Subriemannian geodesics - an introduction

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- How to set differential equations for finding them
- Where subriemannian geodesics might be useful
- What are isoholonomic and (dual) isoperimetric problems
- How they are related to subriemannian geodesics

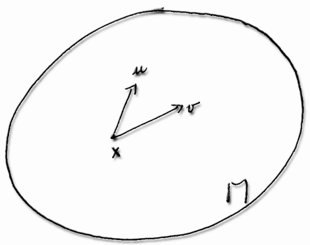
Contents

Riemannian geometry (review)

In Riemannian geometry, at each $x \in M$, there exists

$$g(u, v) \equiv g_{\mu\nu} u^\mu v^\nu$$

for each pair of vectors u, v (at the same point $x \in M$)



Horizontal distribution $(\mathcal{H}, g)$ on M

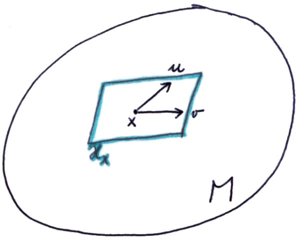
In subriemannian geometry, at each $x \in M$ there is, first, a distinguished k-dimensional subspace

$$\mathcal{H}_x \subset T_x M$$

called **horizontal** subspace in x. Now, the concept of the (positive definite) scalar product

$$g(u, v) \equiv g_{\mu\nu} u^\mu v^\nu$$

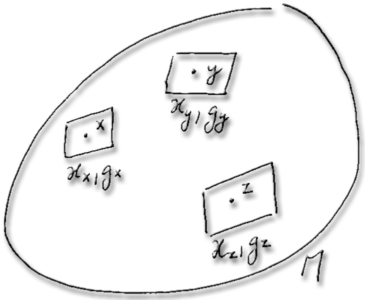
is only defined for u, v from $\mathcal{H}_x$.



Horizontal distribution $(\mathcal{H}, g)$ on M (cont.)

So, altogether,

- a k-dimensional **horizontal distribution** $\mathcal{H}$ on M is given
- the **subriemannian metric**, g, is (only) defined in $\mathcal{H}$



Subriemannian structure on M

The structure $(M, \mathcal{H}, g)$ is known as
the **subriemannian structure** on M.
It is studied in **subriemannian geometry**.
Alternative names:

- **Carnot-Carathéodory** geometry (Gromov, ...)
- **Non-holonomic Riemannian** geometry (Vershik-Gershkovich)
- **Singular Riemannian** geometry (Hermann, ...)

How to describe a distribution on M

Two most frequent ways how a distribution $\mathcal{D}$ is described:

- k (linearly independent) vector fields are given
- $n-k$ (linearly independent) covector fields are given

In particular, let (e_a, e_i) be a frame and (e^a, e^i) the dual coframe. Then the prescriptions

- $\mathcal{D} = \text{Span}\ \{e_a\}$
- $\mathcal{D} =$ common null-space of $e^i = \bigcap \text{Ker}\ e^i$

define the same distribution.

How to describe a distribution on M (cont.)

An alternative way how a distribution $\mathcal{D}$ may be described:

- let a $\binom{2}{0}$-tensor field h (of rank k) be given

Then

- $\mathcal{D} = \operatorname{Im} h$ (the **range** = **image** of h) provides a k-dimensional distribution, where h is regarded as a linear map

$$h_x : T_x^*M \to T_xM \qquad \alpha \mapsto h(\alpha, \ . \ ) \qquad \alpha_\mu \mapsto \alpha_\nu h^{\nu\mu}$$

How to describe a distribution on M (cont.)

We want to "optimize" description of $\mathcal{D}$ in terms of h.
Fix an adapted frame (e_a, e_i), i.e. $e_a \in \mathcal{D}$. Write

$$h = h^{ab} e_a \otimes e_b + h^{ai} e_a \otimes e_i + h^{ia} e_i \otimes e_a + h^{ij} e_i \otimes e_j$$

i.e.

$$h \leftrightarrow \begin{pmatrix} h^{ab} & h^{ai} \\ h^{ia} & h^{ij} \end{pmatrix}$$

Then

$$h(\alpha, \ . \ ) = \cdots = (\alpha_b h^{ba} + \alpha_i h^{ia}) e_a + (\alpha_a h^{ai} + \alpha_j h^{ji}) e_i$$

Since $\mathcal{D} = \text{Span } \{e_a\}$, we get $\boxed{h^{ai} = 0 = h^{ji}}$

How to describe a distribution on M (cont.)

The simplest additional choice is $\boxed{h^{ia} = 0, \ h^{ab} = h^{ba}}$
So, finally, a distribution $\mathcal{D} = \text{Span}\ \{e_a\}$ is encoded in

$$h = h^{ab} e_a \otimes e_b \qquad h^{ab} = h^{ba}$$

with any symmetric, rank k matrix h^{ab}.
Or, equivalently, in

$$h \leftrightarrow \begin{pmatrix} h^{ab} & h^{ai} \\ h^{ia} & h^{ij} \end{pmatrix} = \begin{pmatrix} h^{ab} & 0 \\ 0 & 0 \end{pmatrix} \qquad h^{ab} = h^{ba}$$

$\mathcal{H}$ and g in a single object - the cometric h

Now we need to tell h about the metric g in $\mathcal{H}$.
Let $g_{ab} := g(e_a, e_b)$. It is non-singular $\Rightarrow$ there exist (unique) inverse g^{ab}. Define tensor $g^{ab} e_a \otimes e_b$. Take this as h (i.e. choose $h^{ab} = g^{ab}$)

$$\boxed{h = h^{ab} e_a \otimes e_b \qquad h^{ab} g_{bc} = \delta^a_c}$$

In a component-free language

$$\boxed{g(h(\alpha), v) = \langle \alpha, v \rangle \ , \quad v \in \mathcal{H}}$$

Single tensor h, the cometric on M, carries full information about both $\mathcal{D}$ and g.
Subriemannian geometry is given by a pair (M, h).

Riemannian geometry as a particular case

Riemannian geometry is a particular case, when

$$\mathcal{H}_x = T_x M$$

(so that g operates on all pairs of vectors in x).
Riemannian geometry is usually described by

$$g = g_{ab} e^a \otimes e^b \qquad \text{metric}$$

but (since it is non-degenerate) one can equivalently use

$$g = g^{ab} e_a \otimes e_b \qquad \text{cometric}$$

So the cometric is (in this sense!) more universal object, it works in both Riemannian and subriemannian case.

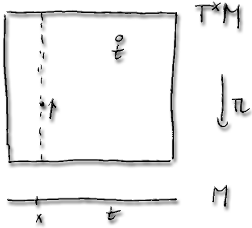
Subriemannian hamiltonian

There is even a simpler object equivalent to h. Recall that we can canonically "lift" completely symmetric contravariant tensors on M into (particular) functions on T^*M. The general formula is

$$\boxed{\overset{\circ}{t}(p) := t(p, \dots, p)}$$

or, in canonical (Darboux) coordinates (x^μ, p_μ) on T^*M

$$\boxed{\overset{\circ}{t}(x, p) := t^{\mu \dots \nu}(x) p_\mu \dots p_\nu}$$



Subriemannian hamiltonian (cont.)

If, say,

$$V = V^{\mu}(x)\partial_{\mu}$$

is a vector field on M, the corresponding function on T^*M

$$\overset{\circ}{V} = V^{\mu}(x)p_{\mu}$$

is called momentum corresponding to V.

Subriemannian hamiltonian (cont.)

In particular, for
the coordinate basis vector fields ∂_μ and
the frame vector fields $e_a = e_a^\mu(x)\partial_\mu$
we get

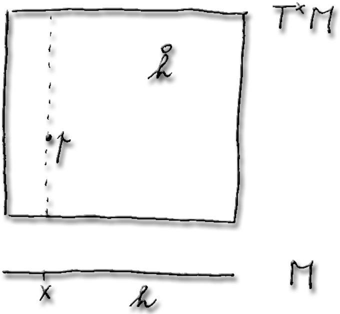
$$\boxed{\begin{array}{lcll} \mathring{\partial}_\mu & = & p_\mu & \text{the } \mu\text{-th } \textbf{canonical momentum} \\ \mathring{e}_a & = & e_a^\mu(x)p_\mu \equiv P_a(x,p) & \text{the } a\text{-th } \textbf{canonical momentum} \end{array}}$$

Subriemannian hamiltonian (cont.)

And, finally, for the cometric h on M

$$h = h^{ab} e_a \otimes e_b$$

we obtain the function on T^*M,
the subriemannian Hamiltonian
corresponding to h



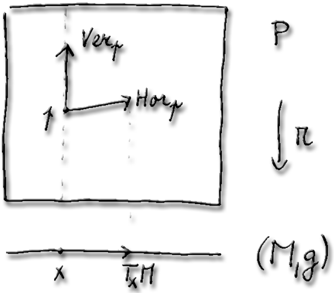
$$\begin{aligned} H(x,p) \equiv \frac{1}{2}\ \mathring{h} &= \frac{1}{2}\ h^{ab} P_a P_b \\ &= \frac{1}{2}\ h^{ab}(x)(e_a^\mu(x)p_\mu)(e_b^\nu(x)p_\nu) \\ &\equiv \frac{1}{2}\ h^{\mu\nu}(x)p_\mu p_\nu \end{aligned}$$

Total space as a subriemannian manifold

Let (M, g) be a Riemannian manifold.
Let $\pi : P \to M$ be a principle bundle with connection ω.
Then, there is natural subriemannian structure on P. It is given in terms of the connection and the metric on M:

$$\mathrm{Hor}_p := \mathrm{Ker}\ \omega_p$$

$$g_p(u, v) := g_x(\pi_* u, \pi_* v)$$



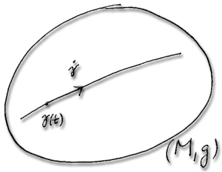
Riemannian geodesics - Lagrangian language

In the Riemannian case, on (M, g), the geodesic equations read

$$(\nabla_{\dot{\gamma}}\dot{\gamma})^{\mu} \equiv \ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = 0$$

They are second order equations on M and turn out to coincide with the **Euler-Lagrange equations** for the (free motion) Lagrangian

$$L(x, v) = \frac{1}{2} g_{\mu\nu}(x) v^{\mu} v^{\nu}$$



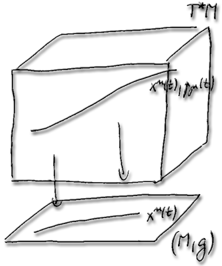
Riemannian geodesics - Hamiltonian language

But

$$L(x,v) = \frac{1}{2} g_{\mu\nu}(x) v^\mu v^\nu \quad \leftrightarrow \quad H(x,p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu$$

so the geodesic equations may also be written as first order equations on T^*M in the form of the **Hamiltonian equations**

$$\begin{aligned} \dot{x}^\mu &= \partial H / \partial p_\mu = g^{\mu\nu} p_\nu \\ \dot{p}_\mu &= -\partial H / \partial x^\mu = -(1/2) g^{\nu\rho}{}_{,\mu} p_\nu p_\rho \end{aligned}$$



Riemannian geodesics - Hamiltonian language (cont.)

A useful observation:
this particular Hamiltonian is nothing but the "lift"
of the "inverse" metric tensor (cometric): if

$$g^{-1} := g^{\mu\nu}\partial_\mu \otimes \partial_\nu$$

then

$$H(x,p) \equiv \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu = \frac{1}{2}\overset{\circ}{g^{-1}}$$

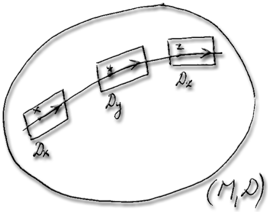
What's nice: this is a coordinate-free expression for
"geodesic" Hamiltonian $H(x,p)$ on T^*M out of g on M.

Horizontal curves

Whenever a distribution $\mathcal{D}$ is defined on a manifold M, there are specific curves $\gamma(t)$ such that

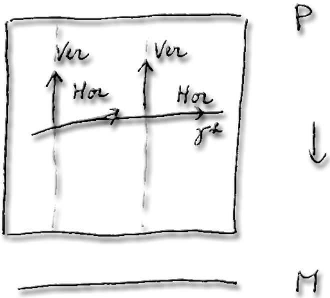
$$\dot{\gamma} \in \mathcal{D}$$

They "reside in the distribution".
In particular, for the horizontal distribution $\mathcal{H}$ on a subriemannian manifold (M, h) the curves are called horizontal curves.



Horizontally lifted curves on the total space P

Important example: let
γ be a curve on M and let
γ^h be its horizontal lift
onto the total space of a principal bundle $\pi : P \to M$ with connection.
Then, the lift is also a horizontal curve in the sense of the subriemannian structure on P.



The length of a horizontal curve

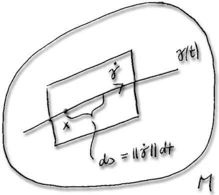
For a horizontal curve on (M, h), the length of the velocity vector

$$||\dot{\gamma}|| := \sqrt{g(\dot{\gamma}, \dot{\gamma})}$$

makes sense. Then, the "standard" definition

$$l[\gamma] := \int_{\gamma} ||\dot{\gamma}|| dt$$

makes sense, too.



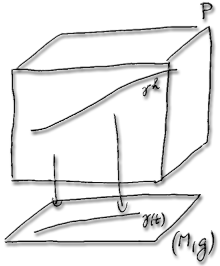
Example - the length of a horizontally lifted curve

In particular, for $\pi : P \to (M, g)$, the length of a horizontally lifted curve γ^h is a well-defined concept and it is given as

$$\boxed{l[\gamma^h] = l[\gamma]}$$

where the latter is (as usual)

$$l[\gamma] = \int_{t_A}^{t_B} \sqrt{g(\dot{\gamma}, \dot{\gamma})}dt$$



This is simply because

$$g_p(\dot{\gamma}^h, \dot{\gamma}^h) := g_{\pi(p)}(\pi_* \dot{\gamma}^h, \pi_* \dot{\gamma}^h) = g_x(\dot{\gamma}, \dot{\gamma})$$

Extremal horizontal curves

Subriemannian distance of two points A and B on M is

$$d(A, B) := \inf l[\gamma]$$

over horizontal curves that connect A and B.
If no such curve exists, we set $d(A, B) = \infty$.
(The existence depends on details of the distribution.)
Subriemannian geodesic is the path which realizes the (finite) distance.

How to find subriemannian geodesics

To find geodesics in the particular case, in Riemannian geometry, we can write down Hamiltonian equations with the "cometric" Hamiltonian

$$H(x,p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu \equiv \frac{1}{2} \overset{\circ}{g}{}^{-1}$$

Remarkable fact: this is also true in subriemannian case !

How to find subriemannian geodesics (cont.)

Namely, if we write down Hamiltonian equations

$$\dot{x}^\mu = \partial H / \partial p_\mu \qquad \dot{p}_\mu = -\partial H / \partial x^\mu$$

with the subriemannian Hamiltonian

$$H(x,p) = \frac{1}{2}\, h^{ab} P_a P_b = \frac{1}{2}\, h^{ab}(x)(e_a^\mu(x)p_\mu)(e_b^\nu(x)p_\nu)$$

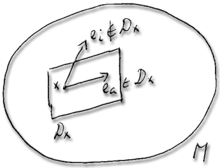
then the **projection** to M of a solution is

1. automatically horizontal (!)
2. extremal (!)

Why this method works

Let (e_a, e_i) be any adapted frame ($e_a \in \mathcal{H}$, orthonormal) on a given subriemannian manifold (M, h).
We promote (e_a, e_i) to become orthonormal with respect to an auxiliary Riemannian metric, i.e. define G such that

$$G := e^a \otimes e^a + e^i \otimes e^i$$



So

$$G \leftrightarrow \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \delta_{ij} \end{pmatrix} \qquad G^{-1} \leftrightarrow \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \delta^{ij} \end{pmatrix}$$

Why this method works (cont.)

On the Riemannian manifold (M, G), the geodesics may be expressed in the standard Hamiltonian language on T^*M, the Hamiltonian being

$$H(x,p) = \frac{1}{2}\overset{\circ}{G}^{-1} = \frac{1}{2}(P_a P_a + P_i P_i)$$

The projections onto M of the solutions are, however, not horizontal, in general.
(There is no reason for the shortest path with respect to G to be horizontal.)

Why this method works (cont.)

We can force the shortest curves to be "willingly" horizontal by economical means - by severe penalization of "bad" (nonhorizontal) motions.
Introduce new metric as

$$G_\lambda \leftrightarrow \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \lambda^2 \delta_{ij} \end{pmatrix} \qquad G_\lambda^{-1} \leftrightarrow \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \frac{1}{\lambda^2}\delta^{ij} \end{pmatrix}$$

with $\lambda^2 >> 1$ (still with respect to (e_a, e_i)).

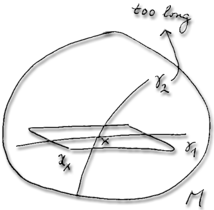
Why this method works (cont.)

If $\dot\gamma(t)$ is not horizontal somewhere, the segment of the curve around this point adds a huge contribution.

So, the curves which are everywhere horizontal, are the only candidates to be reasonably short. Then, in the limit $\lambda^2 \to \infty$, the shortest paths, the solutions of Hamiltonian equations with

$$H_\lambda = \frac{1}{2}(P_a P_a + \frac{1}{\lambda^2} P_i P_i)$$

are automatically horizontal.



Why this method works (cont.)

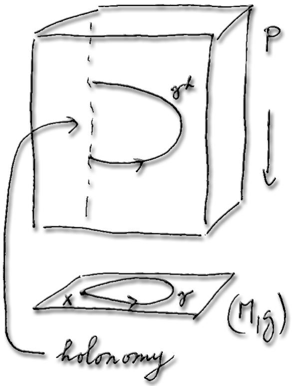
But, in the limit $\lambda^2 \to \infty$,

$$H_\lambda = \frac{1}{2}(P_a P_a + \frac{1}{\lambda^2} P_i P_i) \to \frac{1}{2} P_a P_a$$

which is nothing but the subriemannian Hamiltonian corresponding to (M, h).

What are isoholonomic problems

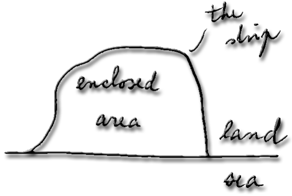
Isoholonomic means that the holonomy of a loop $\gamma(0) = \gamma(1)$ is kept constant. And something else (namely the length) is to be minimized. So one tries to get a given holonomy via the shortest loop.



What are (dual) isoperimetric problems

Isoperimetric means that a perimeter is kept constant.
And something else (namely area) is to be maximized.
The dual problem: keep the area constant and minimize the length.

Isoperimetric classics - the Dido's problem:
Find the shape of a strip
(realized by an ox hide, originally)
enclosing the maximum area,
given the length of the strip constant.
Solution: arc of a circle,
see Prof. Virgil's Aeneid.



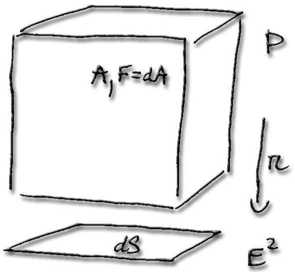
How (dual) isoperimetric becomes isoholonomic

Let E^2 be the standard Euclidean plane.
Construct a principal bundle $\pi : P \to E^2$ with the group $G = (\mathbb{R}, +)$ or $U(1)$, such that

$$F \equiv dA = \pi^* dS$$

where dS is the area 2-form in E^2.
Then, for any section σ,

$$\sigma^* F = (\pi \circ \sigma)^* dS = dS$$



How (dual) isoperimetric becomes isoholonomic (cont.)

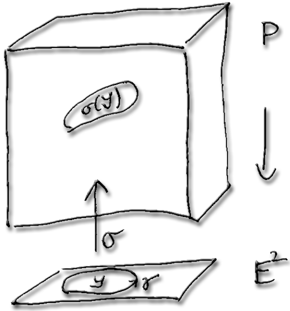
Now, consider a loop

$$\gamma(0) = \gamma(1)$$

which happens to be the boundary of $\mathcal{S}$

$$\gamma = \partial \mathcal{S}$$

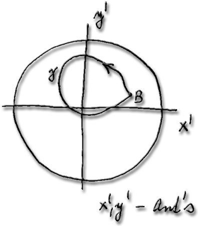
Then, we get

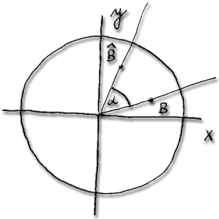


$$\text{area of } \mathcal{S} := \int_{\mathcal{S}} dS = \int_{\mathcal{S}} \sigma^* F = \oint_{\gamma} \sigma^* A = \text{holonomy of } \gamma$$

Ant on a turntable

Consider a gramophone turntable and an Ant living on it. When he performs a loop-shaped walk, at the end the turntable gets rotated.
This can be treated in terms of a $SO(2)$-bundle with connection over the turntable: the net rotation is the holonomy of the loop.
So, we come to a subriemannian structure discussed above.



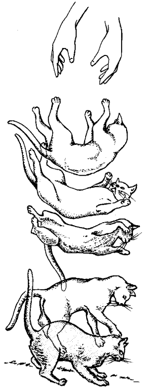


Ant on a turntable - resulting Hamiltonian equations

$$
\begin{aligned}
\dot{r} &= p_r \\
\dot{\varphi} &= \frac{p_\varphi}{r^2} - \frac{p_\alpha}{1+r^2} \\
\dot{\alpha} &= \left(\frac{p_\varphi}{r^2} - \frac{p_\alpha}{1+r^2}\right)\left(-\frac{r^2}{1+r^2}\right) \equiv \dot{\varphi}\left(-\frac{r^2}{1+r^2}\right) \\
\dot{p}_r &= \left(\frac{p_\varphi}{r^2} - \frac{p_\alpha}{1+r^2}\right)\left(\frac{2rp_\alpha}{1+r^2}\right) \equiv \dot{\varphi}\left(\frac{2rp_\alpha}{1+r^2}\right) \\
\dot{p}_\varphi &= 0 \\
\dot{p}_\alpha &= 0
\end{aligned}
$$

Falling cat

Old wisdom: a cat, dropped from upside down, will land on her feet.
Freely falling reference frame - no fall.
So she is able to reorient herself while floating weightless in space.
Physical mechanism: angular momentum equals zero.
Mathematical expression:
there is an $SO(3)$-principal bundle with connection and the resulting net rotation is the holonomy for the loop in the space of abstract shapes.



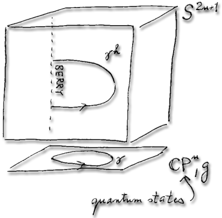
picture from Montgomery

Berry phase

In Quantum Mechanics of $(n+1)$-level systems, there is a natural $U(1)$-principal bundle

$$\pi : S^{2n+1} \to \mathbb{C}P^n \quad \text{Hopf bundle, if } n = 1$$

Here, S^{2n+1} is the space of normed vectors in the Hilbert space $\mathbb{C}^{n+1}$ whereas $\mathbb{C}P^n$ is the space of the states. There is a distinguished metric tensor on $\mathbb{C}P^n$ (Fubiny-Study). In addition, there is also a canonical connection in the bundle. So, we come to a subriemannian structure discussed above.



For Further Reading

R. Strichartz.
Sub-Riemannian Geometry.
J.Diff.Geom. 24, 221-263, 1983.

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R. Montgomery.
A Tour of Subriemannian Geometries, Their Geodesics and Applications.
American Mathematical Society, 2006.

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American Mathematical Society, 2006.

Numerous other authors.
Plenty of other papers and several monographs.
Lot of other journals.

For Further Reading

- R. Strichartz.
  Sub-Riemannian Geometry.
  J.Diff.Geom. 24, 221-263, 1983.
- R. Montgomery.
  Isoholonomic Problems and some Applications.
  Commun.Math.Phys., 128, 565-592, 1990.
- R. Montgomery.
  A Tour of Subriemannian Geometries, Their Geodesics and Applications.
  American Mathematical Society, 2006.
- Numerous other authors.
  Plenty of other papers and several monographs.
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