (it results, for example, from an embedding into E^3, but it is not necessary). There is a smooth loop, γ, on it, which is the boundary $\gamma = \partial S$ of a domain S. In the *whole* domain (including the boundary) there is an orthonormal frame field e_a. To be more specific, imagine that the domain is a *coordinate* patch and the frame field results from the coordinate one (using Gramm-Schmidt procedure). Now, we move along the loop and observe how the tangent vector $\dot{\gamma}$ rotates with respect to the frame field. One can derive how much it rotates within a small piece and then obtain the net angle of rotation (it is sometimes called, when divided by 2π, *rotation index* of the loop) by summing (integrating) all the small contributions. The aim of the computation, however, is not to determine the resulting net angle, since the latter is clear from the very beginning - it is 2π (just a single turn; rotation index is 1). The true reason to perform the computation is to derive an equation saying that a sum of two particular integrals (obtained from summation of small increments) equals 2π, the fact we otherwise did not know. In this way we get a particular case of the Gauss-Bonnet theorem.

Then we will study what changes are to be performed if the boundary *fails* to be smooth (for example if it is a triangle).

Eventually we will study surfaces which may be *decomposed* into triangles. In this point the situation becomes global, the Euler characteristics enters the scene and the general statement of the Gauss-Bonnet theorem emerges.

3.2.1 Expression of the net angle of rotation in terms of total geodesic and total Gaussian curvature

So, let us return to the situation of interest. Consider a *loop* γ, which is the *boundary* of a two-dimensional domain $S \subset (M, g)$, $\gamma = \partial S$. Within the *entire* domain S (including its boundary $\gamma = \partial S$), there is a smooth orthonormal frame field e_a.

Introduce the angle θ between e_1 and $\dot{\gamma}$.

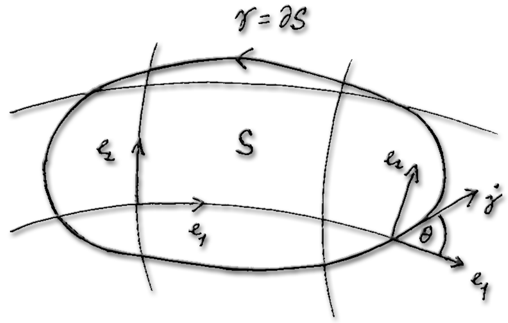

Figure 7: Definition of the angle θ

Then

$$\begin{aligned} \dot{\gamma} &= \cos\theta \; e_1 + \sin\theta \; e_2 \\ w &= -\sin\theta \; e_1 + \cos\theta \; e_2 \end{aligned}$$

Make a small step of length ds along the loop γ. The corresponding increment $d\theta$ of the angle θ may be computed (see the details in Appendix A). What we are interested in is the *net* angle $[\theta]_\circlearrowleft$ of rotation summed up along the *whole* loop. This is given by integrating $d\theta$ along $\gamma \equiv \partial S$. The result turns out to be (see Appendix B.1)

$$[\theta]_\circlearrowleft = \oint_{\partial S} k ds + \int_S K dA \tag{52}$$

At the same time, however, the value of the net angle is *clear* from the outset: the velocity vector is the same at the end as is was at the beginning (it is a smooth vector field on the curve) and it is evident, by inspection, that it undergoes *a single* (counterclockwise) rotation. So the angle is 2π. Therefore we get

$$\oint_{\partial S} k ds + \int_S K dA = 2\pi \tag{53}$$

This is already the statement (38) for $\chi(S) = 1$ (being, as we know from (32), exactly the value of $\chi(S)$ for domains S under consideration here).

3.2.2 Polygon - count in the jumps in vertices

The result (52) refers to a *smooth* boundary. In the case of polygons the boundary consists of smooth pieces (edges) as well as corners (*vertices*). So, it is only *piecewise* smooth. In the vertices, the velocity vector clearly jumps (in order to become tangent to a new edge) and it is the sum of all these jumps which is to be *added* to the r.h.s. of (52) (or, then, the l.h.s. of (53)) in order to obtain the true net angle of rotation.

Consider an n-gon. Let the jump be $\hat{\theta}_k$ in the k-th vertex. Then we are to add $\hat{\theta}_1 + \cdots + \hat{\theta}_n$ to the r.h.s. of (52). Now $\hat{\theta}_k$ is the *exterior* angle in k-th vertex. The corresponding *interior* angle θ_k, by definition, when combined with the exterior angle, sums up to the *flat* angle, i.e. $\hat{\theta}_k + \theta_k = \pi$.

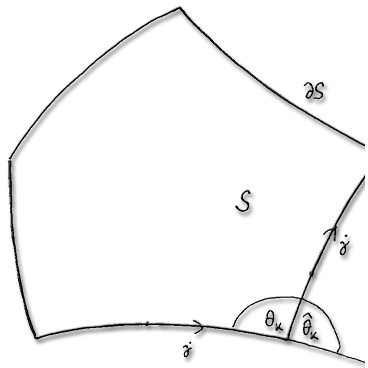

Figure 8: k-th vertex: exterior angle $\hat{\theta}_k$ and interior angle θ_k

Therefore

$$\hat{\theta}_1 + \cdots + \hat{\theta}_n = (\pi - \theta_1) + \cdots + (\pi - \theta_n) = n\pi - \sum(\text{interior angles})$$

So the result (52), for an n-gon, is to be replaced by

$$[\theta]_{\circlearrowleft} = \oint_{\partial S} k ds + \int_S K dA + n\pi - \sum(\text{interior angles})$$

Here, ∂S stands for smooth parts of the boundary alone (edges of the polygon).

However, nothing changes in the observation made in Section 3.2.1 that the net angle $[\theta]_{\circlearrowleft}$ is known from the outset, being 2π (still the velocity vector undergoes, when the jumps are included, a single counterclockwise rotation altogether). Therefore we get the final general result for the net (interior) angle of an n-gon S on a (possibly curved) surface

$$\sum(\text{interior angles}) = \oint_{\partial S} k ds + \int_S K dA + (n-2)\pi \tag{54}$$

Example 3.2.2.1: An important particular case is provided by a *geodesic* n-gon, i.e. an n-gon whose edges happen to be geodesic curves. On a geodesic, the acceleration vanishes (by definition), so $k = 0$, and the line integral contribution becomes vacuous. What remains is the statement

$$\sum(\text{interior angles}) = \int_S K dA + (n-2)\pi$$

Example 3.2.2.2: For example, the net angle of a geodesic *triangle* (so $n = 3$) on a sphere with radius ρ (so $K = 1/\rho^2$) reads

$$\text{net interior angle of a geodesic triangle on a sphere} = \pi + A/\rho^2$$

where A stands for the area of the triangle. This statement is known as *Girard theorem.*

3.2.3 Triangulation - how Euler characteristics appears

The result (54) gives for a triangle (which is a subset of a coordinate patch on the surface)

$$\sum(\text{interior angles } \triangle) = \oint_{\partial\triangle} k ds + \int_{\triangle} K dA + \pi \tag{55}$$

Now consider a surface S. It may not be closed, so it may have a *smooth* boundary ∂S. Let us slice the surface to triangles ("triangulate" it), each of them entirely residing in a coordinate patch.

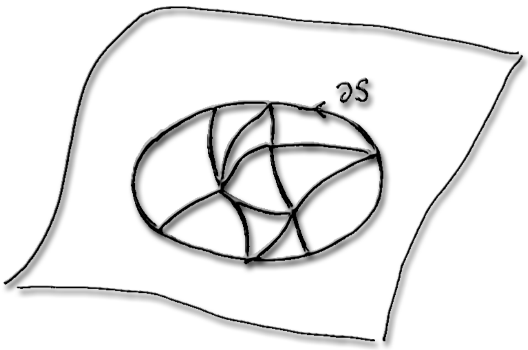

Figure 9: Surface S is sliced into triangles (triangulated)

For each triangle, the result (54) applies. Write down *all* the corresponding equations and *add* them. The following happens in doing so:

- summation of all terms $\int_{\triangle} KdA$ gives simply integral over the whole surface, $\int_S KdA$
- summation of all terms $\oint_{\partial\triangle} kds$ gives simply integral over the whole boundary, $\oint_{\partial S} kds$ (each interior edge contributes twice, with opposite sign, summing up to zero; exterior edges sum up to the total real boundary)

So we get

$$\sum(\text{interior angles of all } \triangle) = \int_S K dA + \oint_{\partial S} k ds + f\pi \tag{56}$$

where f denotes the total number of triangles (*faces*), present in the triangulation.

Now three elementary observations are fairly useful. [4]

First, the sum of the angles of triangles, which meet [5] at a common *interior* vertex is 2π and the sum of the angles, which meet at a common *boundary* vertex is π (since the boundary is smooth).

Consequently, the l.h.s. of the equation (56) may be written as

$$2\pi v_i + \pi v_b$$

where v_i is the number of interior vertices of the triangulation and v_b is the number of vertices on the boundary.

Second, each triangle has three edges. Once more, there are either interior or boundary ones. For the interior ones, each real edge is shared by a pair of triangles, boundary edges count just once. Therefore there is a simple relation between the number of two kinds of edges and the number of faces:

$$3f = 2e_i + e_b$$

Third, the number of vertices sitting on the boundary equals the number of boundary edges (boundary of the boundary vanishes, so that each vertex may be regarded as a starting point of exactly one boundary edge)

$$e_b = v_b$$

Let us write all the relevant relations together:

$$\begin{aligned} 2\pi v_i + \pi v_b &= \textstyle\int_S K dA + \oint_{\partial S} k ds + \pi f \\ 2e_i + e_b &= 3f \\ e_b &= v_b \end{aligned}$$

One can easily deduce from this system of equations (take it as a small exercise), that

$$\int_S K dA + \oint_{\partial S} k ds = 2\pi(f - e + v) \tag{57}$$

where $e \equiv e_i + e_b$ is the total number of edges and $v \equiv v_i + v_b$ is the total number of vertices.

The number (clearly an *integer*) asociated with the surface S by the rule

$$\boxed{\chi(S) := f - e + v} \tag{58}$$

[4]The actual summation in the l.h.s. of (56) may be organized as follows: label each particular angle by "its" *vertex*. Doing this, discriminate between interior vertices and vertices sitting on the boundary. Then sum, first, all angles associated with a fixed vertex and then sum over all vertices (separately interior and boundary ones).

[5]Recall that we consider triangles drawn on a smooth surface S rather than a discretized version of the surface, composed of "flat" triangles. A small surroundings of each vertex is thus a small plane and angles indeed sum up to 2π or π respectively, there.

is (once more, see (29) :-) the celebrated *Euler characteristics* of the surface S. It turns out that it does not depend on the particular triangulation (it should be clear from (57) - the l.h.s. does not know about any triangulation) - it is a *topological invariant* of the surface, it does not feel continuous deformations of the latter.

[Notice, we have as many as *two* independent definitions of Euler characteristic $\chi(S)$ of a two-dimensional surface S - (29) and (58)

$$\begin{aligned} \chi(S) &= b^0 - b^1 + b^2 \\ &= v - e + f \end{aligned}$$

(See also (63) and (64), where further alternative formulas are presented.) They share a common feature of being alternating sum of "0,1 and 2-dimensional objects". Note, however, that whereas individual terms in the first expression have objective meaning in its own right, individual terms in the second expression depend on particular triangulation and it is only the alternating sum which has intrinsic meaning.]

So we finally get the desired result

$$\int_S K dA + \oint_{\partial S} k ds = 2\pi\chi(S) \qquad \textit{Gauss-Bonnet theorem}$$

notified in (38).

3.3 Euler characteristics and critical points of smooth functions

Consider a *closed* surface S and a function on it whose all critical points are non-degenerate (to be defined in a moment). Then it turns out that the knowledge of the number and type of these points (minima, maxima, saddle points) is already enough for computing Euler characteristics of the surface.

3.3.1 How one can compute $\chi(S)$ with the help of critical points

Let $f : S \to \mathbb{R}$ be a function on S. In *critical* points we have (by definition) $df = 0$, so that *outside* critical points the surface is endowed with a global non-vanishing *gradient* (vector) field $\nabla f = \sharp_g df$ (or, in components, $(\nabla f)^i = g^{ij}\partial_j f$). We norm the field to unity and call it e_1.

Non-degenerate critical point of $f : S \to \mathbb{R}$ is a critical point (i.e. $df = 0$) in which the *Hessian* does not vanish (i.e. the matrix of second derivatives is non-singular).

▼ This is a coordinate-free concept. Indeed, the Hessian matrix transforms as follows:

$$\begin{aligned} H_{ij} &= \partial_i \partial_j f = J_i^k \partial'_k (J_j^r \partial'_r f) = J_i^k J_j^r (\partial'_k \partial'_r f) + (\partial_i J_j^r)(\partial'_r f) \\ &= J_i^k H'_{kr} J_j^r + (\partial_i J_j^r)(\partial'_r f) \end{aligned}$$

(where J_j^i stands for the Jacobian matrix $\partial_j x^{i'}$). In *critical point* the second term vanishes and what remains is

$$H_{ij} = (J^T H' J)_{ij}$$

so that for the *determinant* of the Hessian matrix (the *Hessian* H) we have

$$H = (\det J)^2 H'$$

Thus non-degeneracy of the *critical* point *is* an invariant concept. ▲

Let us have a look how Taylor expansion looks like in the vicinity of the non-degenerate critical point. Let P be such a point of a function f and let choose coordinates x^i centered at P (so that $x^i(P) = 0$). Then

$$f(x) = f(P) + (\partial_i f)(P) x^i + \frac{1}{2}(\partial_i \partial_j f)(P) x^i x^j + \cdots = f(P) + \frac{1}{2} H_{ij} x^i x^j + \ldots \quad (59)$$

By a *linear* change of coordinates we can diagonalize the symmetric bilinear form and get a canonical form with just two squares. From this we see that each non-degenerate critical point is necessarily *isolated*.

Therefore we can place each critical point into a small disk R_i (the i-th one) with the boundary $\gamma_i = \partial R_i$.

Call Y the complement to all the disks (so $S = Y + \sum R_i$, see Figure 10).

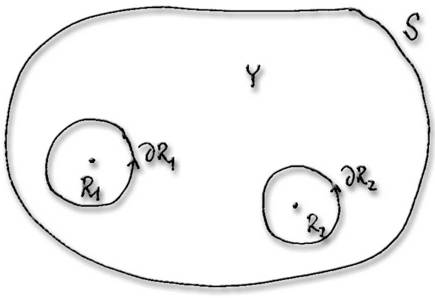

Figure 10: Y is the surface S minus all the disks around the critical points

Recall that, on the entire Y, there is a well-defined unit vector field e_1. Then, we can add the unique second unit field e_2 and form a *right-handed orthonormal frame field* (e_1, e_2) (an orientation on S is assumed here).

Now, some computations similar to those in Section 3.2.1 may be performed.

First, one can compute, for each boundary $\gamma_i = \partial R_i$, the net angle $[\theta_i]_\circlearrowleft$ between the velocity (tangent) vector $\dot{\gamma}_i$ and the gradient field ∇f. Put it differently (when divided by 2π), how many times $\dot{\gamma}_i$ makes a complete turn w.r.t. the frame field (e_1, e_2) while traversing the loop $\gamma_i = \partial R_i$ exactly once. When summed over all disks, one gets (see Appendix B.2)

$$\int_Y K dA = \sum_i \oint_{\gamma_i} (k)_i ds - \sum_i [\theta_i]_\circlearrowleft \tag{60}$$

Second, one can compute, for each boundary $\gamma_i = \partial R_i$, the net angle $[\phi_i]_\circlearrowleft$ between the same $\dot{\gamma}_i$ and an arbitrary (but fixed) frame field $(\hat{e}_1, \hat{e}_2)$ *inside* the disk R_i. When summed over all disks, one gets (see Appendix B.2)

$$\sum_i \int_{R_i} K dA = \sum_i [\phi_i]_\circlearrowleft - \sum_i \oint_{\gamma_i} (k)_i ds \tag{61}$$

Then, summation of (60) and (61) gives

$$\int_S K dA = \sum_i [\phi_i]_\circlearrowleft - \sum_i [\theta_i]_\circlearrowleft \equiv \sum_i [\phi_i - \theta_i]_\circlearrowleft \tag{62}$$

(since line integrals $\oint_{\gamma_i} (k)_i ds$ enter the sum with opposite signs and they pairwise *cancel*).

What is the meaning of the angle $\phi_i - \theta_i$, emerging in this expression? Since ϕ_i is the angle from $\hat{e}_1$ to $\dot{\gamma}_i$ and θ_i is the angle from $e \equiv e_1$ to (the same) $\dot{\gamma}_i$, their difference $\phi_i - \theta_i$ is the angle from $\hat{e}_1$ to $e_1 \equiv e$, i.e. from an arbitrarily chosen frame field in the i-th disk R_i to the gradient field generated by the function f.

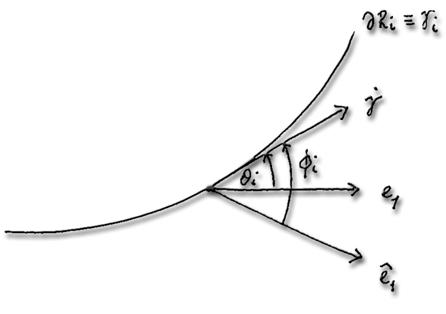

Figure 11: The angle $\phi_i - \theta_i$ is the angle from $\hat{e}_1$ to e_1

Then $[\phi_i - \theta_i]_\circlearrowleft$ is just (the $2\pi-$multiple of) the *winding number* of the gradient field w.r.t. an (arbitrarily chosen) frame field $(\hat{e}_1, \hat{e}_2)$ in the i-th disk R_i (it says how many times the gradient field makes a complete turn w.r.t. the frame field $(\hat{e}_1, \hat{e}_2)$ while traversing the loop $\gamma_i = \partial R_i$ exactly once).

Already this fact alone is remarkable enough since it reveals, that the $(1/2\pi)$-multiple of the total Gaussian curvature (integral over the whole surface of the Gaussian curvature) is necessarily an *integer*

$$\frac{1}{2\pi}\int_S K dA = \text{ sum of some winding numbers } = \text{ an } \textit{integer}$$

If, moreover, this integers might be found somehow, we would be able, according to (39), compute the Euler characteristics of the surface S in a completely new way (without triangulation of the surface or referring to Betti numbers). So let us study the winding numbers in some more detail.

3.3.2 Contribution of a non-degenerate critical point

So, we need to choose (arbitrarily) a frame field $(\hat{e}_1, \hat{e}_2)$ in each R_i and compute the winding number of the gradient field w.r.t. the frame field.

On a two-dimensional surface, a critical point may be minimum (both squares with plus sign), maximum (both with minus sign) or saddle point (both signs present). So there are just *three* types of (non-degenerate) critical points, here.

We need a metric tensor (to raise the index) for construction of gradient field $((\nabla f)^i = g^{ij}\partial_j f)$. Since the disks R_i are expected to be small, geometry on them will be not far from Euclidean one (the smaller is the disk, the better is the approximation). And since our aim is to compute winding number, i.e. an *integer*, there should be no difference between nearly Euclidean and truly Euclidean case. So we choose *exactly* Euclidean geometry in each R_i, $g_{ij}(x) = \delta_{ij} = g^{ij}(x)$.

The (symmetric) matrix H_{ij} may be diagonalized with the help of an *orthogonal* transformation (i.e. not spoiling $g_{ij} = \delta_{ij}$). In these adapted coordinates x, y we have

$$\begin{aligned} \tfrac{1}{2}H_{ij}x^i x^j &= \tfrac{1}{2}(ax^2 + by^2) \\ df &= axdx + bydy \\ \nabla f &= ax\partial_x + by\partial_y \qquad\qquad 0 \neq a, b \in \mathbb{R} \end{aligned}$$

So, the task is as follows: in a small disk R centered in origin of the plane xy we are given the vector field $ax\partial_x + by\partial_y$ and we are to compute its winding number w.r.t. some (orthonormal right-handed) frame field defined in whole disk R. Standard Cartesian frame field meets all the requirements. Moreover, any other frame field which meets the requirements may be obtained by a smooth deformation of the Cartesian one. Then, because of the fact that the resulting number is known to be integer, the choice of the Cartesian frame is as good as is any other choice.

Still another great simplification may be based on the fact that the resulting number is known to be integer. We can smoothly deform the gradient vector field ∇f (or the function f itself). We already know that we have just a 2-parametric family of the fields, so we can smoothly deform parameters a, b with no effect on the resulting winding number. Remember, however, that both parameters should be non-zero (otherwise the critical point cease to be non-degenerate). This means

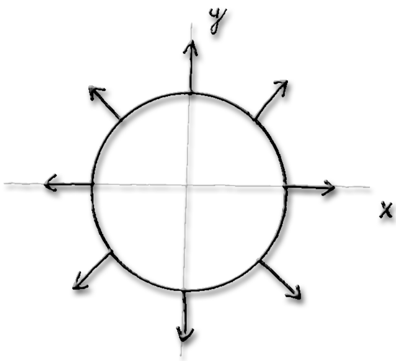

Figure 12: The field $x\partial_x + y\partial_y$ in selected points of the circle

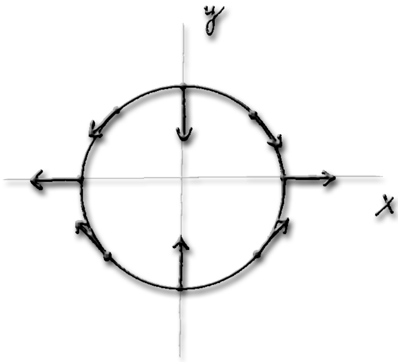

Figure 13: The field $x\partial_x - y\partial_y$ in selected points of the circle

that the pair (point) (a, b) must not lie on the axes in the plane ab. Starting with a given pair (a, b), the deformation has to preserve the *quadrant* of the pair in the plane ab. This enables one to choose obvious "canonical representatives" in each quadrant: twice plus-minus 1. We obtain the following 4 possibilities

(a, b)	$2f$	type of critical point	vector field
$(1, 1)$	$x^2 + y^2$	minimum	$x\partial_x + y\partial_y$
$(-1, -1)$	$-x^2 - y^2$	maximum	$-x\partial_x - y\partial_y$
$(1, -1)$	$x^2 - y^2$	saddle point	$x\partial_x - y\partial_y$
$(-1, 1)$	$-x^2 + y^2$	saddle point	$-x\partial_x + y\partial_y$

Now, the fields in the first two lines only differ in the overall sign and this same is true for the two fields in the third and fourth line. However, if fields only differ in the overall sign (stick in opposite directions), they rotate, when traversing the loop, by the same angle in the same direction. Consequently, their winding numbers coincide. It is then enough to examine the first and the third field. If we draw their values in some selected points on the circle (e.g. on the axes x, y as well as on the axes of all quadrants, to be sure, altogether 8 arrows, see Figures 12 and 13), it is evident that the first field turns once counter-clockwise and the third field also turns once, but clockwise. Consequently, their winding numbers are $+1$ a -1.

type of the critical point	winding number
minimum, maximum	$+1$
saddle point	-1

This means, however, that the sum of winding numbers of the gradient field in all critical points, which is present on the r.h.s. of (62), may be expressed as the number of minima plus the number of maxima minus the number of saddle points. So we obtain the following formula for the Euler characteristics:

$$\boxed{\chi(S) = \text{ number of maxima} - \text{number of saddles} + \text{number of minima}} \qquad (63)$$

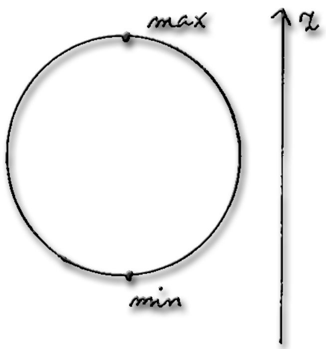

Figure 14: Height function on the sphere: one maximum and one minimum

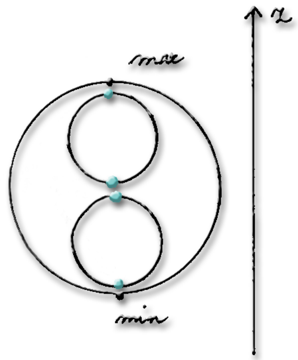

Figure 15: Height function on the button with 2 holes: one maximum, one minimum and 4 saddle points

Recall that this is true for *any* function f on S such that all its critical points are non-degenerate.

How to find these useful functions? It turns out that, for all closed oriented surfaces a possible choice is the *height function.* Simply imagine the surface placed in our three dimensional Euclidean space and choose f to assign the value of the *z-th coordinate* to each point on the surface. Critical points of *this particular* function are easily found *by inspection* and the same "method" also immediately reveals *their type.*

Example 3.3.2.1: Take as a surface S the *sphere* S^2 and as a function f on S the height function (value of the z-coordinate, see Figure 14). It has a single maximum (uppermost), single minimum (undermost) and no saddle points, so that it gives $\chi(S^2) = 2$. In order to make a check, imagine a tetrahedron drawn on the sphere (it *does* provide a *triangulation* of the latter). For the tetrahedron one has $f = 4, e = 6, v = 4$, so that $f - e + v = 2$. Or, do the check via total Gaussian curvature: $K = 1/\rho^2$, so that $\int_{S^2} K dA = (1/\rho^2)A = 4\pi\rho^2/\rho^2 = 4\pi$. This only matches Gauss-Bonnet theorem (39) if $\chi(S^2) = 2$. Or, finally, see (30).

Example 3.3.2.2: Consider $S \equiv S_g$, a surface with *genus* g. Namely, either the sphere S^2 with g handles or, alternatively, a button with g holes. (It turns out the two surfaces are topologically equivalent (homeomorphic)). In particular, $g = 0$ reduces to the sphere itself and $g = 1$ corresponds to the *torus* T^2. For S_g the Euler characteristics is given by a simple formula

$$\boxed{\chi(S_g) = 2(1 - g)} \tag{64}$$

Indeed, consider the button realized so that its holes stand in a vertical column

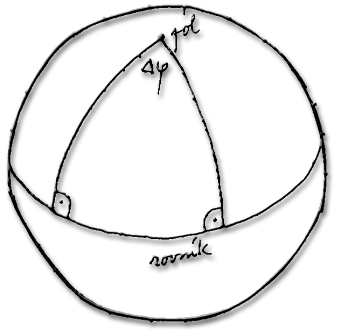

Figure 16: Here the formula works

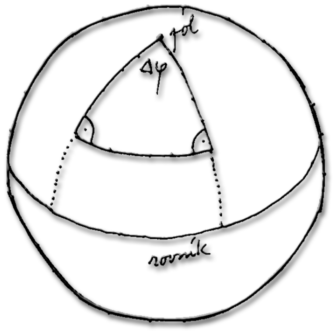

Figure 17: Here one should be careful

(see Figure 15). Choose, again, the function f to be the height function (value of the coordinate z). Now we have a single maximum (uppermost), single minimum (undermost) and, in addition, $2g$ *saddle points* (in *each hole* one saddle up and one saddle down). This gives $\chi(S_g) = 1 - 2g + 1$.

3.4 Few statements closely related to G-B theorem

Here we mention some statements closely related to Gauss-Bonnet theorem. Some of them might be fairly hard to prove without knowing the theorem.

Statement 1: Sum of angles of a triangle in the (Euclidean) plane is π.
Indeed: The formula (55) gives, for the plane ($K = 0$) and for a usual (geodesic!) triangle ($n = 3, k = 0$) just π (extremely surprising :-)

Statement 2: Sum of angles of a geodesic triangle on the sphere with radius ρ is $\pi + A/\rho^2$ (here A stands for the area of the triangle).
This is *Girard's theorem.* The formula (55) gives for a geodesic triangle ($n = 3, k = 0$) on the sphere ($K = 1/\rho^2$) just $\pi + A/\rho^2$.

Statement 3: The formula from the last example is often illustrated on the triangle, whose vertices are the North Pole plus two vertices on the equator, separated by the angle $\Delta\varphi$ (of arbitrary magnitude; the edges are then parts of two meridians and of the equator, see Figure 16). The sum of its angles is $\Delta\varphi + \pi$ (by inspection), the area is $A = \rho^2 \Delta\varphi$ (by the same method), so that $\Delta\varphi + \pi = \pi + A/\rho^2$ indeed holds. If we, however, replaced the equator by the parallel line with ϑ_0 (see Figure 17), the formula cease to be valid. Why? Is there a way to recover the right result?
Solution: The parallel line is *not* a geodesics. So, it is not a geodesic triangle. We are to add the term containing the geodesic curvature. Its computation: on

the parallel line, using the natural parameter, we have $\dot{\gamma} = e_\varphi$, so the acceleration reads $a = \nabla_{e_\varphi} e_\varphi = \omega^\vartheta_\varphi(e_\varphi)e_\vartheta = \langle \alpha, e_\varphi \rangle e_\vartheta$. Since e_ϑ is directed to the right w.r.t. e_φ, geodesic curvature $k = -\langle \alpha, e_\varphi \rangle = \cos\vartheta_0/\rho\sin\vartheta_0$. Its integration along the relevant piece of the parallel line gives $\Delta\varphi\cos\vartheta_0$. The area of the triangle is $A = \rho^2\Delta\varphi(1-\cos\vartheta_0)$, so that $A/\rho^2 = \Delta\varphi(1-\cos\vartheta_0)$. The sum of the angles in the triangle is still $\Delta\varphi + \pi$ (the parallel line is still perpendicular to the meridian). If we forgot the term containing geodesic curvature, we would assert that

$$\Delta\varphi + \pi = \pi + A/\rho^2 = \pi + \Delta\varphi(1-\cos\vartheta_0)$$

which is *not* true, whereas after taking the term seriously we assert that

$$\Delta\varphi + \pi = \pi + A/\rho^2 + \int k ds = \pi + \Delta\varphi(1-\cos\vartheta_0) + \Delta\varphi\cos\vartheta_0$$

which (for a change) *does* hold.

Statement 4: Consider a manifold with topology of T^2 (two-dimensional torus). Define a(rbitrary) metric tensor on it. Compute the corresponding Gaussian curvature K. Then the function K necessarily vanishes in some point on T^2.
Solution: First one has to check that $\chi(T^2) = 0$. (Either via appropriate triangulation or, more easily, using the formula $\chi(S_g) = 2(1-g)$ for a surface with genus g.) Since the torus has no boundary, Gauss-Bonnet theorem (39) says that $\int_{T^2} K dA = 0$. Then, the function K has no other choice than to vanish somewhere.

Statement 5: If, on a simply connected surface with *negative* Gaussian curvature two geodesics rush out, they never meet again. (A fairly sad story, you have to admit.)
Solution: By contradiction. If they met (see Figure 18), a domain S between them would be created (simple connectedness guarantees one can deform the first geodesics to the other one, drawing thus the domain). Apply G-B theorem (38) to S. The boundary consists of two geodesics, so $k = 0$. Topology of S is the one of (say) one triangle, i.e. $\chi(S) = 1$. So l.h.s. is negative whereas r.h.s. is positive. We should never tolerate equations to behave so poorly.

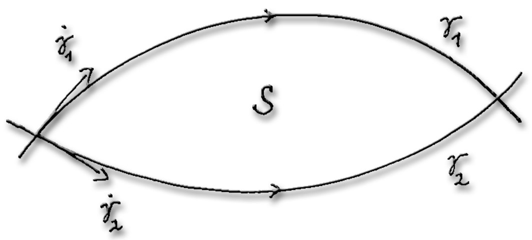

Figure 18: This assumption (that they meet) leads to a contradiction

4 Fiber bundles and connections

Fibre bundles play very important role in topology. They are used in a number of ways. Here we just introduce basic concepts, more or less intuitively, and then focus our attention on how *connections* are treated in this setting. For this, *principal* (and associated vector) bundles play the key role.

4.1 Cartesian products and bundles

Cartesian product in general is a useful concept in mathematics. It enables one to regard a more complicated object as kind of combination ("product") of simpler objects. For example, we can regard a square as a product of two intervals. What does it mean? It means the following - instead of speaking of a *single* point s of the square (the more complicated object) we speak of *two* points, a and b, each one from the corresponding interval (two simpler objects).

In general let A and B be sets. Then $A \times B$ is a new set whose points are *ordered pairs* (a, b) where $a \in A$ and $b \in B$. If the sets happen to be manifolds, one can easily make a manifold from $A \times B$ as well (see (1.3.3) in [2]). There is a natural *projection* map on the first factor [6]

$$\pi_1 : A \times B \to A \qquad (a, b) \mapsto a \tag{65}$$

(so it simply "forgets about" the second entry of the pair (a, b)). This simple structure, the two manifolds $A \times B$ and A, together with the projection map π_1, constitute an extremely useful and powerful tool called *fiber bundle* (its simplest version, the *product* bundle). Where are the "fibers" from the nomenclature? They are the subsets "(a, B)" $\subset A \times B$, that is the pairs (a, b) such that a is fixed and b runs over whole B. Notice that their alternative (and more succinct) description is

$$\text{fiber over } a := \pi^{-1}(a) \equiv \text{ the inverse image of the point } a \tag{66}$$

(those points of $A \times B$ which π_1 maps to a). The *total space* of the bundle, here the manifold $A \times B$, is "fibered", i.e. it is a union of all fibers.

In a general fiber bundle this product structure is still present, but it is only required *locally*. So, there are two manifolds, the *total space* E and the *base* M, plus a (surjective) map, the *projection*, from the total space onto the base

$$\pi : E \to M \tag{67}$$

such that any two fibers ($\pi^{-1}(m_1)$ and $\pi^{-1}(m_2)$) are "equal" (both of them diffeomorphic to some F). What means "local product" structure? For each point m on M such a neighborhood $\mathcal{O}$ should exist (perhaps fairly "small") that its inverse image, $\pi^{-1}(\mathcal{O})$, "is" the product of $\mathcal{O}$ and F. (Note, however, that, contrary to

[6] There is also a map π_2 to the second factor, here, but this map does not survive passing to a general case.

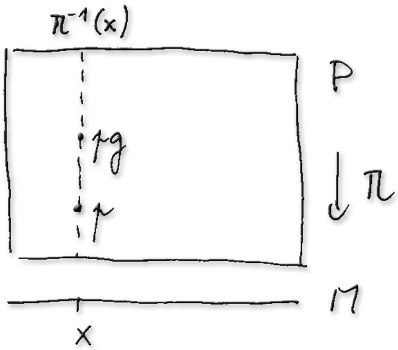

Figure 19: Principal bundle - action of a Lie group in the total space along fibres

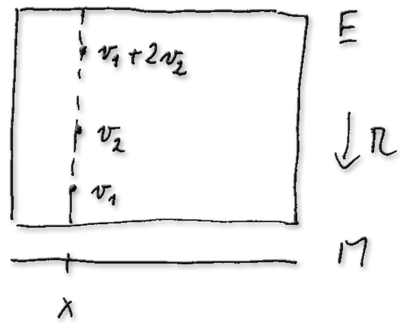

Figure 20: Vector bundle - linear structure available in each fibre

the product bundle, the *whole* total space $E \equiv \pi^{-1}(M)$ is *not* required to be the product $M \times F$.)

Perhaps the simplest way to see the difference is to compare two concrete bundles, the *cylinder* and the *Möbius strip*. They both share the same base (it is a circle S^1) and fibre (an interval), but they clearly substantially differ in topology of the total space. The cylinder *is* a product bundle whereas the Möbius strip *is not*. However, note that even on the Möbius strip, the inverse image of a neighborhood of any point on the base S^1 *is* a product (of this neighborhood and the interval). So, there is *no difference* between inverse images of such neighborhoods on the cylinder and on the Möbius strip. The difference suddenly appears when the inverse images of the *whole* base manifold, the circle, are compared. One can say, that both bundles ale glued from equal pieces (products of two intervals, in this case), but the way *how* they are *glued together* is different. Exactly in this point bundles substantially generalize the concept of Cartesian products. And this particular generalization turns out to be very fruitful.

4.2 Principal and (associated) vector bundles

In interesting cases, the fibers, which are sub-manifolds in the total space E, are endowed with some structure. For connection theory, two kinds of structures in fibers are of great importance.

First, in *principle bundles*, denoted here as $\pi : P \to M$, each fibre is a *principle G-space* for some Lie group G, see Figure 19. This means that the group G *acts* in each fibre (for each group element g there is a transformation R_g of the fibre) and the action enjoys the following properties. It is

- *right* (i.e. $R_{g_1 g_2} = R_{g_2} R_{g_1}$)
- *free* (i.e. each non-unit $G \ni g \neq e \in G$ really moves all points away)

- *transitive* (any two points in the fiber may be connected by action of the group)

From these three properties it follows, that each fibre is diffeomorphic to the group G itself when regarded just as a manifold. (It is, however *not* a group, there is, for example, no multiplication rule for two points. The total space, as a manifold, is then glued from pieces of the product structure $\mathcal{O} \times G$, where $\mathcal{O} \subset M$.)

The actions in all fibers combine into a single action on the *total space* P:

$$R_g : P \to P \qquad p \mapsto pg$$

which is right, free and transitive *in each fibre.*

Second, in *vector bundles*, each fibre is a *vector space*, see Figure 20. (The total space, as a manifold, is then glued from pieces of the product structure $\mathcal{O} \times \mathbb{R}^n$, where $\mathcal{O} \subset M$.)

Example 4.2.1: For each manifold M, there exits the *frame bundle* $\pi : LM \to M$. The points of LM are (by definition) all frames (bases) in all points of the manifold M. (For each point $x \in M$ there is the *tangent space* T_xM in x. It is an n-dimensional vector space ($n = \dim M$) and we can choose, in infinitely many ways, a basis e_a there. The totality of all these bases (frames in x) gives the fiber over x.) There is a natural action of $GL(n, \mathbb{R})$ on LM: $e_a \mapsto A^b_a e_b \equiv (R_A e)_a$. So, the frame bundle of an n-dimensional manifold M provides an example of a principal $GL(n, \mathbb{R})$-bundle.

Example 4.2.2: For each manifold M, there exits the *tangent bundle* $\pi : TM \to M$. The points of TM are (by definition) all (tangent) vectors (elements of T_xM) in all points x of the manifold M. (The totality of all vectors in x (elements of T_xM) gives the fiber over x.) There is a natural linear structure in each fiber of TM (i.e. in each T_xM; clearly each fiber is isomorphic to $\mathbb{R}^n$). So, the tangent bundle of an n-dimensional manifold M provides an example of a vector bundle (with n-dimensional fibre).

Example 4.2.3: The *Hopf map* mentioned in (37)

$$\pi : S^3 \to S^2 \qquad \chi \mapsto \mathbf{n} \equiv \chi^+ \boldsymbol{\sigma} \chi \qquad \chi \in S^3 \subset \mathbb{C}^2 \qquad \chi^+ \chi = 1 \tag{68}$$

may be regarded as the *projection* in a principal $U(1)$-bundle over S^2, known as the *Hopf bundle.* The action of $U(1)$ on S^3 reads $\chi \mapsto e^{i\alpha}\chi$. You can check that $\mathbf{n} \mapsto \mathbf{n}$, so that $U(1)$ indeed acts in fibers. This bundle (and a natural connection in it) may be used in description of magnetic monopoles as well as Berry phase.

Consider a *principal* G-bundle $\pi : P \to M$. In addition, imagine we have a representation ρ of G in a vector space V. Then, there is a construction (see Section 21.4. in [2]) enabling one to "replace" each fiber in P by the vector space V. In this way, a new *vector* bundle over M is obtained. It is said to be *associated* with the original principle bundle. The other way round, one can construct, for a vector bundle whose fibres carry a representation of a group G, a unique principle bundle such, that the original vector bundle is associated with the principal one.

It is important to remark, that the way in which a principal bundle is glued from simple (product) pieces is inherited by the associated vector bundle.

It turns out, as an example, that the tangent bundle (from Example 4.2.2) is associated with the frame bundle (from Example 4.2.1).

4.3 Connections

Consider the following two important structures used in theoretical physics:

- *parallel transports*, semicolons, Riemannian curvature tensor and all this stuff well known from, say, general relativity (or advanced continuum mechanics) and

- the (classical) *gauge field* theory.

Surprisingly enough, both of them may be incorporated into an elegant and (conceptually) simple *common* scheme, called by mathematicians *connection theory.* (It is, unfortunately, not so short and trivial story to show here how exactly this is actually done. The interested reader is referred to chapters 19-21 in [2]).

The playing ground for the theory is a principal G-bundle. In this section, just some essential ideas will be mentioned. We focus on the concept of the *curvature form.* This concept will be then used, in the next chapter, in construction of topological invariants based on "characteristic classes".

In order to define a *connection in a principal G-bundle $\pi : P \to M$*, one has to introduce a *connection form* ω, a one-form of type Ad (see Appendix C.1) on the total space P. So, first, they are one-forms $(\omega^1, \dots, \omega^n)$ on P, where $n =$ the dimension of the Lie algebra $\mathcal{G}$ of the group G. They combine into a *single* object, *Lie algebra valued* one-form

$$\omega = \omega^i E_i \qquad E_i \in \mathcal{G} \tag{69}$$

And finally, under the action of the group G on P the resulting ω behaves as follows

$$R_g^*\,\omega = \mathrm{Ad}_{\,g^{-1}}\,\omega \qquad \text{or, equivalently} \qquad R_g^*\,\omega^i = (\mathrm{Ad}_{\,g^{-1}})^i_j\,\omega^j \tag{70}$$

The form ω (living on the total space P) carries all information about the connection. In terms of ω, say, one can perform *parallel transport* of various objects on (the base) M. Notorious phenomenon of a parallel transport is its *path-dependance.* Information, whether a particular connection *really* leads to path-dependance phenomena, sits in the *curvature form* Ω. It is a *two*-form of type Ad living on the total space P. It may be computed from ω by means of the formula

$$\Omega = \Omega^i E_i \qquad \Omega^i = d\omega^i + \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k \tag{71}$$

where c^i_{jk} are structure constants of the Lie algebra $\mathcal{G}$ w.r.t. the basis E_i (so that $[E_i, E_j] = c^k_{ij} E_k$).

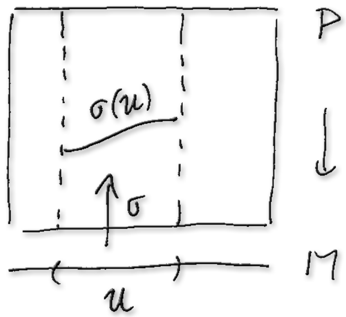

Figure 21: In physics, section is called a *gauge*. It directs upwards, right *against* the projection.

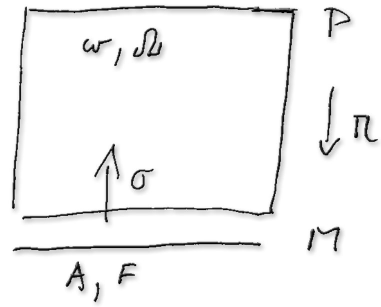

Figure 22: Therefore its pull-back σ^* directs downwards, *along* the projection. One can "download" ω and Ω and get A and F.

Both ω and Ω live "upstairs", on P. But physics usually takes place "downstairs", on M. So a way is needed to somehow get ω and Ω on M. It is achieved with the help of *local sections*.

A local section σ (in physics known as a *gauge*) is a mapping

$$\sigma : M \supset \mathcal{O} \to P \qquad \pi \circ \sigma = \mathrm{id}_{\mathcal{O}} \tag{72}$$

from an open subset $\mathcal{O}$ on the base M to the total space P. (The additional condition $\pi \circ \sigma = \mathrm{id}_{\mathcal{O}}$ simply says that the image of x lies in the fiber over x.) Then, its *pull-back* σ^* "downloads" forms from P to $\mathcal{O}$, see Fig.21 and Fig.22. In this way we get $\sigma^*\omega$ (a one-form on $\mathcal{O}$) and $\sigma^*\Omega$ (a one-form on $\mathcal{O}$). The nomenclature is as follows:

$$\omega^i = \text{connection forms} \tag{73}$$
$$\Omega^i = \text{curvature forms} \tag{74}$$
$$\sigma^*\omega^i \equiv A^i = \text{gauge potentials} \tag{75}$$
$$\sigma^*\Omega^i \equiv F^i = \text{gauge field strengths} \tag{76}$$

So, gauge potentials and gauge field strengths live on M, "where physics takes place", and they depend on the choice of section (they are *gauge dependent*).

Now imagine we have as many as *two* sections (see Figure 23), σ and σ' (both $M \supset \mathcal{O} \to \pi^{-1}\mathcal{O} \subset P$). Quantitatively, they are related by a function $S : \mathcal{O} \to G$ such that $\sigma'(x) = \sigma(x)S(x)$, see Figure 24. Then one can "download" forms from P to $\mathcal{O}$ in two ways, too, and get A and A' as well as F and F'. There is a natural (and useful) question of how the two results are related. (In physics, these are *gauge transformations* of A and F.) Well, the transformation rule for A is

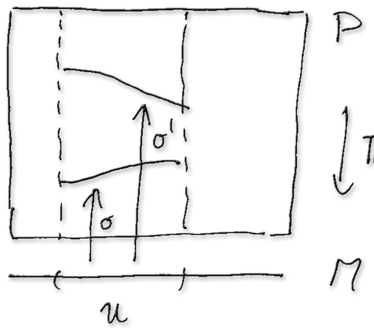

Figure 23: Two different sections (in physics, two different *gauges*).

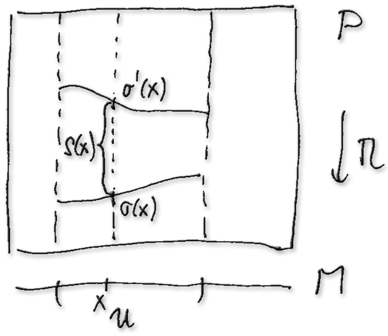

Figure 24: Quantitative measure of how two sections are related is a function $S : \mathcal{O} \to G$ such that $\sigma'(x) = \sigma(x)S(x)$.

more complicated and, fortunately, we do not need it. The formula for $F \mapsto F'$ is, however, simple:

$$F' = \mathrm{Ad}_{S^{-1}} F \equiv S^{-1} F S \tag{77}$$

(see (21.2.3) in [2]; the second expression being only valid for *matrix* groups).

Let x^μ be local coordinates on $\mathcal{O} \subset M$. Then we can write

$$A^i = A^i_\mu(x) dx^\mu \tag{78}$$

$$F^i = \frac{1}{2} F^i_{\mu\nu}(x) dx^\mu \wedge dx^\nu \tag{79}$$

So, in components, A carries two kinds of indices. Lie algebra index i and coordinate tensor index μ. Similarly, F carries (still a single) Lie algebra index i and a *pair* of coordinate tensor indices $\mu\nu$.

Since the most interesting Lie groups (and their Lie algebras) are realized as square *matrices*, instead of the Lie algebra index a *pair* of *matrix* indices may also be used, see Appendix C.2.

Example 4.3.1: Consider a $SU(2)$-connection. Here, the curvature 2-form F is $su(2)$-valued, i.e.

$$F = F^j E_j = -\frac{i}{2} F^j \sigma_j \equiv -\frac{i}{2} \begin{pmatrix} F^3 & F^1 - iF^2 \\ F^1 + iF^2 & -F^3 \end{pmatrix} \equiv \begin{pmatrix} F^1_1 & F^1_2 \\ F^2_1 & F^2_2 \end{pmatrix} \tag{80}$$

5 Characteristic classes

There is a way, invented by Chern and Weil, to characterize *topology* of a fibre bundle in terms of closed forms on the base of the bundle. *Curvature* form of a *connection* is used for the construction. So, de-Rham cohomology classes may be associated with the bundle.

[This construction forms a bridge between algebraic topology and differential geometry. Connection is a geometrical concept defined on the bundle and it is used to construct an object from algebraic topology, the cohomology class. And, what is essential, it turns out that the resulting class actually does not depend on the particular choice of the connection. Thus, geometry is just a means of computing something, which is purely topological.]

The forms are known as *characteristic classes.* Two bundles, whose characteristic classes differ, cannot be equivalent. Trivial bundles have trivial (zero) characteristic classes. In this chapter, we first describe a general scheme of the construction, and then we show explicit examples, which lead to standard characteristic classes (Chern, Pontryagin and Euler).

From the perspective of physics, the logic is a bit reversed. In physics, the integrals come as objects of primary interest (they have direct physical meaning). Useful conclusions then may be deduced by recognizing their topological meaning.

5.1 Chern-Weil theory

Let $\Omega = \Omega^i E_i$ be the curvature 2-form of a connection in a principal G-bundle $\pi : P \to M$. Consider a symmetric, multilinear, Ad-invariant form (with k entries) in the Lie algebra $\mathcal{G}$

$$w : \mathcal{G} \times \cdots \times \mathcal{G} \to \mathbb{R} \qquad w(\mathrm{Ad}_g X, \dots, \mathrm{Ad}_g Y) = w(X, \dots, Y) \tag{81}$$

Then

$$w(\Omega, \dots, \Omega) := w(E_i, \dots, E_j)\ \Omega^i \wedge \cdots \wedge \Omega^j \equiv w_{i \dots j}\ \Omega^i \wedge \cdots \wedge \Omega^j \tag{82}$$

is a $2k$-form on P

$$w(\Omega, \dots, \Omega) \in \Omega^{2k}(P) \tag{83}$$

This form is *closed.* [7]

▼ First notice, that it is invariant with respect to the action R_g of G on P (Ω is of type Ad and w is Ad-invariant) so that $D = d$ here. Then

$$\begin{aligned} d(w(\Omega, \dots, \Omega)) &= D(w(\Omega, \dots, \Omega)) \\ &= w(D\Omega, \dots, \Omega) + \cdots + w(\Omega, \dots, D\Omega) \\ &= w(0, \dots, \Omega) + \cdots + w(\Omega, \dots, 0) \\ &= 0 \end{aligned}$$

[7] It is even *exact*, but this is not important, now.

where the *Bianchi identity* $D\Omega = 0$ was used. ▲

It turns out that this form on P induces a well-defined *global and closed* $2k$*-form on* M

$$\boxed{w(F, \dots, F) \in Z^{2k}(M)} \tag{84}$$

(It is, however, already *not* exact on M, contrary to $w(\Omega, \dots, \Omega)$ on P.) So, it also defines a *cohomology class on* M.

▼ Consider two local sections, σ and σ' (both $M \supset \mathcal{O} \to \pi^{-1}\mathcal{O} \subset P$), related by $\sigma'(x) = \sigma(x)S(x)$, $S : \mathcal{O} \to G$. Recall, that pull-backs of Ω with respect to the two sections, $F := \sigma^*\Omega$ and $F' := \sigma'^*\Omega$ (i.e. field strengths in two gauges, in physical parlance), are related via the simple formula (see (77))

$$F' = \mathrm{Ad}_{S^{-1}} F \equiv S^{-1} F S \tag{85}$$

(*gauge transformation* formula for F). This means, however, that the pull-back of $w(\Omega, \dots, \Omega)$ does *not* depend [8] on the choice of section (= it is gauge invariant):

$$w(F', \dots, F') = w(\mathrm{Ad}_{S^{-1}} F, \dots, \mathrm{Ad}_{S^{-1}} F) = w(F, \dots, F) \in \Omega^{2k}(\mathcal{O}) \tag{86}$$

(since w is Ad-invariant).

Consequently, we get a *global* $2k$*-form on* M: Take any covering of M by open subsets together with a local section on each of these subsets. Then, on any intersection, we get the same form starting with any section.

Closeness: simply because $d\sigma^* = \sigma^* d$. ▲

So we learned that given a symmetric Ad-invariant k-linear form on $\mathcal{G}$, a unique element of the cohomology class $H^{2k}(M)$ is induced. This element is given by the representative $w(F, \dots, F) \in Z^{2k}(M)$ and remember that a connection ω is needed for its construction.

However, it turns out - surprisingly enough - that the resulting cohomology *class* $[w(F, \dots, F)] \in H^{2k}(M)$ does not depend at all on the connection (see Appendix D.1)! It is given by the bundle alone (the way it is glued together from trivial pieces).

[This allows us, from the pragmatic point of view, to proceed as follows: If we need to compute the class $[w(F, \dots, F)]$ and there are *two* connections available for the computation, we choose the one for which the job is *easier* :-)]

Moreover, standard reasoning described in section (2.1) says, that the integral over a $2k$*-cycle* S ($2k$-dimensional closed surface)

$$\int_S w(F, \dots, F) \tag{87}$$

[8] Notice that this is *not* the case for the pull-back of Ω itself.

of the $2k$-form $w(F, \ldots, F)$ does not depend on the particular cycle within the same homology class, i.e. it is not changed if $S \mapsto S + \partial D$ (D being a $2k+1$-dimensional chain (domain)). So, in addition to its insensitivity to the change of connection it does also not depend on a smooth deformation of the cycle.

By this general construction we get a *topological invariant* associated with the bundle $\pi : P \to M$.

$$\boxed{\int_S w(F, \ldots, F) = \text{ topological invariant}} \tag{88}$$

In the next three subsections we will see how some particular mappings of the needed type (81) may be (fairly easily) constructed. Then we will be able to write down explicit expressions for corresponding topological invariants.

5.2 Pontryagin classes

Consider, for any $n \times n$ real matrix A and a real number λ, the following remarkable expression

$$\det(\lambda \mathbb{I} + A) \tag{89}$$

For example, in the case $n = 2$ we have the explicit result

$$\det(\lambda \mathbb{I} + A) = \det \begin{pmatrix} \lambda + A_{11} & A_{12} \\ A_{21} & \lambda + A_{22} \end{pmatrix} = \lambda^2 + \lambda \mathrm{Tr}\, A + \det A \tag{90}$$

For general n, it is clearly a *polynomial in* λ. Its coefficients are real numbers as well. From the way how a determinant is computed one can see, that the coefficient standing by λ^{n-k} is a *polynomial of order* k in matrix elements of A. So one can write

$$\det(\lambda \mathbb{I} + A) = \sum_{k=0}^{n} \lambda^{n-k} P_k(A) \tag{91}$$

and regard this formula as a convenient way to define the polynomials $P_k(A)$.

Once more back to $n = 2$ case - we get explicitly

$$\begin{aligned} P_0(A) &= 1 && (92) \\ P_1(A) &= A_{11} + A_{22} \equiv \mathrm{Tr}\, A && (93) \\ P_2(A) &= A_{11}A_{22} - A_{12}A_{21} \equiv \det A && (94) \end{aligned}$$

Recall that the Ad-representation of any matrix group is given by a simple formula

$$\mathrm{Ad}_B C = BCB^{-1} \tag{95}$$

Therefore it is clear that if we interpret the matrix A in (91) as an element of the *Lie algebra* $gl(n, \mathbb{R})$ (or any of its subalgebras), all the polynomials $P_k(A)$ become *Ad-invariant.*

▼ On the left-hand side we have

$$\begin{aligned}\det(\lambda\mathbb{I} + \mathrm{Ad}_B A) &= \det\left(\lambda\mathbb{I} + BAB^{-1}\right) \\ &= \det(B\,(\lambda\mathbb{I} + A)\,B^{-1}) \\ &= \det(\lambda\mathbb{I} + A)\end{aligned}$$

Therefore the right-hand side has to have the same property:

$$P_k(\mathrm{Ad}_B A) = P_k(A) \tag{96}$$

▲

Imagine now that A belongs to the *subalgebra* $so(n) \subset gl(n, \mathbb{R})$ (we will see in a moment why this case is interesting). Then we find that

$$A \in so(n) \qquad \Rightarrow \qquad P_k(A) = 0 \qquad \text{for } k \text{ odd} \tag{97}$$

▼ Matrices from $so(n) \subset gl(n, \mathbb{R})$ are *skew-symmetric*, $A^T = -A$. Then, on the left-hand side, we have

$$\begin{aligned}\det(\lambda\mathbb{I} + A) &= \det(\lambda\mathbb{I} + A)^T \\ &= \det(\lambda\mathbb{I} - A) \\ &= \det(\lambda\mathbb{I} + (-A))\end{aligned}$$

Therefore the right-hand side has to have the same property:

$$\sum_{k=0}^{n} \lambda^{n-k} P_k(A) = \sum_{k=0}^{n} \lambda^{n-k} P_k(-A) \tag{98}$$

so that

$$P_k(A) = P_k(-A) \tag{99}$$

But $P_k(A)$ is a polynomial of order k so, at the same time,

$$P_k(-A) = (-1)^k P_k(A) \tag{100}$$

from which (97) follows. ▲

For explicit computation (and results) of the polynomials $P_k(A)$ the reader is referred to Appendix D.2.

Ok, come to bundles and connections. Consider a *real* n-dimensional *vector* bundle. There is a (unique) *principal* $GL(n, \mathbb{R})$-bundle $\pi : \hat{P} \to M$ behind it. Let $\hat{\omega}$ be any connection form on $\hat{P}$ and let $\hat{\Omega}$ be the corresponding curvature form. The latter is a $gl(n, \mathbb{R})$-valued 2-form, so we can regard it as a $n \times n$ matrix with 2-form entries.

Now imagine we introduced a metric tensor into each fiber of the vector bundle (a smooth *fiber metric*; one can show it *is always* possible). Then the group which preserves this (stronger) structure is the (smaller) group of orthogonal matrices, $O(n)$, a *subgroup* of the original group $GL(n, \mathbb{R})$. Then, there is a smaller principal bundle $\pi : P \to M$, which is only $O(n)$-bundle (it is a *subbundle* of $\pi : \hat{P} \to M$). The vector bundle endowed with the fiber metric is an associated vector bundle for $\pi : P \to M$. So, we can introduce a new connection in the new bundle, with connection form ω and curvature form Ω, both of them being just $o(n)$-valued; in particular, we can regard the curvature form Ω as an $n \times n$ *skew-symmetric* matrix with 2-form entries.

Introducing of a fibre metric has clearly no consequence at all on the topology of the vector bundle. So, if there is some topological invariant which characterizes the bundle, it has to have the same "value" after introducing metric as it had before. Now, there is a technique (see e.g. section 20.5. in [5]) which enables one to (uniquely) "extend" a connection from the "restricted" bundle P (subbundle) to the original "big" one $\hat{P}$. So, in our case here, a connection appears by this mechanism on $\hat{P}$. This means, however, we have already as many as *two* (in general different) connections on $\hat{P}$. The one introduced before (with curvature form $\hat{\Omega}$) and the connection obtained by "extension" of the connection from the subbundle $\pi : P \to M$. Which one is better to compute topological invariants?

Well, according to the earlier discussion (see the text in the bracket in section 5.1), the result will be the same, but notice, that when using the second one we can profit from the fact that the curvature matrices are *skew-symmetric*. So, we choose *this* possibility.

That is to say, let us write down the expressions (91) with the matrix A replaced by $-F/2\pi$ (which is an antisymmetric matrix with entries = 2-forms on M)

$$\det\left(\lambda \mathbb{I} - \frac{1}{2\pi} F\right) = \sum_{k=0}^{n} \lambda^{n-k} P_k\left(-F/2\pi\right) \equiv \sum_{k=0}^{n} \lambda^{n-k} \mathcal{P}_k(F) \tag{101}$$

so that

$$\mathcal{P}_k(F) = P_k\left(-F/2\pi\right) \tag{102}$$

Since the Ad-invariant polynomials $\mathcal{P}_k(A)$ are sums of products of k matrix elements and the matrix elements are 2-forms, the polynomials $\mathcal{P}_k(F)$ are $2k$-forms, now. They are nothing but particular examples of global an closed forms $w(F, \dots, F) \in Z^{2k}$ mentioned in (84). Then, as we know from the general expression (88), integrals of any of $\mathcal{P}_k(F)$ over $2k$-cycles ($2k$-dimensional closed surfaces) are topological invariants

$$\boxed{\int_{S_{2k}} \mathcal{P}_k(F) = \text{ topological invariant}} \tag{103}$$

(of the bundle $\pi : P \to M$ as well as $\pi : \hat{P} \to M$).

▼ Two issues should be perhaps explained a bit more carefully. First, why we replace A by $-F/2\pi$ and not just F? If we put just F, we obtained a topological invariant as well! The reason is rather deep (but the proof is far beyond the level of exposition in this text). Namely it turns out that the correction factor $1/2\pi$ guarantees that the *numbers* obtained by integration of the polynomial over c_{2k} become *integers*.

Second, we mentioned that we use the "better" connection on P in order to obtain F skew-symmetric. This is really the case provided we only use local sections with image in $\hat{P} \subset P$. Although a *general* section of $\pi : P \to M$ does *not* enjoy this property (so that F *fails* to be skew-symmetric), since we know that the resulting *polynomial* (87) does not depend on particular choices of sections, we can pretend we only work with specific sections with image in $\hat{P} \subset P$. It then greatly helps, for example, to see that $P_k(-F/2\pi)$ necessarily vanishes for k = odd. ▲

Since it is a wasting of ink (electrons) to denote *vanishing* polynomials as $\mathcal{P}_{2k+1}(F)$:-), we rename those which can in principle survive as follows:

$$P_{2k}(-F/2\pi) \equiv \mathcal{P}_{2k}(F) =: p_k(F) \tag{104}$$

so that now

$$\det\left(\lambda\mathbb{I} - \frac{1}{2\pi}F\right) = \lambda^n p_0(F) + \lambda^{n-2}p_1(F) + \lambda^{n-4}p_2(F) + \dots \tag{105}$$

Taking into account the explicit expressions of $P_k(A)$ (see Appendix D.2) and relations $p_k \leftrightarrow \mathcal{P}_k \leftrightarrow P_k$ from (104), we get the first few *Pontryagin classes*:

$$p_0(F) = 1 \tag{106}$$

$$p_1(F) = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{Tr}\, F^2 \tag{107}$$

$$p_2(F) = +\frac{1}{8}\left(\frac{1}{2\pi}\right)^4 ((\mathrm{Tr}\, F^2)^2 - 2\mathrm{Tr}\, F^4) \tag{108}$$

where, in more detail,

$$\begin{aligned} \mathrm{Tr}\, F^2 &\equiv F^a_b \wedge F^b_a \\ (\mathrm{Tr}\, F^2)^2 &\equiv F^a_b \wedge F^b_a \wedge F^c_d \wedge F^d_c \\ \mathrm{Tr}\, F^4 &\equiv F^a_b \wedge F^b_c \wedge F^c_d \wedge F^d_a \end{aligned}$$

Notice, that

$$\begin{aligned} p_1(F) &= \text{4-form on } M \\ p_2(F) &= \text{8-form on } M \\ p_3(F) &= \text{12-form on } M \\ &\text{etc.} \end{aligned}$$

so that we need a sufficiently large dimensional base manifolds (at least dim M = 4 for giving $p_1(F)$ a chance, etc.) in order to profit from these invariants in a non-trivial way.

As we already mentioned above, integrals (over closed $4k$-dimensional surfaces) of Potryagin classes are always *integers*. They are known as Pontryagin *numbers*

$$\boxed{p_k := \int_{S_{4k}} p_k(F) = \ k\text{-th } \textit{Pontryagin number}} \tag{109}$$

▼ Recall that Potryagin classes (and, consequently, Potryagin numbers) are topological characteristics of vector (or corresponding principal) *bundles* and do *not* depend on particular choice of connection in this bundle. So, the (standard) notation $p_k(F)$ (in which F is present) is a bit misleading; $p_k(E)$, where $E \to M$ is the vector bundle, would be more pregnant. In this sense, the Euler class discussed in Example 5.4.1 is a positive exception. ▲

5.3 Chern classes and Chern numbers

Now, consider a *complex* n-dimensional *vector* bundle. There is a (unique) *principal* $GL(n, \mathbb{C})$-bundle $\pi : \hat{P} \to M$ behind it. Let $\hat{\omega}$ be any connection form on $\hat{P}$ and let $\hat{\Omega}$ be the corresponding curvature form. The latter is a $gl(n, \mathbb{C})$-valued 2-form, so we can regard it as a $n \times n$ matrix with *complex valued* 2-form entries (i.e. each entry is an expression $\alpha + i\beta$, where both α and β are "ordinary" (= real valued) 2-forms).

Now, we can repeat, with minor modifications, the steps we did in the case of Potryagin classes. We can introduce a smooth *hermitean* fibre metric. Then the group which preserves this (stronger) structure is the (smaller) group of unitary matrices, $U(n)$, a *subgroup* of the original group $GL(n, \mathbb{C})$. Then, there is a smaller principal bundle $\pi : P \to M$, which is only $U(n)$-bundle (it is a *subbundle* of $\pi : \hat{P} \to M$). The vector bundle endowed with the hermitean fiber metric is an associated vector bundle for $\pi : P \to M$. Again, we can introduce a new connection in the new bundle, with connection form ω and curvature form Ω, both of them being just $u(n)$-valued; in particular, we can regard the curvature form Ω as an $n \times n$ *anti-hermitean* matrix with 2-form entries (i.e. it becomes an anti-hermitean matrix after inserting two tangent vector arguments).

That is to say, we first consider, for real λ and anti-hermitean $n \times n$ matrix A, determinant

$$\det(\lambda \mathbb{I} + iA) = \sum_{k=0}^{n} \lambda^{n-k} P_k(iA) \tag{110}$$

The polynomials $P_k(iA)$ are *real* (in spite of the fact, that they are constructed from a hermitean - i.e. *complex* - matrix iA).

▼ Indeed, complex conjugation of the l.h.s. gives

$$[\det(\lambda \mathbb{I} + iA)]^* = \det(\lambda \mathbb{I} + iA)^+ = \det\left(\lambda \mathbb{I} + (iA)^+\right) = \det(\lambda \mathbb{I} + iA)$$

Therefore also the r.h.s. is real and λ *is* real. ▲

Then we replace the matrix A by $F/2\pi$; we get

$$\det\left(\lambda\mathbb{I} - \frac{1}{2\pi i}F\right) = \sum_{k=0}^{n} \lambda^{n-k} P_k\left(iF/2\pi\right) \equiv \sum_{k=0}^{n} \lambda^{n-k} c_k(F) \tag{111}$$

so that the *Chern classes* are given by the formula

$$c_k(F) = P_k\left(iF/2\pi\right) \tag{112}$$

Since we already derived explicit expressions of first few polynomials $P_k(A)$ (see Appendix D.2), we get the first few Chern classes

$$\begin{aligned} c_0(F) &= 1 && (113)\\ c_1(F) &= \frac{i}{2\pi}\mathrm{Tr}\, F && (114)\\ c_2(F) &= \left(\frac{i}{2\pi}\right)^2 \frac{(\mathrm{Tr}\, F)^2 - \mathrm{Tr}\, F^2}{2!} && (115)\\ c_3(F) &= \left(\frac{i}{2\pi}\right)^3 \frac{(\mathrm{Tr}\, F)^3 - 3(\mathrm{Tr}\, F^2)(\mathrm{Tr}\, F) + 2\mathrm{Tr}\, F^3}{3!} && (116) \end{aligned}$$

and so on, where, in more detail,

$$\begin{aligned} \mathrm{Tr}\, F &\equiv F^a_a \\ \mathrm{Tr}\, F^2 &\equiv F^a_b \wedge F^b_a \\ (\mathrm{Tr}\, F)^2 &\equiv F^a_a \wedge F^b_b \\ \mathrm{Tr}\, F^3 &\equiv F^a_b \wedge F^b_c \wedge F^c_a \\ (\mathrm{Tr}\, F^2)(\mathrm{Tr}\, F) &\equiv F^a_b \wedge F^b_a \wedge F^c_c \end{aligned}$$

Finally, on general grounds (see (88)) we know, that *integrals* of any of $c_k(F)$ over $2k$-cycles ($2k$-dimensional closed surfaces S_{2k}) are topological invariants; they are the celebrated Chern *numbers*

$$\boxed{c_k := \int_{S_{2k}} c_k(F) = \ k\text{-th } \textit{Chern number}} \tag{117}$$

(of the bundle $\pi : P \to M$ or any of its associated complex vector bundle). Because of the factor $1/2\pi$ in the definition (112), they may be shown to be *integers*.

Example 5.3.1: Consider a $U(1)$-connection (say, electromagnetism). Here the curvature 2-form F is $u(1) \equiv i\mathbb{R}$-valued, i.e.

$$F = F^1 e_1 = i\mathcal{F} \qquad e_1 = i, \quad F^1 \equiv \mathcal{F} \tag{118}$$

and, since $U(1)$ is abelian, $\mathcal{F}$ is given simply as

$$\mathcal{F} = d\mathcal{A} = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu \wedge dx^\nu \tag{119}$$

So F is just a 1×1-matrix, now. Then, the defining equation (111) says

$$\det\left(\lambda \mathbb{I} - \frac{1}{2\pi i} F\right) = \lambda - \frac{1}{2\pi}\mathcal{F} = \sum_{k=0}^{1} \lambda^{1-k} c_k(F) = \lambda + c_1(F) \tag{120}$$

and, consequently, there is just a *single nontrivial* Chern number here, namely the *first Chern number*

$$c_1 := \int_{S_2} c_1(F) = -\frac{1}{2\pi}\int_{S_2} \mathcal{F} \equiv -\frac{1}{2\pi}\int_{S_2} d\mathcal{A} \tag{121}$$

(Vanishing of higher Chern numbers is also formally visible from (115)), (116)), since here $\operatorname{Tr} F = F$, $\operatorname{Tr} F^2 = F^2 \equiv F \wedge F$ etc.)

Example 5.3.2: Consider a particular case of Example 5.3.1, when the base manifold is *two-dimensional* (say, the sphere S^2 or the torus T^2). If (x^1, x^2) are local coordinates, then

$$\mathcal{A} = \mathcal{A}_1 dx^1 + \mathcal{A}_2 dx^2 \qquad \text{and} \qquad \mathcal{F} \equiv d\mathcal{A} = (\partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1) dx^1 \wedge dx^2 \tag{122}$$

So, the first Chern number (121) reduces to the surface integral

$$c_1 = -\frac{1}{2\pi}\int_{S_2}\left(\frac{\partial \mathcal{A}_2}{\partial x^1} - \frac{\partial \mathcal{A}_1}{\partial x^2}\right) dx^1 dx^2 \tag{123}$$

Example 5.3.3: Let $|\psi(s)\rangle$ be a (normalized) quantum state, dependent on a set of parameters $s^\mu \equiv (s^1, \dots, s^n)$ (say, components of external magnetic field). Then it turns out that

$$\mathcal{A}_\mu(s) := -i\langle\psi(s)|\partial_\mu \psi(s)\rangle \qquad \partial_\mu \equiv \partial/\partial_{s^\mu} \tag{124}$$

gives a $U(1)$-connection form $A = i\mathcal{A}_\mu ds^\mu$ on the parameter space (playing the role of the *base* space of the $U(1)$-bundle). It is the celebrated *Berry connection.*

▼ Because of $\langle\psi(s)|\psi(s)\rangle = 1$, one easily checks that $\mathcal{A}_\mu$ is *real* so that the one-form $A = i\mathcal{A}_\mu ds^\mu$ is indeed $u(1)$-valued. ▲

Its curvature 2-form is $F = i\mathcal{F}$, where

$$\mathcal{F}_{\mu\nu}(s) = -i(\langle\partial_\mu\psi|\partial_\nu\psi\rangle - \langle\partial_\nu\psi|\partial_\mu\psi\rangle) \tag{125}$$

so that the first Chern number of the corresponding $U(1)$-bundle is

$$c_1 := -\frac{1}{2\pi}\int_{S_2} \mathcal{F} = \frac{i}{2\pi}\int_{S_2} \langle\partial_\mu\psi|\partial_\nu\psi\rangle ds^\mu \wedge ds^\nu \tag{126}$$

Example 5.3.4: On a *two-dimensional base* (here parameter) manifold, the formula (126) simplifies to

$$c_1 = \frac{i}{2\pi} \int_{S_2} (\langle \partial_1 \psi | \partial_2 \psi \rangle - \langle \partial_1 \psi | \partial_2 \psi \rangle) ds^1 ds^2 \tag{127}$$

Now if we rename the coordinates $(s^1, s^2) \mapsto (k^1, k^2)$, the integral (127) becomes (it is exactly (123) for connection (124))

$$c_1 = \frac{i}{2\pi} \int_{S_2} d^2k \left(\left\langle \frac{\partial \psi}{\partial k^1} \middle| \frac{\partial \psi}{\partial k^2} \right\rangle - \left\langle \frac{\partial \psi}{\partial k^2} \middle| \frac{\partial \psi}{\partial k^1} \right\rangle \right) \tag{128}$$

Expressions of this type, or those explicitly specified for the general closed surface S_2 being two-dimensional *torus* T^2

$$c_1 = \frac{i}{2\pi} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \left(\left\langle \frac{\partial \psi}{\partial \alpha} \middle| \frac{\partial \psi}{\partial \beta} \right\rangle - \left\langle \frac{\partial \psi}{\partial \beta} \middle| \frac{\partial \psi}{\partial \alpha} \right\rangle \right) \tag{129}$$

play important role in papers which use the first Chern numbers in *integer quantum Hall effect*, see for example [11], [12].

Example 5.3.5: Consider a $SU(2)$-connection. From (90) we see that

$$\det(\lambda \mathbb{I} + iF/2\pi) = \det \begin{pmatrix} \lambda + iF_1^1/2\pi & iF_2^1/2\pi \\ iF_1^2/2\pi & \lambda + iF_2^2/2\pi \end{pmatrix} = \lambda^2 + \frac{i\lambda \operatorname{Tr} F}{2\pi} - \frac{\det F}{(2\pi)^2} \tag{130}$$

Here, the curvature 2-form F is $su(2)$-valued, i.e. it may also be written using Lie algebra component forms (i.e. F^j instead of F^a_b)

$$F = F^j E_j = -\frac{i}{2} F^j \sigma_j \equiv -\frac{i}{2} \begin{pmatrix} F^3 & F^1 - iF^2 \\ F^1 + iF^2 & -F^3 \end{pmatrix} \equiv \begin{pmatrix} F_1^1 & F_2^1 \\ F_1^2 & F_2^2 \end{pmatrix} \tag{131}$$

We see that

$$\operatorname{Tr} F = 0 \qquad \det F = \frac{1}{4}(F^1 \wedge F^1 + F^2 \wedge F^2 + F^3 \wedge F^3) \equiv \frac{1}{4} F^j \wedge F^j \tag{132}$$

So, in reality,

$$\det(\lambda \mathbb{I} + iF/2\pi) = \lambda^2 - \frac{1}{(4\pi)^2} F^j \wedge F^j \overset{!}{=} \lambda^2 + \lambda c_1(F) + c_2(F) \tag{133}$$

This shows that the *first* Chern number always *vanishes* for $SU(2)$-connection and that the *second* Chern number, given by explicit formula

$$c_2 := \int_{S_4} c_2(F) = -\frac{1}{(4\pi)^2} \int_{S_2} F^j \wedge F^j \tag{134}$$

is the only nontrivial case of interest for $SU(2)$-connection.

5.4 Euler class and Chern-Gauss-Bonnet theorem

In addition to determinant and trace, there is still another (useful) number which may be associated with a square matrix A. Although it is not so generally known as are determinant or trace, its properties are equally remarkable. It is the *pfaffian* $\mathrm{Pf}\,A$. For *even-dimensional* ($2m \times 2m$) and *skew-symmetric* matrix A it is defined by the formula

$$\mathrm{Pf}\,(A) = \frac{1}{2^m m!}\,\varepsilon^{\overbrace{ab\ldots cd}^{2m}}\,\underbrace{A_{ab}\ldots A_{cd}}_{m\text{ matrices}} \tag{135}$$

(see (5.6.8) in [2], where You can find also an alternative definition in terms of forms, which is more convenient for studying general properties of pfaffian). Notice, that it is a *polynomial of order* m of matrix elements (determinant of the same matrix being polynomial of order $2m$). For example, for $m = 1$ we get explicitly

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a \qquad \text{whereas} \qquad \det\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a^2 \tag{136}$$

so that in *this particular* case

$$(\mathrm{Pf}\,(A))^2 = \det A \tag{137}$$

One can show, however, that this holds *in general.* Another important property of pfaffian is, that

$$\mathrm{Pf}\,(B^T A B) = (\det B)\ \mathrm{Pf}\,A \tag{138}$$

Recall that for $B \in SO(2m)$, i.e. for B a *special orthogonal* matrix, one has $B^T = B^{-1}$ and $\det B = 1$, so that for $A \in o(n)$, the Lie algebra of the group, the content of (138) may also be rephrased as

$$\mathrm{Pf}\,(\mathrm{Ad}_B A) = \mathrm{Pf}\,A \tag{139}$$

Thus the pfaffian provides another *Ad-invariant polynomial* in the Lie algebra $o(n)$ (in addition to polynomials $p_k(A)$ given by the receipt (105) and leading to Pontryagin classes) which can be used to construct a new characteristic class.

▼ Note, however, that a *smooth* choice of *orientation* in each fiber of a vector bundle is needed in order to be able to reduce the group of the relevant principal bundle further from $O(2m)$, always possible because of the possibility to introduce a (smooth) fibre *metric*, to $SO(2m)$). Such orientation, contrary to the fiber metric, may not exist (it does not exist, for example, in the Möbius strip regarded as a vector bundle over the circle). ▲

So, let us define the *Euler class* (of a real vector bundle with $2m$-dimensional fibre) as

$$e(F) := \mathrm{Pf}\left(\frac{F}{2\pi}\right) \tag{140}$$

In full details it gives

$$e(F) = \frac{1}{(4\pi)^m m!} \varepsilon^{\overbrace{ab\dots cd}^{2m}} \underbrace{F_{ab} \wedge \dots \wedge F_{cd}}_{m \text{ matrices}} \tag{141}$$

where $F_{ab} \equiv F^a_b$.

▼

$$e(F) = \mathrm{Pf}\left(\frac{F}{2\pi}\right) = \left(\frac{1}{2\pi}\right)^m \mathrm{Pf}\,(F) = \left(\frac{1}{2\pi}\right)^m \frac{1}{2^m m!} \varepsilon^{\overbrace{ab\dots cd}^{2m}} \underbrace{F_{ab} \wedge \dots \wedge F_{cd}}_{m \text{ matrices}}$$

▲

So, for the first two cases, $m = 1$ and $m = 2$ (i.e. for two and four-dimensional fibers respectively) we get

$$e(F) = \frac{1}{4\pi}\epsilon^{ab}F_{ab} \equiv \frac{1}{2\pi}F_{12} \tag{142}$$

$$e(F) = \frac{1}{32\pi^2}\epsilon^{abcd}F_{ab}F_{cd} \equiv \left(\frac{1}{2\pi}\right)^2 (F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23}) \tag{143}$$

Example 5.4.1: The *tangent bundle* $\pi : TM \to M$ is a real vector bundle over M. So, for $2m$-dimensional orientable M (when the fiber dimension of TM is $2m$), one can define Euler class of TM. However, since TM is *canonically* constructed from M, the Euler class, in this particular case, is actually a (topological) characteristics *of M itself* rather than of some bundle over M, as is the case usually. This is the reason why it is often denoted as $e(M)$ (= actually $e(TM)$).

The tangent bundle TM is an associated bundle for the *frame bundle LM*. From the general scheme of this chapter it follows, that the Euler class can be expressed (by the formula (140)) in terms of any connection in LM. This is, however, the common *linear connection* on M (well known, say, from general relativity or continuum mechanics).

In order to distinguish this particular connection, we will denote its curvature 2-forms as R_{ab} (instead of general F_{ab} used for *any* connection). They are related in a standard way (see section (15.6) in [2], where, however, the curvature 2-forms are denoted as Ω_{ab}) with the Riemann (curvature) *tensor* $\mathcal{R}_{abcd}$ on M by

$$R_{ab} =: \frac{1}{2}\mathcal{R}_{abcd}\, e^c \wedge e^d \tag{144}$$

So, for $2m$-dimensional oriented manifold M endowed with *arbitrary linear* connection the Euler class *of M* is given by

$$e(M) = \frac{1}{(4\pi)^m m!} \varepsilon^{\overbrace{ab\dots cd}^{2m}} \underbrace{R_{ab} \wedge \dots \wedge R_{cd}}_{m \text{ matrices}} \tag{145}$$

Example 5.4.2: For a *two*-dimensional manifold ($m = 1$) equation (145) gives

$$e(M) = \frac{1}{2\pi}\ KdA \tag{146}$$

▼

$$e(M) = \frac{1}{4\pi}\ \varepsilon^{ab}\ R_{ab} = \frac{1}{4\pi}\ \varepsilon^{ab}\ \varepsilon_{ab} d\alpha = \frac{1}{2\pi} d\alpha = \frac{1}{2\pi} KdA$$

See (15.6.10) and (15.6.13) in [2], also mentioned at the end of Appendix A.1. ▲

This is a remarkable observation, however, since the r.h.s. of (146) coincides with the expression to be integrated in Gauss-Bonnet theorem (39). Put it differently, the good old *Gauss-Bonnet theorem* (39) may be rewritten as the statement

$$\int_S e(S) = \chi(S) \tag{147}$$

In words: The integral of the Euler *class* over a closed two-dimensional surface S equals Euler *characteristics* of the surface.

What was to be expected here and what was not? Well, the Euler class, when integrated over any closed surface, should necessarily result in a *topological invariant* of the surface. This is the general idea behind (88). What particular invariant? It is *here* where equation (147) comes to help. It says that the invariant turns out to be the Euler characteristics of the surface.

Example 5.4.3: Now for *any* closed (and oriented) $2m$-dimensional manifold M, integral of the Euler class $e(M)$ over M itself is necessarily a topological invariant of M (once more, because of (88)). What particular invariant? According to a theorem due to Chern, it is (still, as for $m = 1$) the *Euler characteristic* of M, defined, however, in terms of alternating sum of *Betti numbers* of M (see (29)). So, the *Chern-Gauss-Bonnet theorem* (alternatively called *generalized* Gauss-Bonnet theorem) holds:

$$\int_M e(M) = \chi(M) \tag{148}$$

where

$$\chi(M) := b^0 - b^1 + b^2 - \dots + b^{2m} \tag{149}$$

▼ The theorem is clearly a generalization of the classical Gauss-Bonnet theorem. It was proved by Chern in 1945 (roughly houndred years after the original

G-B theorem) without assuming the manifold to be a hyper-surface. For hyper-surfaces, the result had been shown first by Allendoerfer and Weil in 1940, see Wikipedia. ▲

Example 5.4.4: Standard *Einstein-Hilbert* action, from which gravitation field equations may be derived, does *not* work for *two-dimensional* gravity.
Indeed: The action reads

$$S[g] \propto \int R\omega \equiv \int R\sqrt{|g|}d^4x \tag{150}$$

where R is the *scalar curvature*. In two dimensions, R becomes (up to a factor 2) the *Gaussian* curvature, $\omega \equiv \sqrt{|g|}d^2x$ becomes dA, so we get

$$S[g] \propto \int K dA \tag{151}$$

This is, however, nothing but the integral of the *Euler class* (see (146)). Upon a change of connection, it acquires just an additive *exact* ("boundary") term (this is true for *any characteristic* class, see (180)). So, under standard "variational" conditions (no variation on the boundary) the integral is *insensitive* to a change of connection. (Put another way, its *variation* with respect to the metric tensor g *vanishes* and no equations of motion emerge from it; the action becomes "topological" in two dimensions).

A Appendix to Chapter 2

A.1 Derivation of the expression (19) in section 2.1

What we want to construct is a *rotation-invariant* and at the same time *closed* $(n-1)$-form in $\mathbb{R}^n$. (Some working knowledge of forms is needed.)

Start with a standard (rotation-invariant) *volume form* in $\mathbb{R}^n$

$$\omega = dx^1 \wedge \cdots \wedge dx^n \tag{152}$$

Take its *interior product* with a vector field V

$$i_V\omega = V^i dS_i \qquad dS_i := \frac{1}{(n-1)!}\epsilon_{ij\ldots k}dx^j \wedge \cdots \wedge dx^k \tag{153}$$

This form is closed iff the divergence of the vector field vanishes.

$$d(i_V\omega) = 0 \qquad \Leftrightarrow \qquad \operatorname{div} V = 0 \tag{154}$$

▼ Indeed, from the definition of the divergence $\mathcal{L}_V\omega =: (\operatorname{div} V)\omega$ (see (8.2.1) in [2]) and the Cartan magic formula $\mathcal{L}_V = i_V d + d i_V$ we have

$$(\operatorname{div} V)\omega = \mathcal{L}_V\omega = d(i_V\omega) + i_V(d\omega) = d(i_V\omega)$$

since $d\omega = 0$ simply because of its (too high :-) degree. ▲

So wee need a rotation-invariant vector field with vanishing divergence. Try the (clearly rotation-invariant) ansatz

$$V^i(x^1, \ldots, x^n) = f(r)x^i \tag{155}$$

then

$$\operatorname{div} V = V^i_{\ ,i} = rf'(r) + nf(r) = 0 \qquad \text{for} \qquad f(r) = k/r^n$$

So the wanted $(n-1)$-form in $\mathbb{R}^n$ is a constant multiple of the form

$$\frac{\mathbf{n}\cdot d\mathbf{S}}{r^{n-1}} \equiv \frac{\mathbf{r}\cdot d\mathbf{S}}{r^n} \equiv \frac{x^i dS_i}{r^n} \tag{156}$$

A.2 Explicit computation of $H^p(S^1)$

Just a sketch. Since S^1 is one-dimensional and connected, the only space to be computed is $H^1(S^1)$. Well, consider a closed 1-form β. If it is exact and $\beta = df$ for a smooth f on S^1, then

$$\int_{S^1}\beta = 0$$

by Stokes theorem. It holds, however, also the converse of the statement: let $\int_{S^1}\beta = 0$, fix a point P and define the function $f(s) = \int_P^s \beta$ (it is only single valued if $\int_{S^1}\beta = 0$). Then (check) $df = \beta$. So

$$\int_{S^1}\beta = 0 \qquad \Leftrightarrow \qquad \beta \text{ is exact} \tag{157}$$

Now let α be *any* closed 1-form on S^1 and let $\int_{S^1} \alpha = k \neq 0$. Let $\hat{\alpha}$ be another such form. Then $\int_{S^1}(\hat{\alpha} - \alpha) = 0$, so $\hat{\alpha} - \alpha = d\chi$ for some smooth χ and therefore the classes coincide

$$[\hat{\alpha}] = [\alpha] \tag{158}$$

So there is a bijection

$$\int_{S^1} \alpha \qquad \leftrightarrow \qquad [\alpha] \tag{159}$$

(actually a linear isomorphism) between (linear space of) real numbers and (linear space of) equivalence classes of closed 1-forms. Therefore $H^1(S^1) = \mathbb{R}$.

Do there exist closed one-forms for each k? Take the one-form "$d\varphi$" (i.e. (13) restricted to the unit circle). It *is* closed and its integral over S^1 is clearly 2π. So, its $k/2\pi$ multiple is closed and corresponds to k. We see, that *any closed* one-form with integral over S^1 equal to k may be written as

$$\alpha = \frac{k}{2\pi} \text{"}d\varphi\text{"} + d\chi \tag{160}$$

for some smooth function χ.

B Appendix to Chapter 3

B.1 Derivation of formula (52) in section 3.2.1

Consider a *loop* γ, which is the *boundary* of a two-dimensional domain $S \subset (M, g)$, $\gamma = \partial S$. The domain is expected to be a coordinate patch. Apply the standard Gramm-Schmidt procedure: first normalize the coordinate field ∂_1 to unity, thus obtaining e_1 and then construct the second field e_2 such that (e_1, e_2) becomes an orthonormal right-handed frame. So, within the *entire* domain S (including its boundary $\gamma = \partial S$), there is a smooth orthonormal frame field e_a.

With respect to the frame, there are well defined 1-forms of (metric and symmetric) connection ω^a_b on the whole S. As usually (see 15.6.10 in [2]), complete information about ω^a_b is actually present in a single 1-form α defined by the parametrization $\omega_{ab} = \epsilon_{ab}\alpha$. In the same way, all information about curvature 2-forms Ω^a_b is in a single 2-form $d\alpha$, since $\Omega_{ab} = \epsilon_{ab} d\alpha$.

Now

$$\begin{aligned} \dot{\gamma} &= \cos\theta \ e_1 + \sin\theta \ e_2 \\ w &= -\sin\theta \ e_1 + \cos\theta \ e_2 \end{aligned}$$

and a simple computation gives

$$a = (\dot{\theta} - \langle \alpha, \dot{\gamma} \rangle) \ w$$

▼ Since

$$\begin{aligned}
a &\equiv \nabla_{\dot\gamma}\dot\gamma \\
&= \nabla_{\dot\gamma}(\cos\theta\ e_1 + \sin\theta\ e_2) \\
&= -\dot\theta\sin\theta\ e_1 + \cos\theta\langle\omega_1^2,\dot\gamma\rangle e_2 + \dot\theta\cos\theta e_2 + \sin\theta\langle\omega_2^1,\dot\gamma\rangle e_1 \\
&= -\dot\theta\sin\theta\ e_1 - \cos\theta\langle\alpha,\dot\gamma\rangle e_2 + \dot\theta\cos\theta e_2 + \sin\theta\langle\alpha,\dot\gamma\rangle e_1 \\
&= (\dot\theta - \langle\alpha,\dot\gamma\rangle)\ (-\sin\theta\ e_1 + \cos\theta\ e_2) \\
&= (\dot\theta - \langle\alpha,\dot\gamma\rangle)\ w
\end{aligned}$$

▲

Comparison with the definition (45) gives the expression

$$k = \dot\theta - \langle\alpha,\dot\gamma\rangle$$

from which

$$\begin{aligned}
d\theta &= kds + \langle\alpha,\dot\gamma\rangle ds \\
&= kds + \gamma^*\alpha
\end{aligned}$$

This $d\theta$ may be regarded as an increment of the angle θ corresponding to a small step of the length ds along the loop (boundary). So this is the angle of rotation of the velocity vector. When traversing the whole loop, the resulting net angle is the sum (integral) of all those small pieces

$$[\theta]_\circlearrowleft = \oint_{\partial S} d\theta = \oint_{\partial S}(kds + \gamma^*\alpha) = \oint_{\partial S} kds + \int_S d\alpha = \oint_{\partial S} kds + \int_S Ke^1\wedge e^2$$

where the definition (in Cartan language) of the *Gaussian curvature* K was used

$$d\alpha =: Ke^1\wedge e^2 \equiv KdA \tag{161}$$

(see 15.6.10 and 15.6.13 in [2]; K is a function on the surface, serving as a factor between the curvature 2-form $d\alpha$ and the canonical (metric) area 2-form $e^1\wedge e^2$). So, the net angle of rotation of the velocity vector may be expressed as

$$[\theta]_\circlearrowleft = \oint_{\partial S} kds + \int_S KdA \tag{162}$$

B.2 Derivation of formula (62) in section 3.3.1

The computations are similar to those from Appendix B.1.

First we concentrate on rotation of $\dot\gamma_i$ about the gradient field ∇f defined in Y. On i-th loop $\gamma_i = \partial R_i$ introduce the angle θ_i between the (unit) vector $\dot\gamma_i$ and the frame vector e_1 (normed gradient field ∇f), i.e. set

$$\begin{aligned}
\dot\gamma_i &= \cos\theta_i\ e_1 + \sin\theta_i\ e_2 \\
w_i &= -\sin\theta_i\ e_1 + \cos\theta_i\ e_2
\end{aligned}$$

(w_i is such that $(\dot\gamma_i, w_i)$ represent a right-handed orthonormal frame field on the loop γ_i). Then

$$a_i = (\dot\theta_i - \langle \alpha, \dot\gamma_i \rangle)\ w_i$$

and

$$d\theta_i = (k)_i ds + \gamma_i^* \alpha$$

(Here, α is the connection form w.r.t. the frame (e_1, e_2) defined in Y.) So, for the net angle of rotation (winding number, when divided by 2π) we get

$$[\theta_i]_{\circlearrowleft} = \oint_{\gamma_i} (k)_i ds + \oint_{\gamma_i} \alpha \tag{163}$$

(just mimic the derivation of (52) performed in Appendix A.1). Now realize that the totality of γ_i provides the *boundary* of Y, more precisely [9]

$$-\partial Y = \sum_i \gamma_i$$

So, when *all* equations (163) are summed up, we get

$$\int_Y d\alpha = \sum_i \oint_{\gamma_i} (k)_i ds - \sum_i [\theta_i]_{\circlearrowleft}$$

i.e.

$$\int_Y K dA = \sum_i \oint_{\gamma_i} (k)_i ds - \sum_i [\theta_i]_{\circlearrowleft} \tag{164}$$

Ok. Now move to (the interior of) the disks R_i. Consider the i-th of them. Its boundary is $\gamma_i = \partial R_i$. Fix *arbitrary* orthonormal right-handed frame field $(\hat e_1, \hat e_2)$ inside R_i. Introduce the angle ϕ_i between the i-th velocity vector $\dot\gamma_i$ and $\hat e_1$, i.e. set

$$\begin{aligned} \dot\gamma_i &= \cos\phi_i\ \hat e_1 + \sin\phi_i\ \hat e_2 \\ w_i &= -\sin\phi_i\ \hat e_1 + \cos\phi_i\ \hat e_2 \end{aligned}$$

If $\hat\alpha$ is the connection form w.r.t. *this* frame, then the counterpart of (163) is

$$[\phi_i]_{\circlearrowleft} = \oint_{\gamma_i} (k)_i ds + \oint_{\gamma_i} \hat\alpha$$

and since $\gamma_i = \partial R_i$, the last term may also be rewritten with the help of Stokes formula, producing a 2-form

$$d\hat\alpha = d\alpha \equiv K dA$$

under the integral. Notice the first equality sign, so that it is *the same* 2-form $K dA$, which is present in the integral (164). So,

$$\int_{R_i} K dA = [\phi_i]_{\circlearrowleft} - \oint_{\gamma_i} (k)_i ds$$

[9] Recall that $S = Y + \sum R_i$ and $\partial S = 0$.

▼ The form KdA lives on the whole surface S. On Y, on the disks, and even right in critical points of f (the metric, and consequently the connection, has nothing to do with the function f :-). It is, however, expressed w.r.t. different frame fields in different parts of S. In general (see (15.6.2) and (15.6.3) in [2]), under the change of frame $e \mapsto eB$, connection forms transform as $\omega \mapsto B^{-1}\omega B + B^{-1}dB$ and the corresponding curvature forms *transform* as $\Omega \mapsto B^{-1}\Omega B$. Here we live on a 2-dimensional manifold and we use orthonormal frames, so that $B \in SO(2)$. We can standardly parametrize B in terms of a single angle χ, the latter being a smooth function in the disk R_i Using $\omega_{ab} = \epsilon_{ab}\alpha$ and $\Omega_{ab} = \epsilon_{ab}\beta$, this gives $\alpha \mapsto \alpha + d\chi$ and $\beta = d\alpha \mapsto d(\alpha + d\chi) = \beta$. Therefore

$$d\hat{\alpha} = \hat{\beta} = \hat{K}\hat{e}^1 \wedge \hat{e}^2 = \beta = Ke^1 \wedge e^2 \qquad \hat{K} = K$$

So, even though the connection form depends on the choice of a frame field, the curvature form does *not* (in this particular case). And since neither does the area element,

$$\hat{e}^1 \wedge \hat{e}^2 = e^1 \wedge e^2 \equiv dA$$

the same is true for the Gaussian curvature K. ▲

C Appendix to Chapter 4

C.1 A simple example of a form of type ρ

Let

$$R_a : x \mapsto x + a$$

be the action of $G = (\mathbb{R}, +)$ on $M = \mathbb{R}$. Then for the two functions $\sin x$ and $\cos x$ we have

$$\cos(x+a) = \cos a \cos x - \sin a \sin x \qquad \sin(x+a) = \sin a \cos x + \cos a \sin x$$

i.e.

$$R_a^* \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \equiv \rho(-a) \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \tag{165}$$

for the 2×2 matrix representation

$$a \mapsto \rho(a) := \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \tag{166}$$

of the group $G = (\mathbb{R}, +)$. So the two functions are actually (the only) members of a secret society created in order to stand against a common enemy - the action of the group (by pull-back). None of them is able to resist the cruel enemy separately. However, as a pair, they are already strong enough. As a matter of fact, as we can see from the formula (165), the group is not a tittle able to harm the pair. The pair spans an *invariant two-dimensional sub-space* (w.r.t. the action of the group) in the infinite-dimensional space of all smooth functions and each attack of the group just results in a simple defence manoeuvre performed by the pair - creation of a *new* linear combination in *the same* sub-space.

Arbitrary element of the subspace is of the form

$$\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} = \cos x \begin{pmatrix}1\\ 0\end{pmatrix} + \sin x \begin{pmatrix}0\\ 1\end{pmatrix} = f^1(x)E_1 + f^2(x)E_2 = f^i E_i \equiv f \tag{167}$$

and the result (165) may be written neatly as

$$R_a^* f = \rho(-a) f \tag{168}$$

▼ In this particular case, the basis is usually denoted as $E_1 = 1, E_2 = i$ so that

$$\cos x \ E_1 + \sin x \ E_2 = \cos x + i \sin x \equiv e^{ix}$$

and (168) is better known in the form

$$e^{i(x+a)} = e^{ia} e^{ix} \qquad\qquad : -)$$

▲

The general case includes
- p-forms (not just functions = 0-forms)
- Lie group G (not just $G = (\mathbb{R}, +)$)
- representation ρ of G in vector space V (not just $\mathbb{R}^2$)

So, a general formula for a *p-form of type* ρ reads

$$R_g^* \alpha = \rho(g^{-1})\alpha \qquad \alpha = \alpha^i E_i \quad E_i \in V \tag{169}$$

Here, the secret society consists of the n-tuple of p-forms $(\alpha^1, \dots, \alpha^n)$.

C.2 Matrix versus Lie algebra indices

Perhaps two types of indices (matrix indices versus Lie algebra indices) might be potentially a source of confusion. The field strength form F is Lie algebra valued, so $F = F^i E_i$, where E_i is a basis of the Lie algebra $\mathcal{G}$. However, for *matrix* Lie groups, elements of the Lie algebra are matrices as well. So, E_i are matrices, too. If we use the natural basis E^a_b for matrices (1 at a-th column and b-th row, 0 elswhere), then

$$E_i = (E_i)^b_a E^a_b \qquad\qquad (E^a_b)^c_d := \delta^a_d \delta^c_b \tag{170}$$

so

$$F = F^a_b E^b_a = F^i (E_i)^b_a E^b_a \tag{171}$$

So, there are two types of components [10] of the field strength, F^a_b and F^i (the same is true for gauge potentials) and they are related as

$$F^a_b = F^i (E_i)^b_a \tag{172}$$

Take, as a simple (yet important) example, group $SU(2)$. Its Lie algebra has a basis

$$E_j = -\frac{i}{2}\sigma_j \tag{173}$$

so that

$$F = F^j E_j = -\frac{i}{2} F^j \sigma_j \equiv -\frac{i}{2}\begin{pmatrix} F^3 & F^1 - iF^2 \\ F^1 + iF^2 & -F^3 \end{pmatrix} \equiv \begin{pmatrix} F^1_1 & F^1_2 \\ F^2_1 & F^2_2 \end{pmatrix} \tag{174}$$

D Appendix to Chapter 5

D.1 Why $[w(\Omega, \ldots, \Omega)]$ does not depend on the choice of connection

In order to see this, one has to study first, how $w(\Omega, \ldots, \Omega)$ is sensitive to a change of ω.

There are infinitely many connections on a given principle fibre bundle. Let ω and ω_1 be any two of them. Then, their difference $\alpha := \omega_1 - \omega$ satisfies

$$R^*_g \alpha = \mathrm{Ad}_{\,g^{-1}} \alpha \qquad \langle \alpha, \xi_X \rangle = 0 \tag{175}$$

So α is a *horizontal 1-form of type Ad*. Conversely, if ω is a connection form and α is a horizontal 1-form of type Ad, then their sum is a connection form as well. This means, that all connection forms on P constitute an *affine* space.

Consider the "linking line" between the two connection forms ω and ω_1 and the corresponding "linking line" between the curvature forms Ω and Ω_1

$$\omega_t := \omega + t\alpha \qquad \Omega_t := d\omega_t + \frac{1}{2}[\omega_t, \omega_t] \qquad 0 \le t \le 1 \tag{176}$$

(For each t, ω_t is a connection form and Ω_t is its curvature form. So we have one-parameter family of connection forms and the corresponding family of curvature forms.) Now, the object of interest is the t-dependent $2k$-form on P, $w(\Omega_t, \ldots, \Omega_t)$. The most interesting fact about this form is that the derivative with respect to the parameter t is an *exact* form [11]

$$\frac{d}{dt}\, w(\Omega_t, \ldots, \Omega_t) = d\left(w(\alpha, \Omega_t \ldots, \Omega_t)\right) \equiv d\hat{\beta}_t \tag{177}$$

[10] Of course, neither of them is related to 2-form-indices $\mu\nu$. In full, F is expressed as $F = (1/2) F^i_{\mu\nu} dx^\mu \wedge dx^\nu$

[11] It's proof is a not so hard routine computation.

And since $\hat{\beta}_t$ is G-invariant (both α and Ω_t are of type Ad and w is Ad-invariant) and at the same time horizontal (both α and Ω_t are horizontal), its pull-back onto M does not depend on the choice of a section and we get a well-defined form β_t on M. So, we get a similar (important) equation on M:

$$\frac{d}{dt}\ w(F_t,\dots,F_t) = d\beta_t \tag{178}$$

Performing integration [12] $\int_0^1 dt$ (and making use of Newton-Leibniz formula) we get

$$w(F_1,\dots,F_1) - w(F,\dots,F) = d\left(\int_0^1 dt\ \beta_t\right) \equiv d\beta \tag{179}$$

But the equation we obtained

$$w(F_1,\dots,F_1) = w(F,\dots,F) + d\beta \tag{180}$$

shows, that the choice of a different connection within the same principal bundle $\pi : P \to M$ results in just *adding* an *exact* form to the closed $2k$-form $w(F,\dots,F)$ on M. So, the forms $w(F,\dots,F)$ and $w(F_1,\dots,F_1)$ are *cohomological* and they induce the *same* cohomological *class*

$$[w(F_1,\dots,F_1)] = [w(F,\dots,F)] \in H^{2k}(M) \tag{181}$$

D.2 Computation of the relevant determinant in (101)

Consider, for any $n \times n$ real matrix A and real number λ, the determinant

$$\det(A + \lambda\mathbb{I})$$

We would like to express coefficients standing by powers of λ

$$\det(A + \lambda\mathbb{I}) = \sum_{k=0}^{n} \lambda^{n-k} P_k(A) \tag{182}$$

Recall a general formula for determinant of a matrix B (see (5.6.5) in [2])

$$\det A = \delta^{c\dots d}_{a\dots b}\ \underbrace{B^a_c \dots B^b_d}_{n \text{ matrices}} \qquad \text{or in brief} \qquad \det A = \delta(n)\ \underbrace{B \dots B}_{n \text{ matrices}}$$

where n-delta is defined as follows

$$\delta^{a\dots b}_{c\dots d} = \delta^a_{[c} \dots \delta^b_{d]} \equiv \delta^{[a}_c \dots \delta^{b]}_d \equiv \delta^{[a}_{[c} \dots \delta^{b]}_{d]}$$

[12]Here, we are not "integrating forms" in the sense usual for forms, since the variable t is a *parameter*. What we do is actually nothing but a *continuous "linear combination"*. At the end we still have a form rather than a number.

Then

$$\begin{aligned}\det(A+\lambda\mathbb{I}) &= \delta(n)\underbrace{(A+\lambda\mathbb{I})\dots(A+\lambda\mathbb{I})}_{n\text{ matrices}} \\ &= \textstyle\sum_{k=0}^{n}\lambda^{n-k}\binom{n}{k}\delta(n)\underbrace{A\dots A}_{k}\overbrace{\mathbb{I}\dots\mathbb{I}}^{n-k}\end{aligned}$$

so that

$$P_k(A)=\binom{n}{k}\delta(n)\underbrace{A\dots A}_{k}\overbrace{\mathbb{I}\dots\mathbb{I}}^{n-k}$$

Unit matrices at the end produce $(n-k)$ contractions of n-delta, in indices

$$\delta^{c\dots du\dots v}_{a\dots br\dots s}\delta^{r}_{u}\dots\delta^{s}_{v}=\delta^{c\dots du\dots v}_{a\dots bu\dots v}=\frac{1}{\binom{n}{k}}\delta^{c\dots d}_{a\dots b}\qquad\text{or in brief}\qquad\delta(n)\overbrace{\mathbb{I}\dots\mathbb{I}}^{n-k}=\frac{1}{\binom{n}{k}}\delta(k)$$

(for the last equality, see (5.6.4) in [2]), so that

$$P_k(A)=\delta(k)\underbrace{A\dots A}_{k}=\delta^{c\dots d}_{a\dots b}\underbrace{A^a_c\dots A^b_d}_{k}=\delta^{c}_{[a}\dots\delta^{d}_{b]}\underbrace{A^a_c\dots A^b_d}_{k}=\underbrace{A^a_{[a}\dots A^b_{b]}}_{k}$$

Thus we obtained a simple final expression

$$P_k(A)=\underbrace{A^a_{[a}\dots A^b_{b]}}_{k}\tag{183}$$

In this way we can compute, from the general formula (183), the following concrete expressions:

$$\begin{aligned}P_0(A) &= 1 &(184)\\ P_1(A) &= \mathrm{Tr}\,A &(185)\\ 2!P_2(A) &= (\mathrm{Tr}\,A)^2-\mathrm{Tr}\,A^2 &(186)\\ 3!P_3(A) &= (\mathrm{Tr}\,A)^3-3(\mathrm{Tr}\,A^2)(\mathrm{Tr}\,A)+2\mathrm{Tr}\,A^3 &(187)\\ 4!P_4(A) &= (\mathrm{Tr}\,A)^4-6\mathrm{Tr}\,A^2(\mathrm{Tr}\,A)^2+3(\mathrm{Tr}\,A^2)^2 &(188)\\ &\quad+8(\mathrm{Tr}\,A^3)\mathrm{Tr}\,A-6\mathrm{Tr}\,A^4 &(189)\end{aligned}$$

▼ As an example,

$$P_2(A)=A^a_{[a}A^b_{b]}=\frac{1}{2}(A^a_aA^b_b-A^a_bA^b_a)=\frac{1}{2}((\mathrm{Tr}\,A)^2-\mathrm{Tr}\,(A^2))$$

▲

Notice, that for skew-symmetric matrices, $A^T = -A$ (and, consequently, $\mathrm{Tr}\, A = 0 = \mathrm{Tr}\, A^3$), the results simplify to

$$
\begin{aligned}
P_0(A) &= 1 && (190)\\
P_1(A) &= 0 && (191)\\
2!P_2(A) &= -\mathrm{Tr}\, A^2 && (192)\\
3!P_3(A) &= 0 && (193)\\
4!P_4(A) &= 3(\mathrm{Tr}\, A^2)^2 - 6\mathrm{Tr}\, A^4 && (194)
\end{aligned}
$$

For an alternative method of computation, using eigenvalues of matrices, the reader is referred to [8].

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References

[1] B.A. Dubrovin, S.P. Novikov, A.T. Fomenko: Modern geometry - methods and applications, Parts 2,3, New York, Springer-Verlag, 1985, 1990

[2] M.Fecko: Differential geometry and Lie groups for physicists, Cambridge University Press 2006 (paperback 2011)

[3] M.Fecko: Differential Geometry in Physics, lecture notes from Regensburg 2007, http://sophia.dtp.fmph.uniba.sk/~fecko/regensburg_2007.html

[4] N.Hitchin: Geometry of surfaces, b3 course 2004, www2.maths.ox.ac.uk/~hitchin/hitchinnotes/Geometry_of_surfaces

[5] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry, II, Wiley Interscience, New-York 1969

[6] I.Madsen, J.Tornehave: From Calculus to Cohomology, Cambridge University Press, 1997

[7] J.Moore: Notes for "Geometry and topology in many-particle systems", Physics 250, University of California, Berkeley, http://socrates.berkeley.edu/~jemoore

[8] M.Nakahara: Geometry, topology and physics, Taylor & Francis, 2003

[9] Ch.Nash, S.Sen: Topology and geometry for physicists,
Dover Publications (Reprint Edition), 2011

[10] S.Sternberg: Semi-Riemannian Geometry and General Relativity, 2003

[11] D.J.Thouless, M.Kohmoto, M.P.Nightingale, and M.den Nijs:
Quantized Hall Conductance in a Two-Dimensional Periodic Potential,
Phys. Rev. Lett. 49, 405-408 (1982)

[12] D.J.Thouless: Topological quantum numbers in non-relativistic physics,
World Scientific Publishing, (1998)