Poincaré (non-holonomic Lagrange) Equations

Marián Fecko

Department of Theoretical Physics
Comenius University
Bratislava
fecko@fmph.uniba.sk

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- What is interesting about their structure
- What happens, in particular, on Lie groups
- How they are related to Euler (rigid body motion) equations
- How Četajev repeated the story with Hamilton equations

Contents

Coordinate components of the velocity vector

In standard Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \tag{1}$$

we encounter variables x^i (generalized coordinates) and $\dot{x}^i$ (generalized velocities). From the geometrical point of view the coordinates x^i represent the curve $\gamma(t)$ whereas $v^i = \dot{x}^i$ are the coordinate components of its velocity vector:

$$\begin{aligned} x^i(t) &\quad\leftrightarrow\quad \gamma(t) \\ v^i(t) \equiv \dot{x}^i(t) &\quad\leftrightarrow\quad \dot{\gamma}(t) \end{aligned}$$

Frame components of the velocity vector

Consider a frame field e_a

$$e_a(x) = e_a^i(x)\partial_i \qquad \partial_i = e_i^a(x)e_a(x) \tag{2}$$

(any - orhonormal, left invariant, even the original coordinate one). Then the velocity vector may be expressed in two ways

$$\dot{\gamma}(t) = v^i(t)\partial_i = v^a(t)e_a(x(t)) \tag{3}$$

The variables v^a are frame ("vielbein") components of the velocity vector ("pseudovelocities"). So, we can also write

$$\begin{array}{ccc} x^i(t) & \leftrightarrow & \gamma(t) \\ v^a(t) & \leftrightarrow & \dot{\gamma}(t) \end{array}$$

Relation between the two kinds of components

From (39) we see that

$$v^i \partial_i = v^a e_a \qquad \Rightarrow \qquad v^a = e^a_i v^i \qquad v^i = e^i_a v^a \tag{4}$$

This means that Lagrangian may also be expressed in terms of the mixed variables (x^i, v^a)

$$L = L(x^i, v^a) \qquad \text{rather than} \qquad L = L(x^i, v^i) \tag{5}$$

Variation of Lagrangian

In order to derive equations of motion via the least action principle in terms of (x^i, v^a), we need the variation

$$\delta L(x^i, v^a) = \frac{\partial L}{\partial x^i}\,\delta x^i + \frac{\partial L}{\partial v^a}\,\delta v^a \tag{6}$$

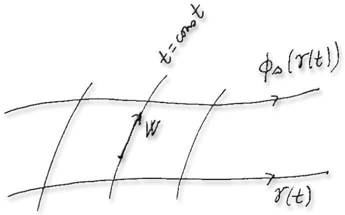
As is the case for $\delta \dot{x}^i$, the variation δv^a is related to δx^i.
The explicit formula is, however, more subtle.
Therefore we prefer to gain some geometrical insight into the variational procedure.
(In his paper, Poincaré simply states the result
with laconic comment
"*Or on trouve aisément*" (one easily finds)).

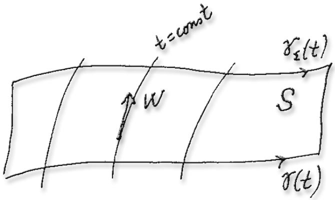
Variational field W

We want to perform a small variation of the curve $\gamma(t)$.
Consider a vector field W an let Φ_ϵ denote its flow. Then the variation may be realized as

$$\gamma(t) \mapsto \Phi_\epsilon(\gamma(t)) \equiv \gamma_\epsilon(t) \qquad \Phi_s \leftrightarrow W \tag{7}$$

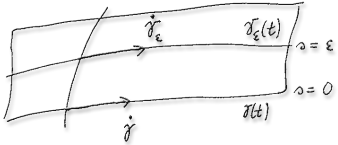
In this way the flow produces a two-dimensional (narrow) surface S.
The field W is tangent to the surface.

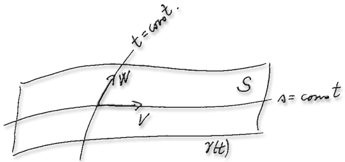




Velocity field V

There is another vector field living on the surface S.
Each curve $\gamma_\epsilon(t)$ induces in all its points velocity vector $\dot{\gamma}_\epsilon(t)$.
Altogether, collection of all velocity vectors of all curves results in a vector field on S - the velocity field V.
In each point of the surface S, the two fields,
V and W,
span the tangent space to the surface.





Commutator of V and W vanishes

The field V is Lie-dragged with respect to W.
Then the Lie derivative vanishes, $\mathcal{L}_W V = 0$, or, put another way, the two fields commute

$$[W, V] = 0 \tag{8}$$

In terms of frame components of the fields this reads

$$0 = [W, V]^a = WV^a - VW^a + c^a_{bc} W^b V^c \tag{9}$$

where $c^c_{ab}(x)$ are anholonomy coefficients of the frame field e_a

$$[e_a, e_b] =: c^c_{ab}(x) e_c \tag{10}$$

Variation δx^i in terms of W

In coordinates, the variation of $x^i(t)$ reads

$$x^i(t) \mapsto x^i(t) + \delta x^i(t)$$

But we know (see (7)) that the new curve arises by means of the flow of the original curve along W. So

$$\begin{aligned} \text{abstractly} \qquad \gamma(t) &\mapsto \Phi^W_\epsilon(\gamma(t)) && (11) \\ \text{in coordinates} \qquad x^i(t) &\mapsto x^i(t) + \epsilon W^i(t) && (12) \end{aligned}$$

and, consequently

$$\delta x^i = \epsilon W^i \qquad (13)$$

Variation δv^a in terms of W (1)

We have

$$v^a e_a = \dot{\gamma} \tag{14}$$

$$(v^a + \delta v^a) e_a = \dot{\gamma}_\epsilon = V(\gamma_\epsilon(t)) \equiv V(\Phi_\epsilon(\gamma(t))) \tag{15}$$

(the two e_a's on the left sit in different points) hence

$$\delta v^a = \epsilon W V^a \tag{16}$$

▼

$$\begin{aligned} V(\Phi_\epsilon(\gamma(t))) &= V^a(\Phi_\epsilon(\gamma(t)))e_a \\ &= (\Phi_\epsilon^* V^a)(\gamma(t))e_a \\ &= (\hat{1} + \epsilon \mathcal{L}_W) V^a)(\gamma(t))e_a \\ &= V^a(\gamma(t)) + \epsilon (W V^a)(\gamma(t))e_a \\ &\equiv v^a e_a + \epsilon (W V^a) e_a \end{aligned}$$

▲

Variation δv^a in terms of W (2)

Now the vanishing of the commutator $[V, W]$ (8, 9) comes in handy and we get

$$\delta v^a = \epsilon(\dot{W}^a + c^a_{bc} v^b W^c) \tag{17}$$

▼

$$\begin{aligned} WV^a &= VW^a - c^a_{bc} W^b V^c \\ &= VW^a + c^a_{bc} V^b W^c \\ &= \dot{W}^a + c^a_{bc} v^b W^c \end{aligned}$$

Everything is evaluated at the original curve $\gamma(t)$, so that $Vf = \dot{\gamma} f = \dot{f}$ and also V^a reduce to v^a. ▲

Using of the results for δx^i, δv^a for performing δL

Using results (13) and (17), we get

$$\begin{aligned}
\delta L(x^i, v^a) &= \frac{\partial L}{\partial x^i}\ \delta x^i + \frac{\partial L}{\partial v^a}\ \delta v^a \\
&= \epsilon \left(\frac{\partial L}{\partial x^i}\ W^i + \frac{\partial L}{\partial v^a}\ (\dot{W}^a + c^a_{bc} v^b W^c) \right) \\
&= \epsilon \left(W^a e_a L + \frac{\partial L}{\partial v^c}\ (\dot{W}^c + c^c_{ba} v^b W^a) \right) \\
&= \epsilon W^a \left(e_a L + c^c_{ba} v^b \frac{\partial L}{\partial v^c} \right) + \epsilon \dot{W}^a \frac{\partial L}{\partial v^a} \\
&= \epsilon W^a \left(e_a L + c^c_{ba} v^b \frac{\partial L}{\partial v^c} - \frac{d}{dt}\ \frac{\partial L}{\partial v^a} \right) + \frac{d}{dt} \left(\epsilon W^a \frac{\partial L}{\partial v^a} \right)
\end{aligned}$$

Variation δS of the action functional $S[\gamma]$

Then

$$\delta S = \epsilon \int_{t_1}^{t_2} dt \left(e_a L + c^c_{ba} v^b \frac{\partial L}{\partial v^c} - \frac{d}{dt} \frac{\partial L}{\partial v^a} \right) W^a + \epsilon \left[\frac{\partial L}{\partial v^a} W^a \right]_{t_1}^{t_2} \quad (18)$$

Using standard arguments (δx^i vanishing at t_1 and t_2 and "arbitrary" inside the interval) we at last get

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial v^a} - e_a L + c^c_{ab} v^b \frac{\partial L}{\partial v^c} = 0 \qquad \text{Poincaré equations}} \quad (19)$$

Just n first-order equations

For $L(x^i, v^a)$, Poincaré equations

$$\frac{d}{dt}\frac{\partial L}{\partial v^a} - e_a L + c^c_{ab} v^b \frac{\partial L}{\partial v^c} = 0 \tag{20}$$

represent just n first-order equations. We know that Lagrange equations represent n second-order equations.

Where is the origin of the discrepancy?

Don't panic. The remaining information sits in (4)

$$\dot{x}^i = e^i_a(x) v^a \tag{21}$$

Complete set of equations to be solved

In order to use the equations for actual computation one has to write down the complete set of equations (Poincaré, 1901)

$$\frac{d}{dt}\frac{\partial L}{\partial v^a} - e_a L + c^c_{ab} v^b \frac{\partial L}{\partial v^c} = 0 \tag{22}$$

$$v^a = e^a_i(x)\dot{x}^i \tag{23}$$

Note that in general they form a coupled system of $2n$ first-order differential equations.
In this respect they resemble standard Hamilton, rather then standard Lagrange, equations.

Particular case - holonomic (coordinate) frame $e_a = \partial_a$

In the particular case of holonomic (coordinate) frame $e_a = \partial_a$

1. $e^a_{bc} \mapsto 0$
2. $e_a L \equiv e^i_a(x)\partial_i L \mapsto \partial_i L \equiv \frac{\partial L}{\partial x^i}$
3. $\frac{\partial L}{\partial v^a} \mapsto \frac{\partial L}{\partial v^i}$

The complete system reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = 0 \qquad (24)$$

$$v^i = \dot{x}^i \qquad (25)$$

equivalent to standard Lagrange equations

Elementary example - free motion in Euclidean plane

In polar orthonormal frame we have

$$e_r = \partial_r \qquad e_\varphi = (1/r)\partial_\varphi \qquad \dot\gamma = \dot r\partial_r + \dot\varphi\partial_\varphi = v^r e_r + v^\varphi e_\varphi \tag{26}$$

$$[e_r, e_\varphi] = (-1/r)e_\varphi \qquad \Rightarrow \qquad c^\varphi_{r\varphi} = -c^\varphi_{\varphi r} = -1/r \tag{27}$$

$$L(r, \varphi, v^r, v^\varphi) = \frac{1}{2}\,((v^r)^2 + (v^\varphi)^2) \equiv (1/2)(\dot r^2 + r^2\dot\varphi^2) \tag{28}$$

$$\dot v^r - \frac{1}{r}\, v^\varphi v^\varphi = 0 \qquad \dot v^\varphi + \frac{1}{r}\, v^r v^\varphi = 0 \tag{29}$$

$$v^r = \dot r \qquad v^\varphi = r\dot\varphi \tag{30}$$

(True, but no profit here :-)

Specific features on Lie groups

There are important examples when

1. the configuration space has the structure of a Lie group

Then it is natural to choose left invariant frame field e_a.

Position-dependent anholonomy coefficients $c^c_{ab}(x)$
reduce to position-independent structure constants c^c_{ab}
of the corresponding Lie algebra.

Left invariant kinetic energy

2. Kinetic energy is left invariant

Then $e_a L = e_a(T - U) = -e_a U$.

Moreover, left invariant metric tensor reads

$$g = I_{ab} e^a \otimes e^b \qquad I_{ab} = \text{ const.} \tag{31}$$

Therefore

$$T = \frac{1}{2} g(\dot{\gamma}, \dot{\gamma}) = \frac{1}{2} I_{ab} v^a v^b \tag{32}$$

Resulting complete set of equations

Under these conditions, the complete set (22) and (23) reduces to

$$I_{ab}\dot{v}^b + c_{dab}v^b v^d = -e_a U \quad (33)$$

$$v^a = e^a_i(x)\dot{x}^i \quad (34)$$

where

$$c_{dab} := I_{dc}c^c_{ab} \quad (35)$$

Note the structure similarity of this set of equations to Euler (rigid body motion) equations.

Euler dynamical and kinematical equations

And indeed, for $G = SO(3) =$ configuration space of rigid body and $I_{ab} = \text{diag}\,(I_1, I_2, I_3) =$ moments of inertia of the body, the set of equations

$$I_{ab}\dot{v}^b + c_{dab}v^b v^d = -e_a U \quad (36)$$

$$v^a = e^a_i(x)\dot{x}^i \quad (37)$$

is nothing but well-known Euler dynamical (36) and kinematical (37) equations.
Here, frame components v^a play the role of "body components" of angular momentum of the rotating body.

Coframe components of the momentum vector

In (39) we introduced frame components of the velocity vector.
In the same way one can define coframe components of the momentum covector.
Consider a coframe field e^a (dual to e_a)

$$e^a(x) = e^a_i(x)dx^i \qquad dx^i = e^i_a(x)e^a(x) \tag{38}$$

Then the momentum covector may be expressed in two ways

$$p(t) = p_i(t)dx^i = p_a(t)e^a(x(t)) \tag{39}$$

The variables p_a are coframe components of the momentum covector.

Frame coordinates on TM and T^*M

Consider a frame field e_a and the dual coframe field e^a.
Then any vector on M may be expressed as $v = v^i \partial_i = v^a e_a$
and any covector on M may be expressed as $p = p_i dx^i = p_a e^a$.
This means that one can use as local coordinates either

$$(x^i, v^i) \text{ on } TM \qquad (x^i, p_i) \text{ on } T^*M \tag{40}$$

or

$$(x^i, v^a) \text{ on } TM \qquad (x^i, p_a) \text{ on } T^*M \tag{41}$$

Clearly

$$v^a = e^a_i v^i \quad v^i = e^i_a v^a \qquad p_a = e^i_a p_i \quad p_i = e^a_i p_a \tag{42}$$

Expressing $\int p_i dx^i - Hdt$ in frame components

In order to derive Hamilton-like equations from the least action principle we need to express the term $\int p_i dx^i$ in terms of frame components. We have

$$p_i \dot{x}^i = p_i e^i_a v^a = p_a v^a \tag{43}$$

Then the action functional reads

$$S[\tilde{\gamma}] = \int_{t_1}^{t_2} (p_a v^a - H(x^i, p_a))dt \tag{44}$$

Here $\tilde{\gamma}(t) \leftrightarrow (x^i(t), p_a(t))$ is a curve in phase space.

Variation of $\int p_i dx^i - Hdt$ in frame components (1)

We have

$$\delta S = \int_{t_1}^{t_2} (\delta p_a v^a + p_a \delta v^a - \delta H(x^i, p_a))dt \tag{45}$$

where

$$\delta H = \frac{\partial H}{\partial x^i}\,\delta x^i + \frac{\partial H}{\partial p_a}\,\delta p_a \tag{46}$$

$$\delta x^i = \epsilon W^i \tag{47}$$

$$\delta v^a = \epsilon(\dot{W}^a + c^a_{bc} v^b W^c) \tag{48}$$

Since the curve $(x^i(t), p_a(t))$ is regarded as a curve in phase space, variations δx^i and δp_a are to be treated as independent.

Variation of $\int p_i dx^i - Hdt$ in frame components (2)

After usual *per partes* we get

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta p_a \left(v^a - \frac{\partial H}{\partial p_a} \right) dt && (49) \\ &+ \int_{t_1}^{t_2} \epsilon W^a \left(-\dot{p}_a - e_a H - c^c_{ab} v^b p_c \right) dt && (50) \\ &+ \epsilon \left(p_a W^a \right)_{t_1}^{t_2} && (51) \\ & && (52) \end{aligned}$$

Remember standard variational Hamiltonian folklore:
δx^i vanishing and no restrictions on δp_a at t_1 and t_2; both "arbitrary" inside the interval.

Variation of $\int p_i dx^i - Hdt$ in frame components (3)

Then one obtains

$$v^a = \frac{\partial H}{\partial p_a} \tag{53}$$

$$\dot{p}_a = -e_a H - c^c_{ab} v^b p_c \tag{54}$$

or, using the kinematical equations $\dot{x}^i = e^i_a(x) v^a$,

$$\dot{x}^i = e^i_a(x) \frac{\partial H}{\partial p_a} \tag{55}$$

$$\dot{p}_a = -e_a H - c^c_{ab} \frac{\partial H}{\partial p_b} p_c \tag{56}$$

(Četajev, 1927).

How Poisson bracket looks in frame components

If $f(x^i, p_a)$ is any function on phase space, then

$$\begin{aligned} \dot{f} &= \frac{\partial f}{\partial x^i}\,\dot{x}^i + \frac{\partial f}{\partial p_a}\,\dot{p}_a \\ &= \frac{\partial H}{\partial p_a}\,(e_a f) - \frac{\partial f}{\partial p_a}\,(e_a H) + p_c c^c_{ab}(x)\frac{\partial H}{\partial p_a}\frac{\partial f}{\partial p_b} \\ &\equiv \{H, f\} \end{aligned}$$

So, Poisson bracket explicitly reads

$$\{F, G\} = \frac{\partial F}{\partial p_a}\,(e_a G) - \frac{\partial G}{\partial p_a}\,(e_a F) + p_c c^c_{ab}(x)\frac{\partial F}{\partial p_a}\frac{\partial G}{\partial p_b} \qquad (57)$$

Example - Hamilton equations for Dido problem (1)

Finding subriemannian geodesics needs solving Hamilton equations, often fairly complicated (see my Stará Lesná 2009 lectures).
It might be instructive to confront both pictures - standard holonomic versus non-holonomic.
Easy example - Dido problem. The frame is

$$\begin{aligned} e_1 &= \partial_x - (y/2)\partial_z \\ e_2 &= \partial_y + (x/2)\partial_z \\ e_3 &= \partial_z \end{aligned}$$

so the frame components of momentum read

$$p_1 = p_x - (y/2)p_z \qquad p_2 = p_y + (x/2)p_z \qquad p_3 = p_z$$

Example - Hamilton equations for Dido problem (2)

Resulting equations:

Hamilton equations

$$
\begin{aligned}
\dot{x} &= p_x - y(p_z/2) \\
\dot{y} &= p_y + x(p_z/2) \\
\dot{z} &= (x\dot{y} - y\dot{x})/2 \\
\dot{p}_x &= -(p_y + x p_z/2)(p_z/2) \\
\dot{p}_y &= -(p_x - y p_z/2)(p_z/2) \\
\dot{p}_z &= 0
\end{aligned}
$$

Četajev equations

$$
\begin{aligned}
\dot{x} &= p_1 \\
\dot{y} &= p_2 \\
\dot{z} &= (x\dot{y} - y\dot{x})/2 \\
\dot{p}_1 &= -p_2 p_3 \\
\dot{p}_2 &= +p_1 p_3 \\
\dot{p}_3 &= 0
\end{aligned}
$$

Which system would you prefer to solve?

For Further Reading

H. Poincaré.
Sur une forme nouvelle des équations de la méchnique.
C.R.Acad.Sci.Paris, v.132, p.369-371, 1901

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Sur une forme nouvelle des équations de la méchnique.
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M. Četajev.
Sur les équations de Poincaré.
C.R.Acad.Sci.Paris, v.185, p.1577-1578, 1927

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  Springer, Berlin Heidelberg, 2009

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