Differential geometry for Physicists

(Exercises done on recitations)

Marián Fecko*
Comenius University, Bratislava

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$\boxed{1.3.1}$ On a circle S^1 of radius R we introduce local coordinates x, x' as shown on the figure (this is called the *stereographic projection*). On higher-dimensional spheres S^2,...,S^n a natural generalization of this idea results in coordinates $\mathbf{r}, \mathbf{r}'$. Verify that
i) on the intersection of the patches, where the primed and unprimed coordinates are in operation, we get for S^1 and S^n respectively the following explicit transition relations

$$x' = \frac{(2R)^2}{x} \qquad \mathbf{r}' = \frac{(2R)^2}{r}\frac{\mathbf{r}}{r}$$

ii) in this way an analytic atlas comprised of two charts has been constructed on S^n; the sphere S^n is thus an n-dimensional analytic manifold
iii) if the complex coordinates z and z' are introduced on S^2

$$\mathbf{r} \leftrightarrow (x, y) \leftrightarrow z \equiv x + iy \qquad \mathbf{r}' \leftrightarrow (x', y') \leftrightarrow z' \equiv x' + iy'$$

then the transition relations look

$$z' = (2R)^2/\bar{z} \qquad \bar{z} \equiv x - iy$$

Hint: on S^n a projection is to be performed onto n-dimensional mutually parallel *planes*, touching the North and South Poles respectively (in these planes $\mathbf{r} \equiv (x^1, \dots, x^n)$ represent common Cartesian coordinates centered at the poles). Then $\mathbf{r}' = \lambda \mathbf{r}$ and one easily finds λ from the observation that in the (two-dimensional) plane given by the poles and the point P the situation reduces to S^1.
(More information to this exercise, in particular a detailed treatment of the n-dimensional case, You can find in *Additional material* file on my web-page.)

*e-mail: `fecko@fmph.uniba.sk`

$\boxed{1.3.2}$ The real *projective space* $\mathbb{R}P^n$ is the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin. The complex projective space $\mathbb{C}P^n$ is introduced similarly - one should replace $\mathbb{R} \mapsto \mathbb{C}$ in the preceding definition. (Here, a complex line consists of all *complex* multiples of a fixed (non-vanishing) complex vector (point of $\mathbb{C}^{n+1}$) z, so that it is a *two-dimensional* object from a real point of view.)
i) Introduce the structure of an n-dimensional smooth manifold (= local coordinates) on $\mathbb{R}P^n$
ii) the same for $\mathbb{C}P^n$ (it is $2n$-dimensional)
iii) show that the states of an n-level system in quantum mechanics are in one-to-one correspondence with the points of $\mathbb{C}P^{n-1}$
iv) show that $\mathbb{C}P^1 = S^2$ (in the sense of (1.4.7)) $\Rightarrow$ (pure) states of spin 1/2 correspond to unit vectors $\mathbf{n}$ in $\mathbb{R}^3$.
Hint: i) one line (a point from $\mathbb{R}P^n$) consists of those points of $\mathbb{R}^{n+1}$ which may be obtained from a fixed $(x^0, x^1, ..., x^n)$ using the freedom $(x^0, x^1, ..., x^n) \sim (\lambda x^0, ..., \lambda x^n)$; in the part of $\mathbb{R}^{n+1}$ where $x^0 \neq 0$ the freedom enables one to make 1 from the first entry of the array (visually this means that the point of intersection of the line with the plane $x^0 = 1$ has been used as a representative of the line); the other n numbers are to be used as local coordinates on $\mathbb{R}P^n$ (they are the coordinates in the plane $x^0 = 1$ mentioned above; see the figure for $n = 1$, try to draw the case $n = 2$): $(x^0, x^1, ..., x^n) \sim (\lambda x^0, ..., \lambda x^n) \sim (1, \xi^1, ..., \xi^n)$ for $x^0 \neq 0$, $\Rightarrow (\xi^1, ..., \xi^n)$ are coordinates (there); in this way obtain step-by-step $(n+1)$ charts, [1] with the last one coming from $(x^0, x^1, ..., x^n) \sim (\lambda x^0, ..., \lambda x^n) \sim (\eta^1, ..., \eta^n, 1)$ for $x^n \neq 0$; ii) in full analogy, $\xi, \dots, \eta$ are now *complex* n-tuples, giving rise to $2n$ real coordinates; iii) two non-vanishing vectors in a Hilbert space, one of them being a complex constant multiple of the other, correspond to a single state; iv) spin 1/2 is a 2-level system
(More information to this exercise, in particular a detailed treatment of the identification of S^2 with $\mathbb{C}P^1$, You can find in *Additional material* file on my web-page.)

$\boxed{1.4.3}$ Let

$$f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \qquad\qquad f(z, w) = zw$$

be the map induced by the multiplication of complex numbers. Check whether it is a C^∞-map.

$\boxed{1.4.4}$ Let $M = \mathbb{R}^2 \diagdown (0, 0)$ and consider the map defined in terms of complex coordinates as follows

$$f : M \to M \qquad\qquad f(z) = z^{-1}$$

Is this a C^∞-map?

[1] In this context the coordinates $(x^0, x^1, ..., x^n)$ in $\mathbb{R}^{n+1}$ are said to be the *homogeneous coordinates* (of the points in $\mathbb{R}P^n$). Note that they are *not* local coordinates on $\mathbb{R}P^n$ in the sense of the definition of a manifold, since they are not in *one-to-one* correspondence with the points (they *are* official coordinates only in $\mathbb{R}^{n+1}$).

$\boxed{2.1.1}$ Show that the prescription

$$A \mapsto \det\ A \equiv f(A)$$

defines a smooth function on the manifold of all real $n \times n$ matrices ($\sim \mathbb{R}^{n^2}$).
Hint: The determinant is a polynomial in the matrix elements.

$\boxed{2.3.3}$ Find integral curves of the field $V = \partial_x + 2\partial_\varphi$ on $\mathbb{R}[x] \times S^1[\varphi]$ (the surface of a cylinder). Draw the results.

$\boxed{\text{new}}$ Find integral curves of the field $V = y\partial_x + x\partial_y$ on $\mathbb{R}^2$. Draw the results and compare the picture with that for the field $W = -y\partial_x + x\partial_y$. What happens if we study, say, $\hat{W} = -4y\partial_x + x\partial_y$ or $\hat{V} = 4y\partial_x + x\partial_y$?
Hint (for the "4" version): rescale the axes

$\boxed{2.3.7}$ Find a vector field V on $\mathbb{R}^{2n}[q^1, \dots, q^n, p_1, \dots, p_n]$, which corresponds to the *Hamilton equations*

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a} \qquad\qquad a = 1, \dots n$$

$[V = (\partial H/\partial p_a)\partial/\partial q^a - (\partial H/\partial q^a)\partial/\partial p_a]$

$\boxed{2.3.9}$ Express the results of exercises (2.3.3) and ("new") in the form of a flow $\Phi_t : x^i \mapsto x^i(t) \equiv \Phi_t(x^i)$.
$[(r, \varphi) \mapsto (r, \varphi + t)$ or $(x, y) \mapsto (x \cos t - y \sin t, x \sin t + y \cos t)$; $(x, \varphi) \mapsto (x + t, \varphi + 2t)]$

$\boxed{2.4.3}$ Prove that the space $(L^*)^*$ is canonically isomorphic to the space L.
Hint: the canonical isomorphism $f : L \to (L^*)^*$ is $\langle f(v), \alpha \rangle := \langle \alpha, v \rangle$.

$\boxed{2.4.5}$ Check that
ii) some special instances are given by

$$T^0_0(L) = \mathbb{R} \qquad T^0_1(L) = L^* \qquad T^1_0(L) \approx L$$

$$T^1_1(L) \approx \text{Hom}\,(L, L) \approx \text{Hom}\,(L^*, L^*) \qquad\qquad T^0_2(L) = \mathcal{B}_2(L)$$

where $\text{Hom}\,(L_1, L_2)$ denotes all linear maps from L_1 into L_2, $\mathcal{B}_2(L)$ are bilinear forms on L and $\approx$ denotes canonical isomorphism.
Hint: $\binom{0}{0}$, $\binom{0}{1}$ and $\binom{0}{2}$ definitions, $\binom{1}{0}$ (2.4.3); $\binom{1}{1}$: the isomorphisms $\text{Hom}\,(L, L) \to T^1_1(L)$ and $\text{Hom}\,(L^*, L^*) \to T^1_1(L)$ read

$$t(v; \alpha) := \langle \alpha, A(v) \rangle \qquad \text{and} \qquad t(v; \alpha) := \langle B(\alpha), v \rangle$$

or, equivalently (in the opposite direction)

$$A(v) := t(v;\ .\ ) \qquad B(\alpha) := t(\ .\ ; \alpha)$$

$\boxed{3.2.2}$ Induce a metric tensor on a torus T^2 from its embedding (1.5.7) into E^3

$$x = (a + b\sin\psi)\cos\varphi \qquad y = (a + b\sin\psi)\sin\varphi \qquad z = b\cos\psi$$

$[g = (a + b\sin\psi)^2 d\varphi \otimes d\varphi + b^2 d\psi \otimes d\psi]$

$\boxed{3.2.3}$ Induce a metric tensor on a torus T^2 from its embedding (1.5.8) into E^4 (*flat torus*)

$$x^1 = \cos\alpha \qquad x^2 = \sin\alpha \qquad x^3 = \cos\beta \qquad x^4 = \sin\beta$$

$[g = d\alpha \otimes d\alpha + d\beta \otimes d\beta]$

$\boxed{3.2.4}$ Induce a metric tensor on a sphere S^2 from its embedding into E^3

$$x = R\sin\vartheta\cos\varphi \qquad y = R\sin\vartheta\sin\varphi \qquad z = R\cos\vartheta$$

$[g = R^2(d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi)]$

$\boxed{3.2.5}$ Induce a metric tensor on a sphere S^3 from its embedding into E^4

$$x = R\sin\vartheta\cos\varphi \qquad y = R\sin\vartheta\sin\varphi \qquad z = R\cos\vartheta\cos\psi \qquad w = R\cos\vartheta\sin\psi$$

Show that the coordinates $(\vartheta, \varphi, \psi)$ (they are called *biharmonic coordinates*) are orthogonal.
$[g = R^2(d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi + \cos^2\vartheta d\psi \otimes d\psi)]$

$\boxed{2.4.8}$ Operations, producing tensors from tensors, are said to be *tensor operations.* So far we have met linear combination and tensor product. One further important tensor operation is provided by *contraction.* It is defined (for $p, q \geq 1$) as follows:

$$C : T^p_q(L) \to T^{p-1}_{q-1}(L) \qquad\qquad t \mapsto Ct := t(\dots, e_a, \dots; \dots, e^a, \dots)$$

where the exact position of arguments e_a and e^a is to be specified - it forms a part of the definition (there are several (pq) various possible contractions, in general, and one has to state *which one* is to be performed).

Check that
$i)$ the result is indeed a tensor (multilinearity)
$ii)$ C does not depend on the choice of the basis e_a (when e_a has been fixed, however, e^a is to be the dual)
$iii)$ in components the rule for C looks like [2]

$$t^{\dots\dots}_{\dots\dots} \mapsto t^{\dots a.}_{.a\dots}$$

[2]Each contraction thus unloads a tensor by two indices. It breathes with fewer difficulties immediately (fewer indices = fewer worries), it feels like after a rejuvenation cure. This human aspect of the matter is reflected sensitively in German terminology, where the word *Verjüngung* (rejuvenescence) is used.

i.e. as a *summation* with respect to a pair of upper and lower indices
iv) independence of a choice of basis results from the component formula, too
Hint: *ii*) (2.4.2); *iv*) (2.4.6)

$\boxed{3.2.6}$ Let r, z, φ be cylindrical coordinates in E^3 and consider a rotational surface $\mathcal{S}$ given by both expressions $r(z)$ and $z(r)$. Induce a metric tensor (in coordinates z, φ and r, φ respectively) on $\mathcal{S}$. Specify for the surface of a cylinder and a cone as well as for both kinds of rotational hyperboloids and rotational paraboloid.
[$g = (1 + (r'(z))^2)dz \otimes dz + r^2(z)d\varphi \otimes d\varphi = (1 + (z'(r))^2)dr \otimes dr + r^2 d\varphi \otimes d\varphi$]

$\boxed{4.3.7}$ Let D_1, D_2 be two derivations of the tensor algebra. Check that
i) their linear combination as well as the commutator

$$D := D_1 + \lambda D_2 \qquad \text{resp.} \qquad D := [D_1, D_2] \equiv D_1 D_2 - D_2 D_1$$

happen to be derivations of the tensor algebra, too
ii) if D_1, D_2 commute with contractions, then this is true for linear combination and the commutator, too.

$\boxed{4.3.8}$ Prove that
i)

$$\mathcal{L}_{V+\lambda W} = \mathcal{L}_V + \lambda \mathcal{L}_W \qquad \mathcal{L}_{[V,W]} = [\mathcal{L}_V, \mathcal{L}_W] \equiv \mathcal{L}_V \mathcal{L}_W - \mathcal{L}_W \mathcal{L}_V$$

ii) the mapping

$$\mathcal{L} : \mathfrak{X}(M) \to \text{Der } \mathcal{T}(M) \qquad V \mapsto \mathcal{L}_V$$

is a homomorphism of Lie algebras
Hint: *i*) according to (4.3.7) we are to prove the equality of two derivations of the tensor algebra which commute with contractions, or equivalently (after reshuffling of all terms to one side of equation), that a certain derivation of this type *vanishes.* By (4.3.2) it is enough to verify this on functions and vector fields, which is easy (4.3.6); *ii*) just this is asserted in *i*).

$\boxed{4.6.7}$ Find Killing vectors and the corresponding flows for the ordinary Euclidean plane.
Hint: denote $\xi^1(x, y) \equiv A(x, y), \xi^2(x, y) \equiv B(x, y)$. Then the Killing equations read

$$A_{,x} = 0 = B_{,y} \qquad \Rightarrow \qquad A(y), B(x)$$
$$A_{,y} = -B_{,x} \qquad \Rightarrow \qquad A'(y) = -B'(x) = \text{const.}$$

so that the general solution is

$$\xi \equiv A\partial_x + B\partial_y = k_1 e_1 + k_2 e_2 + k_3 e_3 \ ,$$

e_1, e_2 and e_3 being three linearly independent solutions

$$e_1 = \partial_x \qquad e_2 = \partial_y \qquad e_3 = -y\partial_x + x\partial_y$$

(they are linearly independent *over* $\mathbb{R}$; this is a basis of the Lie algebra of Killing fields, *not* to be confused with a basis (in the sense of a frame field) of vector fields in $\mathbb{R}^2[x, y]$!). Their flows are translations along the x and y directions and rotations around the origin $(0, 0)$ respectively.

$\boxed{4.6.8}$ Let $x' = x - x_0, y' = y - y_0$ be the coordinates in $\mathbb{R}^2$ with respect to the origin, which is translated into (x_0, y_0).
i) Check that a general Killing vector, expressed in the initial coordinates (x, y) as well as the new coordinates (x', y'), reads

$$\xi = k_1\partial_x + k_2\partial_y + k_3(-y\partial_x + x\partial_y)$$
$$= (k_1 - k_3 y_0)\partial_{x'} + (k_2 + k_3 x_0)\partial_{y'} + k_3(-y'\partial_{x'} + x'\partial_{y'})$$

ii) give an interpretation of this computation
Hint: *ii*) unless the isometry (which may be obtained by the deformation of the identity) is a pure translation (i.e. if $k_3 \neq 0$), it may be regarded as a *pure rotation* around the appropriate point (x_0, y_0) (this point is obtained by equating the coefficients of the generators of translations $\partial_{x'}, \partial_{y'}$ to zero, or using (4.1.6)).

$\boxed{4.6.9}$ Guess (and then test your intuition by plugging the guess into Killing equations) a Killing vector for a general rotational surface discussed in (3.2.6).
Hint: the surface is symmetric with respect to rotations around the z-axis; (4.1.7)

$\boxed{4.6.10}$ Find all Killing vectors for the *(pseudo)-Euclidean space*, i.e. for $E^{p,q} \equiv (\mathbb{R}^n, \eta)$, where η is the Minkowskian metric with the signature $(p, q), p + q = n$. Show that there are three types of flows - translations, rotations and hyperbolic rotations (for $p = 1, q = 3$ they are known as *Poincaré transformations*, for $q = 0$ *Euclidean transformations*, see also (10.1.15) and (12.4.8)).
Hint: in Cartesian coordinates the Killing equations read

$$\xi_{i,j} + \xi_{j,i} = 0 \qquad\qquad \xi_i \equiv \eta_{ij}\xi^j$$

Differentiation with respect to x^k gives

$$\xi_{i,jk} + \xi_{j,ik} = 0$$

In full analogy we get

$$\xi_{i,kj} + \xi_{k,ij} = 0 \qquad \xi_{j,ik} + \xi_{k,ji} = 0$$

Then

$$\xi_{i,jk} = -\xi_{j,ik} = \xi_{k,ij} = -\xi_{i,kj} \qquad \Rightarrow \qquad \xi_{i,jk} = 0$$

$\Rightarrow$

$$\xi^i = A^i_j x^j + a^i \qquad A, a = \text{const.}$$

and plugging into the initial equations leads to the restriction for the matrix A

$$(\eta A)+(\eta A)^T=0 \qquad\Rightarrow\qquad A\in so(p,q)$$

(see (11.7.6)), i.e.

$$\xi=(A^i_jx^j+a^i)\partial_i\equiv\xi^{(A,a)} \qquad (A,a)\in so(p,q)\ \ltimes\ \mathbb{R}^n$$

One can check that

$$\xi^{(A,a)}\ \leftrightarrow\ -\begin{pmatrix}A & a\\ 0 & 0\end{pmatrix}$$

is an isomorphism of the Killing algebra with the semidirect sum $so(p,q)\ltimes\mathbb{R}^{p+q}$ (see (12.4.9)).
We can verify as well that the field $\xi^{(A,a)}$ may be written in the form

$$\xi^{(A,a)}=\frac{1}{2}(A\eta)^{ij}M_{ji}+a^iP_i$$

($(A\eta)^{ij}=-(A\eta)^{ji}$ being a consequence of $(\eta A)+(\eta A)^T=0$) where the vector fields

$$M_{ij}\equiv-M_{ji}\equiv x_i\partial_j-x_j\partial_i \qquad P_i\equiv\partial_i \qquad x_i\equiv\eta_{ij}x^j$$

constitute a basis of the Killing algebra. Flows: solve the equations for the flow of M_{ij} and P_j respectively. The fields P_j correspond to translations, M_{ij} yield rotations and hyperbolic rotations in the plane (ij), depending on the sign of the product $\eta_{ii}\eta_{jj}$ (not to be summed; $+1$ rotations, -1 hyperbolic rotations (*boosts*)).

$\boxed{4.6.24}$ In the mechanics of elastic (continuous) media one introduces the *strain tensor* in the following way: when the points in the continuum are (infinitesimally) displaced according to $\mathbf{r}\mapsto\mathbf{r}+\mathbf{u}(\mathbf{r})$ (a vector field [3] $\mathbf{u}(\mathbf{r})$ is called the *displacement* (field)), the corresponding deformation is encoded in a second rank tensor (field) with components (in Cartesian coordinates)

$$\varepsilon_{ij}:=\frac{1}{2}(\partial_iu_j+\partial_ju_i)$$

Check that the coordinate-free expression of this tensor reads

$$\varepsilon=\frac{1}{2}\mathcal{L}_{\mathbf{u}}g$$

where g is the (standard) metric tensor in E^3 and that it follows from the definition of the Lie derivative as well as from the context that a *deformation* of the medium (a shift of points, which *alters* distances between them) is measured by the Lie derivative of a metric tensor, indeed ($\varepsilon=0\Leftrightarrow$ a deformation did not take place $\Leftrightarrow$

[3] A shift $\mathbf{r}\mapsto\mathbf{r}+\mathbf{u}(\mathbf{r})$ is interpreted as an infinitesimal *flow* generated by a vector field $\mathbf{u}(\mathbf{r})$.

it is an *isometry*).
Hint: (4.6.5)

$\boxed{6.1.3}$ Let $M = \mathbb{R}^3[x,y,z]$, $\alpha = xdy - ydz$, $\beta = z^2 dx \wedge dz - dy \wedge dx$, $V = (xy)^2\partial_x + \partial_y$. Compute

$$\alpha \wedge \beta \qquad i_V\alpha \qquad i_V\beta \qquad i_V(\alpha \wedge \beta)$$

Hint: the calculation before (5.2.10), (5.4.2); $[-(xz^2+y)dx \wedge dy \wedge dz,\ x,\ -dx + (xy)^2 dy + (xyz)^2,\ (xz^2+y)dx \wedge dz - (xz^2+y)(xy)^2 dy \wedge dz]$

$\boxed{6.1.7}$ Let A be a $\mathbb{Z}$-graded and graded commutative algebra and let D_k and D_l be its derivations of degree k and l respectively; so there holds

$$A = \bigoplus_{i=-\infty}^{\infty} A_i \qquad a_i a_j = (-1)^{ij} a_j a_i \quad a_i \in A_i, a_j \in A_j$$

$$D_k : A_i \to A_{i+k} \qquad D_k(a_i b) = (D_k a_i)b \ + \ (-1)^{ik} a_i (D_k b) \quad a_i \in A_i, b \in A$$

Show that their *graded commutator*

$$[D_k, D_l] := D_k D_l \ - \ (-1)^{kl} D_l D_k$$

(being actually a commutator, unless both derivations are of *odd* degree, when it becomes the *anti*commutator) [4] is a derivation of the algebra A (of degree $k+l$), too.
Hint: brute force (apply $[D_k, D_l]$ on the product $a_i b$ and make use of the definitions)

$\boxed{6.2.8}$ Show that the Lie derivative of *differential forms* may be expressed in the following (very useful) form [5]

$$\mathcal{L}_V = i_V \ d + d \ i_V \qquad\qquad \textit{Cartan's identity}$$

Hint: according to (6.1.7) this is an equality of two derivations (of degree 0) of the algebra $\Omega(M) \Rightarrow$ it suffices to verify it in degrees 0 and 1, where it is easy (e.g. in components)

$\boxed{6.2.9}$ Prove the validity of the (fairly useful) identity

$$[\mathcal{L}_V, i_W] \equiv \mathcal{L}_V \ i_W - i_W \ \mathcal{L}_V = i_{[V,W]}$$

Hint: just like in (6.2.8)

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[4] Although it is written as an *ordinary* commutator, in graded algebra this means automatically the *graded* commutator (since the latter is much more important than the former).

[5] The operators which enter this formula may be given a visual meaning in *integral calculus* of forms and this identity itself may be interpreted in terms of Stokes' theorem, see (7.8.2).

$\boxed{6.2.10}$ Prove that the exterior derivative commutes with the Lie derivative (along an arbitrary vector field)

$$[d, \mathcal{L}_V] \equiv d\ \mathcal{L}_V - \mathcal{L}_V\ d = 0$$

Hint: just like in (6.2.8), or use the *result* of (6.2.8)

$\boxed{14.1.6}$ Let $V_f \in \text{Ham}\,(M)$, so that it is a Hamiltonian field generated by the function f. Check that
$i)$ the following definitions turn out to be equivalent

$$i_{V_f}\omega = -df \qquad \Leftrightarrow \qquad V_f = \mathcal{P}(df,\ .\ ) \equiv \sharp_{\mathcal{P}} df$$

$ii)$ there holds

$$V_{(f+\text{const.})} = V_f$$

$iii)$ Hamiltonian fields may also be regarded as the analogues of the Killing vectors from Riemannian geometry, since they preserve ω in just the same way as Killing vectors preserve g

$$\mathcal{L}_{V_f}\omega = 0$$

$iv)$ the collection of all Hamiltonian fields is closed with respect to linear combinations (over $\mathbb{R}$) as well as the commutator; namely

$$V_f + \lambda V_g = V_{f+\lambda g} \qquad [V_f, V_g] = V_{\{f,g\}}$$

so that they constitute an (∞-dimensional) *Lie algebra* $\text{Ham}\,(M) \subset \mathfrak{X}(M)$
Hint: $iii)$ (6.2.8); $iv)$ making use of (6.2.9) and the preceding items here we get

$$i_{[V_f,V_g]}\omega = \mathcal{L}_{V_f} i_{V_g}\omega - i_{V_g}\mathcal{L}_{V_f}\omega = d(i_{V_f} i_{V_g}\omega) = ...(14.1.8)... = -d\{f,g\}$$

$\boxed{\text{new}}$ Show that from $(d\omega)(V_f, V_g, V_h) = 0$ follows $d\omega = 0$, i.e. that from $d\omega = 0$ on (all) *Hamiltonian* fields we can deduce that $d\omega = 0$ on *all* fields.
Hint: Try V_f etc. for *coordinate* functions

$\boxed{14.1.9}$ Consider the algebra of observables of the classical mechanics $\mathcal{A}(M)$. Check that
$i)$ the two "products" $\mathcal{A}(M) \times \mathcal{A}(M) \to \mathcal{A}(M)$ involved are interconnected by the identity

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

$ii)$ the prescription

$$\zeta : \mathcal{A}(M) \to \text{Ham}\,(M) \qquad f \mapsto V_f$$

is a homomorphism of Lie algebras, its kernel being constituted by the constant functions on M.

Hint: $i)$ $\{f, \ .\ \} = V_f(\ .\ ) \Rightarrow$ it is a vector field, i.e. a *derivation* of the (associative) algebra $\mathcal{F}(M)$; $ii)$ (14.1.6)

$\boxed{4.4.2}$ Let Φ_t be the flow generated by a vector field V. Starting from the definition

$$\mathcal{L}_V := \left.\frac{d}{dt}\right|_0 \Phi_t^*$$

prove that
$i)$

$$\frac{d}{dt}\Phi_t^* = \Phi_t^* \mathcal{L}_V$$

$ii)$ for C^ω tensor fields there holds

$$\Phi_t^* = e^{t\mathcal{L}_V} \equiv 1 + t\mathcal{L}_V + \frac{t^2}{2!}\mathcal{L}_V\mathcal{L}_V + \dots$$

Hint: $i)$ $\frac{d}{dt}\Phi_t^* = \left.\frac{d}{ds}\right|_{s=0} \Phi_{t+s}^*$, (4.1.2); $ii)$ $(\frac{d}{dt})^n \Phi_t^* = \Phi_t^*(\mathcal{L}_V)^n$

$\boxed{14.1.10}$ Let $\mathcal{A}(M)$ be the algebra of observables of classical mechanics. Since its elements (observables) are the *functions* on the phase space M, there is a natural action of the group of diffeomorphisms on the algebra ($f \mapsto \Phi^* f$). Check that
$i)$ the structure of the algebra is preserved just by the *symplectomorphisms* of (M, ω), i.e. by such diffeomorphisms of M to itself, which preserve the symplectic form ω (or equivalently the Poisson tensor $\mathcal{P}$)

$$\Phi^*\omega = \omega$$

$ii)$ the *flows* of such transformations are generated by the symplectic (in particular by the Hamiltonian) fields
$iii)$ the action of the flow of a Hamiltonian field on the algebra $\mathcal{A}(M)$

$$U_t^f : \mathcal{A}(M) \to \mathcal{A}(M) \qquad\qquad U_t^f := (\Phi_t^f)^* \quad\ \Phi_t^f \leftrightarrow V_f,\ f \in \mathcal{A}(M)$$

can also be expressed in the form of the series

$$U_t^f g = g + t\{f, g\} + \frac{t^2}{2!}\{f, \{f, g\}\} + \frac{t^3}{3!}\{f, \{f, \{f, g\}\}\} + \dots$$

$iv)$ the *Jacobi identity* for the Poisson bracket is just the infinitesimal version of the condition that the Poisson bracket (of two arbitrary functions) is *preserved* by the flow of an arbitrary Hamiltonian field, i.e. of the condition

$$U_t^f\{g, h\} = \{U_t^f g, U_t^f h\} \qquad\qquad f, g, h \in \mathcal{A}(M)$$

$v)$ the map

$$U_t^f : \mathcal{A}(M) \to \mathcal{A}(M)$$

is for each t an *automorphism* of the algebra of observables $\mathcal{A}(M)$ (it preserves its linear structure as well as *both products*) and the prescription $t \mapsto U_t^f$ is the one-parameter group of such automorphisms

Hint: $i)$ $\Phi^*\{f,g\} \equiv \Phi^*(\mathcal{P}(df,dg)) = (\Phi^*\mathcal{P})(d\Phi^* f, d\Phi^* g) \overset{!}{=} \mathcal{P}(d\Phi^* f, d\Phi^* g)$ so that $\Phi^*\mathcal{P} \overset{!}{=} \mathcal{P}$ and consequently $\Phi^*\omega \overset{!}{=} \omega$; $ii)$ in the standard way $\Phi_t^*\omega \overset{!}{=} \omega \;\Rightarrow\; \mathcal{L}_W\omega = 0$; $iii)$ by definition $\frac{d}{dt}\big|_0 U_t^f = \mathcal{L}_{V_f}$, (4.4.2) and $\mathcal{L}_{V_f} g \equiv V_f g = \{f,g\}$; $iv)$ the differentiation of $U_t^f\{g,h\} = \{U_t^f g, U_t^f h\}$ with respect to t in $t=0$ gives $\mathcal{L}_{V_f}\{g,h\} = \{\mathcal{L}_{V_f} g, h\} + \{g, \mathcal{L}_{V_f} h\}$; $v)$ preserving of the pointwise product and linear combinations trivial (this holds for each Φ^*), preserving of the Poisson bracket solves item $iv)$

$\boxed{14.3.6}$ Let (M,ω) be a symplectic manifold, dim $M = 2n$. Check that
$i)$ the n-fold product of the form ω (as well as its arbitrary non-zero multiple) defines on M the *volume form*
$iii)$ in canonical coordinates (q^a, p_a) the form $\tilde{\Omega}$ reads

$$\tilde{\Omega} = (-1)^{\frac{n(n+1)}{2}} n! dq^1 \wedge \dots \wedge dq^n \wedge dp_1 \wedge \dots \wedge dp_n \equiv (-1)^{\frac{n(n+1)}{2}} n! dqdp$$

Therefore one usually adopts its appropriate constant multiple, the *Liouville form*

$$\Omega_\omega \equiv \Omega := (-1)^{\frac{n(n+1)}{2}} \frac{1}{n!}\tilde{\Omega} = dqdp$$

as *the volume form* on a symplectic manifold and as the *phase volume* of the domain $\mathcal{D} \subset M$ we mean the expression $\int_{\mathcal{D}} \Omega$ (i.e. the volume of the domain $\mathcal{D}$ in the sense of the Liouville volume form).
$iv)$ the *Liouville's theorem* holds: the phase volume of an arbitrary ($2n$-dimensional) domain $\mathcal{D}$ is preserved under the time development Φ_t of the phase points (more generally under the flow of an arbitrary Hamiltonian field ζ_f) [6]

$$\Phi_t \leftrightarrow \zeta_H \qquad \Rightarrow \qquad \int_{\Phi_t(\mathcal{D})} \Omega = \int_{\mathcal{D}} \Omega$$

$v)$ any symplectic manifold is orientable
Hint: $i)$ the fact that $\omega \wedge \dots \wedge \omega$ is everywhere non-zero is clear from its coordinate presentation; $ii)$ (5.6.8); $iv)$ (14.3.4); $v)$ (6.3.5)

$\boxed{8.1.1}$ The most natural definition of the integral of a 0-form (function) f over a 0-simplex (point) is given by [7]

$$\int_P f := f(P)$$

[6]If we regard the flow Φ_t as the flow of a fluid, then the result says that the fluid is *incompressible.*

[7]The intuitive meaning of the integral, as is well known, is the sum of the values of the function in infinitesimal domains (resulting from the division of the total domain of

Check that the *Newton-Leibniz formula*

$$\int_a^b f'(x)dx = f(b) - f(a)$$

may then be regarded as a particular case [8] of Stokes' theorem.
Hint: $M = \mathbb{R}^1$, $c = [a, b]$

$\boxed{8.1.2}$ Let D be the domain in the xy plane which is bounded by the straight lines $x = a, x = b, y = 0$ and by the curve $y = f(x)$ from above. In full analogy let V be the domain in the space xyz which lies above the domain S in the plane xy and which is bounded from above by the surface $z = f(x, y)$. Then we know that area of the domain D may be computed in two (completely different) ways, namely either as

$$\int_a^b f(x)dx \quad \text{or as} \quad \int_D dxdy$$

Similarly, the volume of the domain V may be computed in two (completely different) ways, namely as $\int_S f(x, y)dxdy$, but also as $\int_V dxdydz$. Show that both cases may be regarded as a manifestation of Stokes' theorem.
Hint: $dx \wedge dy = d(-ydx)$, $dx \wedge dy \wedge dz = d(zdx \wedge dy)$

$\boxed{8.1.3}$ Let α, β be two forms on an n-dimensional manifold M, with their degrees being such that $\deg \alpha + \deg \beta + 1 = n$ and let D be an n-dimensional domain. Check that
$i)$ the following identity holds

$$\int_D d\alpha \wedge \beta = -\int_D \hat{\eta}\alpha \wedge d\beta + \int_{\partial D} \alpha \wedge \beta$$

$ii)$ the formula representing the "by parts" method of integration

$$\int_a^b f'(x)g(x)dx = -\int_a^b f(x)g'(x)dx + [fg]_a^b$$

is but a simple special case of this identity
Hint: $i)$ (6.2.5), (7.6.7); $ii)$ $M = \mathbb{R}, D = [a, b], \alpha = f, \beta = g$

$\boxed{8.1.4}$ Given two functions $f(x, y), g(x, y) \in \mathcal{F}(\mathbb{R}^2)$ and a (two-dimensional) domain D, let $C \equiv \partial D$ be its (oriented) boundary (closed curve, the *contour*). Show that there holds

$$\oint_C fdx + gdy = \int_D (\partial_x g - \partial_y f)dx \wedge dy$$

integration) multiplied by the volumes of these domains. If the total domain reduces to *a single point* P, there is nothing to be divided and it suffices to take the value of the function right at this point. Note that in doing this the volume of the point P is effectively regarded to be 1 ($\int_P 1 = 1(P) = 1$).

[8]although in a sense a *tautological* one - the definition of the integral of a 0-form has been *extended* in such a way as to make the theorem hold

Hint: set $\alpha = f dx + g dy$ in (7.6.7)

$\boxed{5.5.1}$ Let $E(L)$ be the set of all bases of a vector space L and let $f \in E(L)$. Then any basis e may be uniquely expressed as

$$e_a = f_b A^b_a \qquad \text{i.e.} \qquad e = fA \qquad\qquad A \in GL(n, \mathbb{R})$$

($GL(n, \mathbb{R})$ being the set of all non-singular $n \times n$ real matrices, see (10.1.3) and beyond). Show that
i) each basis falls either into $E(L)_+$ or to $E(L)_-$, i.e.

$$E(L) = E(L)_+ \ \cup \ E(L)_- \qquad\quad E(L)_+ \ \cap \ E(L)_- = \emptyset$$

$$E(L)_\pm = \{e \in E(L); \det A \gtrless 0\}$$

ii) $E(L)_+$ and $E(L)_-$ are "equally large", i.e. there exists a bijection of $E(L)_+$ onto $E(L)_-$
iii) dividing of $E(L)$ into $E(L)_+$ and $E(L)_-$ does not depend on the choice of $f \in E(L)$, i.e. if e and $\tilde{e}$ share the same half with respect to f, they share the same half with respect to any other reference basis $\hat{f} \in E(L)$.
Hint: *ii*) $(e_1, e_2, \dots, e_n) \leftrightarrow (-e_1, e_2, \dots, e_n)$; *iii*) $\det(AB) = \det A \det B$

$\boxed{5.7.1}$ Let (e^a) be an arbitrary basis in L^*. Check that (for n pieces multiplied)
i)

$$e^a \wedge \dots \wedge e^b = \epsilon^{a \dots b} \ e^1 \wedge \dots \wedge e^n$$

ii) the most general n-form ω may be expressed as

$$\omega = \lambda e^1 \wedge \dots \wedge e^n \qquad\quad \lambda \in \mathbb{R}$$

iii) if $e_a \mapsto \hat{e}_a \equiv e_b A^b_a$, then

$$\omega \equiv \lambda e^1 \wedge \dots \wedge e^n = \hat{\lambda} \hat{e}^1 \wedge \dots \wedge \hat{e}^n$$

where

$$\hat{\lambda} = (\det A) \ \lambda$$

A quantity which transforms in this way under a change of basis is called a *scalar density* (of weight -1; see the text after (6.3.7) and problem (21.7.10))
Hint: *ii*) (5.2.9), (5.6.1), $\lambda = \omega_{1 \dots n}$; *iii*) (2.4.2), (5.6.5).

$\boxed{5.7.3}$ Let (L, g, o) be an n-dimensional vector space endowed with a metric tensor g and an orientation o, $e \equiv (e_a)$ and $\hat{e} \equiv (\hat{e}_a)$ two right-handed orthonormal bases respectively, $f \equiv (f_a)$ an arbitrary basis and $\omega(f) := f^1 \wedge \dots \wedge f^n$ (5.7.2). Prove that
i)

$$\omega(e) = \omega(\hat{e})$$

i.e. $\omega_g := \omega(e)$ does not depend on the choice of right-handed orthonormal basis
ii) its expression in terms of the *arbitrary* basis f reads

$$\omega_g \equiv \omega(e) = o(f)\sqrt{|g|}\ \omega(f)$$

where $o(f)$ is $+1$ or -1 depending on whether f is right-handed or left-handed and $|g| \equiv |\det\ g(f_a, f_b)|$
iii)

$$\omega_g(f_1, ..., f_n) = o(f)\sqrt{|g|} \qquad \text{so that} \qquad \omega_{a...b} = o(f)\sqrt{|g|}\ \varepsilon_{a...b}$$

iv)

$$\omega^{a...b} = o(f)\ \text{sgn}\ g\frac{1}{\sqrt{|g|}}\ \varepsilon^{a...b}$$

where sgn g is the sign of the determinant of the matrix $g_{ab} \equiv g(f_a, f_b)$ (for a metric tensor with signature (r, s) (see the text before (2.4.12)) this is $(-1)^s$)
v) a change of orientation in L results in the change

$$\omega_g \mapsto -\omega_g \qquad \text{i.e.} \qquad \omega_{g,-o} = -\omega_{g,o}$$

if $(-o)$ is the orientation which is opposite with respect to o.
Hint: *ii*) let f_a be the arbitrary basis, $f_a = e_b B^b_a$ (i.e. $f = eB$). Then

$$g(f_a, f_b) \equiv g_{ab} = B^c_a \eta_{cd} B^d_b \equiv (B^T \eta B)_{ab} \Rightarrow\ \det\ g = (\det\ B)^2\ \det\ \eta$$

$$\Rightarrow \det\ B = \pm\sqrt{|\det\ g|}$$

The sign is given (since e is right-handed) by the orientation of the basis f, so that $\det\ B = o(f)\ \sqrt{|\det\ g|}$. According to (5.7.2) then

$$\omega(e) = \omega(fB^{-1}) = \det B\ \omega(f) = o(f)\sqrt{|\det g|}\ \omega(f)$$

iv) $\omega^{a...b} \equiv g^{ac} \dots g^{bd} \omega_{c...d} = \dots$ (5.6.5); *v*) the only change is $o(f) \mapsto\ - o(f)$

$\boxed{16.2.1}$ Check that
i) the second series (homogeneous half) of the Maxwell equations may be written in the form

$$dF = 0 \qquad\qquad \text{where} \qquad F := dt \wedge \mathbf{E}.d\mathbf{r} - \mathbf{B}.d\mathbf{S}$$

is a 2-*form of the electromagnetic field*
ii) an explicit expression of its (Cartesian) components in terms of (Cartesian) components of vectors of electric and magnetic field reads

$$F_{0i} = E_i$$

$$F_{ij} = -\epsilon_{ijk} B_k \qquad \text{i.e.} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

iii) a transition to the dual form may be expressed in terms of the fields $\mathbf{E}, \mathbf{B}$ as

$$F \mapsto *F \qquad \Leftrightarrow \qquad (\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{E})$$

Hint: (16.1.4)

$\boxed{16.2.2}$ Check that
i) the first series (inhomogeneous half) of the Maxwell equations may be written in the form

$$\delta F = -j \qquad \text{or equivalently} \qquad d * F = -J \equiv - * j$$

where the three-dimensional quantities ρ (electric *charge density*) and $\mathbf{j}$ (electric *current density*) are built into a single object living in Minkowski space, the 1-*form of current* or alternatively its dual 3-*form of current*

$$j = \rho dt - \mathbf{j}.d\mathbf{r} \equiv j_\mu dx^\mu$$

$$J = dt \wedge (-\mathbf{j}.d\mathbf{S}) + \rho dV \equiv j^\mu d\Sigma_\mu \equiv *j$$

Hint: (16.1.5), (16.1.6)

$\boxed{16.2.3}$ Check that the total electric charge in a spatial domain $\hat{\mathcal{D}}_3$ is given by the integral

$$Q = \int_{\hat{\mathcal{D}}_3} J \equiv \int_{\hat{\mathcal{D}}_3} *j$$

Hint: according to (16.2.2) we have $J = dt \wedge (-\mathbf{j}.d\mathbf{S}) + \rho dV$ and the first term does not contribute to the integral over $\hat{\mathcal{D}}_3$ due to the factor dt

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