Differential geometry for Physicists

(What we discussed in the course of lectures)

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Contents

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1 Smooth manifolds

1.1 Manifolds

Local (as opposed to global) coordinates x^i, change of coordinates $x^{i'}(x)$ on overlaps of domains, charts, atlas, configuration spaces in mechanics as manifolds (double plane and spherical pendulum), (smooth) mapping of manifolds, injective, surjective and bijective mappings, coordinate presentation $y^a(x)$ of a mapping of manifolds, a curve γ on a manifold M (as a mapping

$$\gamma : \mathbb{R} \to M$$

of manifolds), its coordinate presentation $x^i(t)$, a function ψ on a manifold M (as a mapping

$$\psi : M \to \mathbb{R}$$

of manifolds), its coordinate presentation $\psi(x^i)$

1.2 Vectors and vector fields on manifolds

Curves tangent at a point, tangency as an equivalence relation $\gamma_1 \sim \gamma_2$, equivalence class $[\gamma]$ (of curves; γ is a representative), linear combination of such equivalence classes, directional derivative, algebra $\mathcal{F}(M)$ of functions on M, four (equivalent) definitions of a vector in $m \in M$ (vector as: equivalence class of curves, derivation of algebra of functions plus Leibniz rule, expression $a^i\partial_i|_x$, set of numbers a^i), the tangent space T_mM at $m \in M$ - the (linear) space of vectors in $m \in M$ (in any of the four versions), vector field V as a first order differential operator $V^i(x)\partial_i$, transformation law for its components $V^i(x)$ (from $V^i\partial_i = V^{i'}\partial_{i'}$), integral curve of a vector field, equations

$$\dot{x}^i = V^i$$

for finding integral curves, flow

$$\Phi_t : M \to M$$

of a vector field,

$$m \equiv \gamma(0) \mapsto \gamma(t) =: \Phi_t(m)$$

1.3 Tensors in linear algebra

Finite dimensional linear space L, the dual space L^*, a basis $e_a \in L$, the dual basis $e^a \in L^*$, defined by

$$\langle e^a, e_b \rangle = \delta^a_b \ ,$$

tensor of type (p, q) in L (as a multi-linear map), identification of well known examples, components $t^{a\dots b}_{c\dots d}$, space $T^p_q(L)$ of tensors of type (p, q) in L, various roles played by the same tensor, tensor product $\otimes$ and its properties, the basis

$$e^a \otimes \dots \otimes e^b \otimes e_c \otimes \dots \otimes e_d$$

in $T^p_q(L)$ induced by a basis e_a in L, the unit tensor

$$e^a \otimes e_a \leftrightarrow \delta^a_b \ ,$$

metric tensor

$$g = g_{ab} e^a \otimes e^b \ ,$$

the inverse metric tensor (co-metric)

$$g^{ab} e_a \otimes e_b \ ,$$

lowering of indices

$$\flat : L \to L^* \qquad (v^a \mapsto v_a := g_{ab} v^b) \ ,$$

raising of indices

$$\sharp : L^* \to L \qquad (\alpha_a \mapsto \alpha^a := g^{ab} \alpha_b) \ ,$$

contraction operation as a mapping

$$C : T^p_q(L) \to T^{p-1}_{q-1}(L)$$

1.4 Tensors and tensor fields on manifolds

A tensor in m on M: take T_mM as L from linear algebra, the space $\mathcal{T}^p_q(M)$ of tensor fields of type (p,q) on M, the gradient $d\psi$ of a function ψ as a covector field on M, dx^i as the dual basis to ∂_i, the (coordinate) basis

$$dx^i \otimes \cdots \otimes dx^j \otimes \partial_k \otimes \cdots \otimes \partial_l$$

in $\mathcal{T}^p_q(M)$ (on a coordinate patch) induced by a (coordinate) basis ∂_i for vector fields, metric tensor $g = g_{ij} dx^i \otimes dx^j$ on a manifold, Riemannian manifold, transformation of components under change of coordinates from

$$g = g_{ij} dx^i \otimes dx^j = g_{i'j'} dx^{i'} \otimes dx^{j'}$$

and

$$dx^{i'} = J^{i'}_k dx^k \ ,$$

where

$$J^{i'}_k(x) \equiv \partial x^{i'} / \partial x^j$$

is the (always square) Jacobi matrix of the transformation of coordinates, the Euclidean metric tensor in the plane

$$g_{ij} = \delta_{ij} \ ,$$

i.e.

$$g = dx \otimes dx + dy \otimes dy = dr \otimes dr + r^2 d\varphi \otimes d\varphi \ ,$$

the length functional

$$\gamma \mapsto \int_{t_1}^{t_2} |\dot{\gamma}| dt \equiv \int_{t_1}^{t_2} \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt \equiv \int_{t_1}^{t_2} \sqrt{g_{ij} \dot{x}^i \dot{x}^j} dt \ ,$$

the gradient as a vector field

$$\nabla \psi := \sharp d\psi$$

i.e.

$$(\nabla \psi)^i = g^{ij} \partial_j \psi$$

1.5 Mapping of tensors induced by mapping of manifolds

Push-forward

$$v \mapsto f_* v$$

of a vector in $x \in M$ to a vector in $f(x) \in N$ (induced by $f: M \to N$), explicitly

$$v^i \partial_i \mapsto (J^a_i v^i) \partial_a \ ,$$

or, on the coordinate basis,

$$f_* : \partial_i \mapsto f_* \partial_i = J_i^a \partial_a \ ,$$

where

$$J_i^a(x) \equiv \partial y^a(x)/\partial x^j$$

is the Jacobi matrix (in general non-square) of the mapping f, pull-back of a function ψ on N, the result being a function

$$f^* \psi = \psi \circ f$$

on M (in coordinates $\psi(y) \mapsto \psi(y(x))$), pull-back

$$\alpha \mapsto f^* \alpha$$

of a covector in $f(x) \in N$ to a covector in $x \in M$ (induced by $f : M \to N$), given as

$$\langle f^* \alpha, v \rangle := \langle \alpha, f_* v \rangle \ ,$$

on the coordinate basis

$$dy^a \mapsto f^* dy^a = dy^a(x) = J_i^a dx^i \ ,$$

where

$$J_i^a(x) \equiv \partial y^a(x)/\partial x^j$$

is the Jacobi matrix of the mapping f (the same as for push-forward of vectors), an alternative (equivalent) formula is

$$f^*(\alpha_a(y) dy^a) = \alpha_a(y(x)) dy^a(x) = \alpha_a(y(x)) J_i^a(x) dx^i \ ,$$

pull-back of a metric tensor - the induced metric tensor, its example in Lagrangian mechanics, "curved" and "flat" torus, general properties of pull-back - behavior on tensor product and linear combination, commutation with taking of the gradient ($f^* d = d f^*$ on functions)

1.6 Lie derivative, isometries, Killing vectors

Pull-back of a metric tensor w.r.t. the infinitesimal flow generated by a vector field V, comparison of a tensor in x with that pulled-back from $\Phi_\epsilon(x)$, definition of Lie derivative, component computation of Lie derivative, general properties of Lie derivative, isometry = such

$$f : M \to M$$

which preserves length of any curve, one-parameter group (flow) of isometries, Killing equations

$$\mathcal{L}_V g = 0 \ ,$$

commutator of vector fields

$$[V, W] \ ,$$

Lie algebra of solutions of Killing equations, explicit solution for the common plane, explicit solution for (pseudo)-Euclidean spaces, Lorentz algebra (rotations and boosts = hyperbolic rotations = pseudo-rotations), Poincaré algebra (rotations, boosts and translations), strain tensor (small deformations of continuous media)

2 Differential forms

2.1 Volumes of parallelepipeds and forms in linear algebra

Parallelepiped associated with several vectors, degenerate parallelepiped, a p-form in L, anti-symmetrization operation

$$t_{i\dots j} \mapsto t_{[i\dots j]} \ ,$$

the wedge product

$$\alpha \wedge \beta$$

(bilinear, associative, graded commutative), basis forms

$$e^a \wedge \dots \wedge e^b \ ,$$

expression

$$\alpha = (1/p!)\alpha_{a\dots b} e^a \wedge \dots \wedge e^b$$

of a general p-form, how it helps in practical wedge multiplication, interior product

$$i_v \alpha \ ,$$

its nice properties

2.2 Differential calculus of forms on manifolds

Comma operation does not work as tensor operation, it works as an operation on forms (when followed by square brackets), a good idea

$$t_{i\dots j} \mapsto t_{[i\dots j,k]} \ ,$$

exterior derivative

$$d : \Omega^p \to \Omega^{p+1} \ ,$$

it is nilpotent

$$dd = 0$$

and it obeys graded Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

for $\alpha =$ a p-form, Cartan's (magic) formula

$$\mathcal{L}_V = di_V + i_V d \ ,$$

Cartan's formulas for

$$d\alpha(U, V, \dots) \ ,$$

Lie derivative commutes with the exterior one

$$[\mathcal{L}_V, d] = 0 \ ,$$

formula

$$[\mathcal{L}_V, i_W] = i_{[V,W]}$$

2.3 Closed and exact forms

Closed form - such α that

$$d\alpha = 0 \ ,$$

exact form - such α that

$$\alpha = d\beta$$

for some β (if it exists, β is called potential), the freedom

$$\beta \mapsto \beta + d(\dots) \ ,$$

exact $\Rightarrow$ closed always holds because of $dd = 0$, closed $\Rightarrow$ exact not always holds, but it does hold locally, e.g. in a coordinate patch (more detailed information about the relation between closed and exact forms on M is given by "deRham cohomology" theory)

2.4 Forms on manifolds - integral calculus

An intuitive picture why exactly differential forms are integrated, Stokes formula

$$\int_{\partial D} \alpha = \int_D d\alpha \ ,$$

some special cases (Newton-Leibniz formula, area under the graph of a function, Green's formula, integration by parts), the formula

$$\int_{f(D)} \alpha = \int_D f^* \alpha$$

3 Hamiltonian mechanics and symplectic manifolds

3.1 How Poisson tensor emerges

Hamiltonian equations

$$\dot{x}^a = \partial H/\partial p_a \qquad \dot{p}_a = -\partial H/\partial x^a$$

as linear relations between dots of (all) coordinates and (all) components od dH, most easily visible when (x^a, p_a) coordinates are relabeled to $z^A, \ A = 1, \dots 2n$, we get

$$\dot{z}^A = (dH)_B \mathcal{P}^{BA}$$

with

$$\mathcal{P}^{AB} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ,$$

it defines (skew-symmetric) Poisson tensor $\mathcal{P}$ of type $\binom{2}{0}$, which raises index on the gradient dH. Poisson bracket is then

$$\{f, g\} = \mathcal{P}(df, dg)$$

and Hamilton equations read

$$\dot{\gamma} = V_H \qquad V_H = \mathcal{P}(dH, \ . \ )$$

3.2 How symplectic form emerges

Poisson tensor $\mathcal{P} \leftrightarrow \mathcal{P}^{AB}$ of type $\binom{2}{0}$ defines a (skew-symmetric) tensor $\omega \leftrightarrow \omega_{AB}$ of type $\binom{0}{2}$ as its "inverse" via

$$\mathcal{P}^{AC} \omega_{CB} = -\delta^A_B \ ,$$

$$\omega_{AB} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ,$$

so we obtain a 2-form

$$\omega = (1/2) \omega_{AB} dz^A \wedge dz^B = dp_a \wedge dx^a$$

which is closed ($d\omega = 0$) and non-degenerate ($\det \omega_{AB} \neq 0$); any closed and non-degenerate 2-form is called symplectic form; Hamilton equations read

$$\dot{\gamma} = V_H \qquad i_{V_H} \omega = -dH$$

3.3 Hamiltonian fields and Poisson brackets

Hamiltonian field V_f generated by a function f is defined either as

$$V_f = \mathcal{P}(df,\ .\ )$$

(i.e. via raising of index on df with the help of $\mathcal{P}$), or, in terms of the symplectic form, as V_f obeying

$$i_{V_f}\omega = -df$$

(definitions are, for non-degenerate $\mathcal{P}$, equivalent); Poisson brackets may be expressed in several ways, too, as

$$\{f, g\} = \mathcal{P}(df, dg) = \omega(V_f, V_g) = V_f g = -V_g f\ ;$$

closure of Hamiltonian fields w.r.t. the commutator:

$$[V_f, V_g] = V_{\{f,g\}}\ ;$$

Jacobi identity for Poisson bracket is equivalent to

$$d\omega = 0$$

(i.e. to the fact that ω is closed), invariance of ω w.r.t. (any) Hamiltonian flow

$$\mathcal{L}_{V_f}\omega = 0\ ,$$

comparison with isometries and Killing vectors

3.4 Symmetries and conserved quantities (no action integral)

What is a symmetry of a Hamiltonian triple (M, ω, H), Cartan symmetries, exact Cartan symmetries, i.e. Hamiltonian fields V_f, whose generators f obey

$$\{H, f\} = 0$$

and the bijection

$$V_f \leftrightarrow f$$

onto conserved quantities f, new solutions

$$\gamma_s(t) \equiv \Phi^f_s(\gamma(t))$$

from an old solution $\gamma(t)$ and a symmetry

$$V_f \leftrightarrow \Phi^f_s$$

3.5 Canonical transformations

Darboux theorem (canonical "appearance" of a closed 2-form), its manifestation in symplectic case, transformations of coordinates which preserve canonical "appearance" of ω,

$$dp_a \wedge dq^a = dP_a \wedge dQ^a \equiv dP_a(q,p) \wedge dQ^a(q,p)$$

their manifestation on the appearance of Hamilton equations, two ways of explicit work with canonical transformations (generators and generating functions)

3.6 Poincaré integral invariants

A form invariant w.r.t. to a vector field, consequences for their integrals; on a phase space,

$$\omega \qquad \omega \wedge \omega \qquad \omega \wedge \omega \wedge \omega \qquad \dots$$

are forms invariant w.r.t. any Hamiltonian field, Poincaré (absolute) integral invariants

$$\int_{D_2} \omega \qquad \int_{D_4} \omega \wedge \omega \qquad \int_{D_6} \omega \wedge \omega \wedge \omega \qquad \dots ,$$

Liouville theorem as a particular case, canonical Liouville volume form

$$\Omega \equiv \omega \wedge \dots \wedge \omega ,$$

Poincaré (relative) integral invariants

$$\oint_{c_1} \theta \qquad \oint_{c_3} \theta \wedge \omega \qquad \oint_{c_5} \theta \wedge \omega \wedge \omega \qquad \dots ,$$

where

$$\theta = p_a dx^a \qquad d\theta = \omega \qquad \partial c_1 = \partial c_3 = \dots = 0$$

3.7 Volume form and (the corresponding) divergence of a vector field

Volume form - a non-zero n-form on n-dimensional manifold, coordinate expression

$$\Omega = f(x) dx^1 \wedge dx^1 \wedge \dots \wedge dx^n ,$$

divergence defined as

$$\mathcal{L}_V \Omega =: (\operatorname{div} V) \Omega ,$$

its geometrical meaning (it measures how volumes are changing due to the flow of V), divergence-less fields (the flow preserves volumes), examples - metric divergence, symplectic divergence (Hamiltonian fields are divergence-less)

3.8 Algebra of observables of classical mechanics

Pure states ($m \in (M, \omega)$) and observables ($f \in \mathcal{F}(M)$) in Hamiltonian mechanics, prediction of the result of measurement of an observable in a state ($f(m) \in \mathbb{R}$), time development of pure states

$$m \mapsto \Phi_t(m) \ ,$$

(Schrödinger-like picture), time development of observables

$$f \mapsto f_t := \Phi_t^* f \ ,$$

(Heisenberg-like picture), equivalence of the two pictures

$$f(\Phi_t(m)) = (\Phi_t^* f)(m) \ ,$$

two products in the algebra $\mathcal{F}(M)$ (fg and $\{f, g\}$), Hamiltonian flows preserve whole structure of the algebra, more general states (probabilistic distributions ρ on M), their time evolution

$$\rho \mapsto \rho_t := \Phi_{-t}^* \rho \ ,$$

equivalence of the two pictures

$$\int_M (\Phi_{-t}^* \rho) f \Omega = \int_M \rho (\Phi_t^* f) \Omega \ ,$$

equations of motion

$$\partial_t f_t = \{H, f_t\} \qquad \partial_t \rho_t = -\{H, \rho_t\}$$

3.9 Cotangent bundle T^*M as a phase space

Construction of T^*M from M (in our context, construction of the phase space T^*M associated with a configuration space M), canonical coordinates (x^a, p_a), canonical (exact) symplectic form on T^*M, fiber bundle, base space, total space, fiber over x, projection

3.10 Time-dependent Hamiltonians - what changes are needed

Extended phase space, Hamilton equations in the form

$$\alpha_a = 0 \qquad \beta^a = 0 \ ,$$

Hamilton equations in the form

$$i_{\dot{\gamma}} d\sigma = 0 \ ,$$

where

$$d\sigma = \alpha_a \wedge \beta^a = d(p_a dx^a - H dt)$$

3.11 Action integral, Hamilton's principle

How action integral for Hamiltonian mechanics can be constructed:

$$S[\gamma] = \int_\gamma \sigma \ ,$$

how one can easily see that the action is stationary on solutions of Hamilton equations, the difference between endpoints fixation in Lagrangian and Hamiltonian cases

3.12 Symmetries and conserved quantities (based on action integral)

If the infinitesimal flow Φ_ϵ of a vector field ξ on extended phase space does not change the action, i.e.

$$S[\Phi_\epsilon \gamma] = S[\gamma] \ ,$$

we speak of a symmetry of the action; then it turns out that the function

$$i_\xi \sigma$$

is the corresponding conserved quantity; e.g. for

$$\xi = \partial_t$$

we get that it is a symmetry if

$$\partial_t H = 0$$

and that

$$i_\xi \sigma = -H$$

(so, the time translation is a symmetry if $\partial_t H = 0$ and the corresponding conserved quantity is H itself, the energy)

3.13 Poincaré-Cartan integral invariants

On extended phase space, integrals of

$$\sigma \qquad \sigma \wedge d\sigma \qquad \sigma \wedge d\sigma \wedge d\sigma \qquad \dots \ ,$$

over any two closed surfaces (i.e. $\partial c = 0$) of (corresponding) odd dimension encircling the same tube of trajectories (solutions of Hamilton equations) are equal (relative Poincaré-Cartan integral invariants, integrals

$$\oint_{c_1} \sigma \qquad \oint_{c_3} \sigma \wedge d\sigma \qquad \oint_{c_5} \sigma \wedge d\sigma \wedge d\sigma \qquad \dots \ ,$$

where

$$\sigma = p_a dx^a - H dt \ ;$$

if the surfaces lie in hyper-planes of constant time, we return to relative Poincaré integral invariants); similarly, integrals of

$$d\sigma \qquad d\sigma \wedge d\sigma \qquad d\sigma \wedge d\sigma \wedge d\sigma \qquad \dots$$

over any two domains of (corresponding) even dimension connected by trajectories are equal, absolute Poincaré-Cartan integral invariants = integrals

$$\int_{D_2} d\sigma \qquad \int_{D_4} d\sigma \wedge d\sigma \qquad \int_{D_6} d\sigma \wedge d\sigma \wedge d\sigma \qquad \dots$$

4 Field theory in the language of forms

4.1 The Hodge star operator

Metric volume form (fixation of $f(x)$ in

$$\Omega = f(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \ ),$$

$*$ as a linear isomorphism from p-forms to $(n-p)$-forms, it holds

$$** = \pm \hat{1}$$

Important scalar product on forms

$$\langle \alpha, \beta \rangle := \int_D \alpha \wedge *\beta$$

From

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + \hat{\eta}\alpha \wedge d * \beta$$

one gets

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_{\partial D} \alpha \wedge *\beta$$

where

$$\delta := *^{-1} d * \hat{\eta}$$

If the surface integral vanishes, we obtain

$$d^+ = \delta$$

4.2 Forms in E^3 and how vector analysis drops out

In E^3, the most general 0,1,2 and 3-forms read

$$f \qquad \mathbf{A}.d\mathbf{r} \qquad \mathbf{B}.d\mathbf{S} = *(\mathbf{B}.d\mathbf{r}) \qquad hdV = *h \ ,$$

so all of them are parametrized either by scalar (0- and 3-forms) or by vector (1- and 2-forms) fields, therefore the exterior derivative

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3$$

produces effectively three differential operators, of type scalar $\mapsto$ vector, vector $\mapsto$ vector and vector $\mapsto$ scalar, respectively; they are just well-known grad, rot=curl and div operations from vector analysis; then (the general form-version of the) Stokes theorem gives standard integral identities relating integrals of neighboring dimensions

4.3 Forms in Minkowski space $E^{1,3}$

In $E^{1,3}$, each form may be decomposed as

$$\alpha = dt \wedge \hat{s} + \hat{r}$$

where $\hat{s}$ and $\hat{r}$ are "spatial", i.e. they do not contain dt; that's why spatial forms may be expressed as forms in E^3, but with t present in components, e.g. a 2-form reads

$$\alpha = dt \wedge \mathbf{a}.d\mathbf{r} + \mathbf{b}.d\mathbf{S}$$

with

$$\mathbf{a}(t, \mathbf{r}) \qquad \mathbf{b}(t, \mathbf{r}) \ ;$$

similarly, operations on forms can be expressed through "spatial" operations; in particular

$$d\alpha \equiv d(dt \wedge \hat{s} + \hat{r}) = dt \wedge (\partial_t \hat{r} - \hat{d}\hat{s}) + \hat{d}\hat{r}$$

and

$$*\alpha \equiv *(dt \wedge \hat{s} + \hat{r}) = dt \wedge (\hat{*}\hat{r}) + \hat{*}\hat{\eta}\hat{s} \ ;$$

e.g. on 2-forms we get

$$d(dt \wedge \mathbf{a}.d\mathbf{r} + \mathbf{b}.d\mathbf{S}) = dt \wedge (\partial_t \mathbf{b} - \operatorname{curl} \mathbf{a}).d\mathbf{S} + (\operatorname{div} \mathbf{b})dV$$

and

$$*(dt \wedge \mathbf{a}.d\mathbf{r} + \mathbf{b}.d\mathbf{S}) = dt \wedge \mathbf{b}.d\mathbf{r} - \mathbf{a}.d\mathbf{S}$$

4.4 Maxwell equations in terms of forms

Introducing

$$F = dt \wedge \mathbf{E}.d\mathbf{r} - \mathbf{B}.d\mathbf{S}$$

we get

$$dF = dt \wedge (-\partial_t \mathbf{B} - \operatorname{curl} \mathbf{E}).d\mathbf{S} + (-\operatorname{div} \mathbf{B})dV$$

so that the homogeneous half of Maxwell equations simply reads

$$dF = 0 \ ;$$

now

$$*F = *(dt \wedge \mathbf{E}.d\mathbf{r} - \mathbf{B}.d\mathbf{S}) = dt \wedge (-\mathbf{B}).d\mathbf{r} - \mathbf{E}.d\mathbf{S}$$

and so

$$d * F = dt \wedge (-\partial_t \mathbf{E} + \operatorname{curl} \mathbf{B}).d\mathbf{S} - (\operatorname{div} \mathbf{E})dV$$

Therefore the inhomogeneous half of Maxwell equations reads

$$d * F = -J$$

where

$$J := dt \wedge (-\mathbf{j}.d\mathbf{S}) + \rho dV$$

is the source 3-form. Equivalently,

$$\delta F = -j$$

where

$$j := *J$$

4.5 Some immediate consequences

Existence of 4-potential:

$$dF = 0 \qquad \Rightarrow \qquad F = dA$$

for some 1-form A. Gauge freedom

$$A \sim A' \equiv A + d\psi$$

(Then $F' := dA' = dA = F$.) Consistency of $d * F = -J$ needs

$$dJ = 0$$

(= the continuity equation, local conservation of the charge)

4.6 The action formulation of Maxwell equations

From

$$S[A] := -\frac{1}{2}\langle dA, dA\rangle - \langle A, j\rangle$$

we get

$$S[A + \epsilon a] := S[A] + \epsilon(-\langle a, \delta dA + j\rangle)$$

So the variation principle gives the equation

$$\delta dA = -j$$

When denoting

$$F = dA$$

we get (both halves of) Maxwell equations

$$dF = 0 \qquad \delta F = -j$$

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