Ostrogradsky theorem (from 1850)

Marián Fecko

Department of Theoretical Physics
Comenius University
Bratislava
fecko@fmph.uniba.sk

Student Colloqium and School on Mathematical Physics,
Stará Lesná, Slovakia, August 16 - 22, 2015

We will learn:

- What the phase space of higher order Lagrange equations looks like

We will learn:

- What the phase space of higher order Lagrange equations looks like
- What is Ostrogradsky's proposition for canonical coordinates

We will learn:

- What the phase space of higher order Lagrange equations looks like
- What is Ostrogradsky's proposition for canonical coordinates
- What is Ostrogradsky's proposition for the Hamiltonian

We will learn:

- What the phase space of higher order Lagrange equations looks like
- What is Ostrogradsky's proposition for canonical coordinates
- What is Ostrogradsky's proposition for the Hamiltonian
- What is the most important feature of the Hamiltonian system

We will learn:

- What the phase space of higher order Lagrange equations looks like
- What is Ostrogradsky's proposition for canonical coordinates
- What is Ostrogradsky's proposition for the Hamiltonian
- What is the most important feature of the Hamiltonian system
- How one could "come to" Ostrogradsky's variables

We will learn:

- What the phase space of higher order Lagrange equations looks like
- What is Ostrogradsky's proposition for canonical coordinates
- What is Ostrogradsky's proposition for the Hamiltonian
- What is the most important feature of the Hamiltonian system
- How one could "come to" Ostrogradsky's variables
- Why it might be of interest today (just a touch)

Contents

1-st order Lagrangian $L(x, \dot{x})$

We have standard Lagrange equations

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^a} = 0 \qquad a = 1, \dots, n \tag{1}$$

It is a system of 2-nd order equations

$$Q_a(x, \dot{x}, \ddot{x}) = 0 \qquad a = 1, \dots, n \tag{2}$$

We would like to isolate the highest (= 2-nd) derivative, i.e. write it in the form

$$\ddot{x}^a = F^a(x, \dot{x}) \qquad a = 1, \dots, n \tag{3}$$

1-st order Lagrangian $L(x, \dot{x})$ (2)

Well, in more detail, the system has the structure

$$A_{ab}(x, \dot{x})\ddot{x}^b = B_a(x, \dot{x}) \qquad a = 1, \dots, n \tag{4}$$

where

$$A_{ab}(x, \dot{x}) := \frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}$$

So, the "Newtonian" form (3) is possible iff the matrix A_{ab} is invertible. We speak then of non-degenerate Lagrangian:

$$\boxed{\det \frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b} \neq 0} \qquad \text{non-degenerate Lagrangian (1-st order)} \tag{5}$$

2-nd order Lagrangian $L(x, \dot{x}, \ddot{x})$

We have (a bit less standard) Lagrange equations

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^a} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^a} = 0 \qquad a = 1, \dots, n \tag{6}$$

It is a system of 4-th order equations

$$Q_a(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = 0 \qquad a = 1, \dots, n \tag{7}$$

We would like to isolate the highest (= 4-th) derivative, i.e. write it in the form

$$\ddddot{x}^a = F^a(x, \dot{x}, \ddot{x}, \dddot{x}) \qquad a = 1, \dots, n \tag{8}$$

2-nd order Lagrangian $L(x,\dot{x},\ddot{x})$ (2)

Well, in more detail, the system has the structure

$$A_{ab}(x,\dot{x},\ddot{x})\,\ddddot{x}^{b} = B_a(x,\dot{x},\ddot{x},\dddot{x}) \qquad a = 1,\dots,n \qquad (9)$$

where

$$A_{ab}(x,\dot{x},\ddot{x}) := \frac{\partial^2 L}{\partial \ddot{x}^a \partial \ddot{x}^b}$$

So, the "Newtonian" form (8) is possible iff the matrix A_{ab} is invertible. We speak then of non-degenerate Lagrangian:

$$\boxed{\det \frac{\partial^2 L}{\partial \ddot{x}^a \partial \ddot{x}^b} \neq 0} \qquad \text{non-degenerate Lagrangian (2-nd order)} \qquad (10)$$

3-rd order Lagrangian $L(x, \dot{x}, \ddot{x}, \dddot{x})$

We have (still less standard) Lagrange equations

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^a} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^a} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{x}^a} = 0 \qquad a = 1, \dots, n \quad (11)$$

It is a system of 6-th order equations

$$Q_a(x, \dot{x}, \ddot{x} \equiv x^{a(2)}, \dots, x^{a(6)}) = 0 \qquad a = 1, \dots, n \qquad (12)$$

We would like to isolate the highest (= 6-th) derivative, i.e. write it in the form

$$x^{a(6)} = F^a(x, \dots, x^{(5)}) \qquad a = 1, \dots, n \qquad (13)$$

3-rd order Lagrangian $L(x, \dot{x}, \ddot{x}, \dddot{x})$ (2)

Well, in more detail, the system has the structure

$$A_{ab}(x, \dot{x}, \ddot{x}, \dddot{x})x^{b(6)} = B_a(x, \dot{x}, \dots, x^{a(5)}) \qquad a = 1, \dots, n \quad (14)$$

where

$$A_{ab}(x, \dot{x}, \dots, x^{a(3)}) := \frac{\partial^2 L}{\partial \dddot{x}^a \partial \dddot{x}^b}$$

So, the "Newtonian" form (13) is possible iff the matrix A_{ab} is invertible. We speak then of non-degenerate Lagrangian:

$$\boxed{\det \frac{\partial^2 L}{\partial \dddot{x}^a \partial \dddot{x}^b} \neq 0} \qquad \text{non-degenerate Lagrangian (3-rd order)} \quad (15)$$

k-th order Lagrangian $L(x, \dots, x^{(k)})$

Hopefully all boring details not needed. Clearly in the case of

$$\boxed{\det \frac{\partial^2 L}{\partial x^{a(k)} \partial x^{b(k)}} \neq 0} \qquad \text{non-degenerate Lagrangian } (k\text{-th order}) \tag{16}$$

Lagrange equations may be written in the "Newtonian" form

$$x^{a(2k)} = F^a(x, \dots, x^{(2k-1)}) \qquad a = 1, \dots, n \tag{17}$$

Phase space for Lagrange equations

Let's adopt the definition: Points of the phase space are given as initial conditions corresponding to Lagrange equations.
In natural coordinates

$$L(x,\dot{x}) \quad : \quad \text{phase space} \;\leftrightarrow\; (x_0,\dot{x}_0) \tag{18}$$

$$L(x,\dot{x},\ddot{x}) \quad : \quad \text{phase space} \;\leftrightarrow\; (x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0) \tag{19}$$

$$L(x,\dot{x},\ddot{x},\dddot{x}) \quad : \quad \text{phase space} \leftrightarrow (x_0,x_0^{(1)},\dots,x_0^{(5)}) \tag{20}$$

$$L(x,\dot{x},\dots,x^{(k)}) \quad : \quad \text{phase space} \leftrightarrow (x_0,\dot{x}_0,\dots,x_0^{(2k-1)}) \tag{21}$$

Phase space for Lagrange equations (2)

Then trivially holds: If we know our position in the phase space now, at time t, we are, in principle, able to predict our position in the phase space in the (near) future, at time $t + \epsilon$.
And iterate the procedure.

(We simply solve Lagrange equations using initial conditions.)

This means, however, that the dynamics may be written by sure as a system of first order equations.

Phase space for Lagrange equations (3)

Now recall that the phase space is always even-dimensional.
So there is a chance that what we really get as the first order system will be system of Hamiltonian equations

$$\dot{x}_i = \partial H/\partial p_i \qquad \dot{p}_i = -\partial H/\partial x_i \tag{22}$$

We all know this is the case for 1-st order Lagrangians: Just set

$$x^a(x,\dot{x}) := x^a \tag{23}$$

$$p_a(x,\dot{x}) := \partial L/\partial \dot{x}^a \tag{24}$$

$$H := p_a \dot{x}^a - L \tag{25}$$

Phase space for Lagrange equations (4)

However, it is far from being clear for higher order case since a generic first order system is far from always Hamiltonian.

What Ostrogradsky explicitly demonstrated as long ago as in 1850 is that the dynamics given by Lagrange equations is always Hamiltonian. (Plus a strange feature of this particular dynamics.)

So, he found higher order analogues of (23), (24) and (25) such that equations (22) hold.

Ostrogradsky variables for $L(x, \dot{x}, \ddot{x})$

Ostrogradsky proposes new coordinates on the phase space:

$$(x, \dot{x}, \ddot{x}, \dddot{x}) \mapsto (x^1, x^2, p_1, p_2) \tag{26}$$

where

$$x^1 := x \qquad p_1 := \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} \tag{27}$$

$$x^2 := \dot{x} \qquad p_2 := \frac{\partial L}{\partial \ddot{x}} \tag{28}$$

Hamiltonian reads

$$H := p_1 \dot{x} + p_2 \ddot{x} - L \tag{29}$$

Then, one can check that (6) is equivalent to (22)

Is it simply the usual Legendre transformation?

In Woodard [3] one can read:

"Ostrogradsky's Hamiltonian is obtained by
Legendre transforming on $\dot{x} = x^{(1)}$ and $\ddot{x} = x^{(2)}$ ..."

Well, general formulas for the Legendre transformation read

$$(x_i, y_a) \overset{L}{\mapsto} (x_i, z_a) \qquad z_a(x, y) = \partial L(x, y)/\partial y_a \tag{30}$$

$$(x_i, z_a) \overset{H}{\mapsto} (x_i, y_a) \qquad y_a(x, z) = \partial H(x, z)/\partial z_a \tag{31}$$

where

$$H(x_i, z_a) = z_a y_a - L \equiv (\text{new})_a(\text{old})_a - L \tag{32}$$

Is it simply the usual Legendre transformation? (2)

We see from (26) that the old-new pair is

$$(\ddot{x}, \dddot{x}) \mapsto (p_1, p_2) \tag{33}$$

whereas $(x, \dot{x})$ are nothing but "spectators" (x_i in (30)).
However, from the Hamiltonian (29), the old-new pair should be

$$(\dot{x}, \ddot{x}) \mapsto (p_1, p_2) \tag{34}$$

In addition, the expression $p_1 = \dots$ in (27) is not of the needed form (30) (the second term is excessive).
So, even though the structure of the Ostrogradsky Hamiltonian (29) resembles Legendre transformation, actually it is not the case.

Ostrogradsky variables for $L(x, \dot{x}, \ddot{x}, \dddot{x})$

Ostrogradsky proposes new coordinates on the phase space:

$$(x, \dot{x}, \dots, x^{(5)}) \mapsto (x^1, x^2, x^3, p_1, p_2, p_3) \tag{35}$$

where

$$x^1 := x \qquad p_1 := \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dddot{x}}\right)\right) \tag{36}$$

$$x^2 := \dot{x} \qquad p_2 := \frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dddot{x}}\right) \tag{37}$$

$$x^3 := \ddot{x} \qquad p_2 := \frac{\partial L}{\partial \dddot{x}} \tag{38}$$

Hamiltonian reads

$$H := p_1\dot{x} + p_2\ddot{x} + p_3\dddot{x} - L \tag{39}$$

Then, one can check that (11) is equivalent to (22)

Main feature of the dynamics: instability

$L(x, \dot{x})$:

$$H(x_1, p_1) = p_1 \dot{x} - L \quad (40)$$

$$\equiv p_1 \dot{x}(x_1, p_1) - L(x_1, \dot{x}(x_1, p_1)) \quad (41)$$

$L(x, \dot{x}, \ddot{x})$:

$$H(x_1, x_2, p_1, p_2) = p_1 \dot{x} + p_2 \ddot{x} - L \quad (42)$$

$$\equiv p_1 x_2 + h(x_1, x_2, p_2) \quad (43)$$

$L(x, \dot{x}, \ddot{x}, \dddot{x})$:

$$H(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 \dot{x} + p_2 \ddot{x} + p_3 \dddot{x} - L \quad (44)$$

$$\equiv p_1 x_2 + p_2 x_3 + h(x_1, x_2, x_3, p_3) \quad (45)$$

Instability: H is linear in almost all momenta

We see that, starting from the case $L(x, \dot{x}, \ddot{x})$
(the first "higher-order" case),

Hamiltonian depends on all momenta but one linearly.
(This causes various kinds of problems.)

Notice, that
- this inevitably holds for all higher-order Lagrangians
- usual, 1-st order, Lagrangian is (in this sense) exceptional

First hint for $L(x, \dot{x}, \ddot{x})$

Lagrange equation - old look:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0 \tag{46}$$

Lagrange equation - a new look:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right) = 0 \tag{47}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right) = 0 \tag{48}$$

First hint for $L(x, \dot{x}, \ddot{x})$ (2)

This might serve as an inspiration to introduce (p_1, p_2) in Ostrogradsky fashion. Then Lagrange equation becomes

$$\dot{p}_1 = \frac{\partial L}{\partial x} \qquad \dot{p}_2 = \frac{\partial L}{\partial \dot{x}} - p_1 \tag{49}$$

and after some effort one can find (x^1, x^2) as well as $H(x^1, x^2, p_1, p_2)$.

First hint for $L(x, \dot{x}, \ddot{x}, \dddot{x})$

Lagrange equation - old look:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{x}} = 0 \tag{50}$$

Lagrange equation - a new look:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dddot{x}}\right)\right)\right) = 0 \tag{51}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dddot{x}}\right)\right)\right) = 0 \tag{52}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dddot{x}}\right)\right)\right) = 0 \tag{53}$$

First hint for $L(x, \dot{x}, \ddot{x}, \dddot{x})$ (2)

This might serve as an inspiration to introduce (p_1, p_2, p_3) in Ostrogradsky fashion. Then Lagrange equation becomes

$$\dot{p}_1 = \frac{\partial L}{\partial x} \qquad \dot{p}_2 = \frac{\partial L}{\partial \dot{x}} - p_1 \qquad \dot{p}_3 = \frac{\partial L}{\partial \ddot{x}} - p_2 \tag{54}$$

and after some effort one can find (x^1, x^2, x^3) as well as $H(x^1, x^2, x^3, p_1, p_2, p_3)$.

Second hint: recall good old $L(x, \dot{x})$

Start with the action integral

$$S[x] := \int_{t_1}^{t_2} L(x, \dot{x}) dt \tag{55}$$

Then the variation of S turns out to be

$$\delta S = \cdots = \int_{t_1}^{t_2} dt \ \mathcal{E}^L \delta x + [p \delta x]_{t_1}^{t_2} \tag{56}$$

Here the Euler-Lagrange expression is defined as

$$\mathcal{E}^L := \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \tag{57}$$

and the canonical momentum is

$$p := \frac{\partial L}{\partial \dot{x}} \tag{58}$$

Second hint: recall good old $L(x, \dot{x})$ (2)

This is standardly treated as a means to infer

$$\mathcal{E}^L = 0 \qquad \text{Euler-Lagrange equation} \tag{59}$$

from the assumptions

$$\delta S = 0 \qquad \text{and} \qquad \delta x|_{t_1} = \delta x|_{t_2} = 0 \tag{60}$$

Second hint: recall good old $L(x, \dot{x})$ (3)

One can, however, also use the result (56)

to identify the canonical pair (x, p):

It quietly sits at the end of the expression (56):

$$\delta S = \int_{t_1}^{t_2} dt\ \mathcal{E}^L \delta x + [p \delta x]_{t_1}^{t_2} \tag{61}$$

Apply the trick to $L(x, \dot{x}, \ddot{x})$

Now, we have

$$S[x] := \int_{t_1}^{t_2} L(x, \dot{x}, \ddot{x}) dt \qquad (62)$$

Then (check!)

$$\delta S = \int_{t_1}^{t_2} dt \; \mathcal{E}^L \delta x + [p_1 \delta x]_{t_1}^{t_2} + [p_2 \delta \dot{x}]_{t_1}^{t_2} \qquad (63)$$

where the canonical momenta (p_1, p_2) are nothing but the Ostrogradsky variables

$$p_1 := \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} \qquad p_2 := \frac{\partial L}{\partial \ddot{x}} \qquad (64)$$

Apply the trick to $L(x, \dot{x}, \ddot{x})$ (2)

This might be treated as a means to infer
Euler-Lagrange equation (6)

$$\mathcal{E}^L = 0 \tag{65}$$

from the assumptions

$$\delta S = 0 \qquad \text{and} \qquad \delta x|_{t_1} = \delta x|_{t_2} = \delta \dot{x}|_{t_1} = \delta \dot{x}|_{t_2} = 0 \tag{66}$$

Apply the trick to $L(x, \dot{x}, \ddot{x})$ (3)

One can, however, also use the result (63)

to identify the canonical pairs (x^1, p_1) and (x^2, p_2):

They quietly sit at the end of the expression (63):

$$\delta S = \int_{t_1}^{t_2} dt\ \mathcal{E}^L \delta x + [p_1 \delta x]_{t_1}^{t_2} + [p_2 \delta \dot{x}]_{t_1}^{t_2} \tag{67}$$

So we see, that the remaining Ostrogradsky variables are

$$x^1 = x \qquad x^2 = \dot{x} \tag{68}$$

Apply the trick to $L(x, \dot{x}, \ddot{x})$ (4)

Notice: the condition

$$\delta x|_{t_1} = \delta x|_{t_2} = \delta \dot{x}|_{t_1} = \delta \dot{x}|_{t_2} = 0 \tag{69}$$

means that both x and $\dot{x}$ are to be fixed at both t_1 and t_2 in the course of the variation. This is OK. Our equation is of the 4-th order, it needs initial conditions

$$(x, \dot{x}, \ddot{x}, \dddot{x})|_{t_1} = \text{ given} \tag{70}$$

Instead, one can (for small enough $(t_2 - t_1)$) prescribe

$$(x, \dot{x})|_{t_1} = \text{ given, plus } (x, \dot{x})|_{t_2} = \text{ given} \tag{71}$$

Etc ... apply the trick to $L(x, \dot{x}, \ddot{x}, \dddot{x})$

Here one gets

$$\delta S = \int_{t_1}^{t_2} dt\ \mathcal{E}^L \delta x + [p_1 \delta x]_{t_1}^{t_2} + [p_2 \delta \dot{x}]_{t_1}^{t_2} + [p_3 \delta \ddot{x}]_{t_1}^{t_2} \qquad (72)$$

So we can, again,

identify the canonical pairs (x^1, p_1), (x^2, p_2) and (x^3, p_3)

at the end of the expression (72). They are exactly Ostrogradsky variables (36), (37), (38).

The Hamiltonian: e.g. from conservation of energy

Standard computation: consider $L(x, \dot{x}, \ddot{x}, t)$. Then, using Lagrange equation in the form (49),

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial \ddot{x}}\dddot{x} + \frac{\partial L}{\partial t} \\ &= \dot{p}_1\dot{x} + (p_1 + \dot{p}_2)\ddot{x} + p_2\dddot{x} + \frac{\partial L}{\partial t} \\ &= \frac{d}{dt}(p_1\dot{x} + p_2\ddot{x}) + \frac{\partial L}{\partial t} \end{aligned}$$

or

$$\frac{dH}{dt} \equiv \frac{d}{dt}(p_1\dot{x} + p_2\ddot{x} - L) \quad = \quad -\frac{\partial L}{\partial t}$$

The Hamiltonian: e.g. from conservation of energy (2)

So, if the system is invariant w.r.t.

$$t \mapsto t + \text{ const.} \tag{73}$$

i.e. if

$$L(x, \dot{x}, \ddot{x}, t) = L(x, \dot{x}, \ddot{x}) \tag{74}$$

we get conservation of energy

$$\frac{dH}{dt} = 0 \qquad H := p_1 \dot{x} + p_2 \ddot{x} - L \tag{75}$$

Action as a function of coordinates

Still another view:
Consider action integral

$$S(x, t) = \int_{x_0, t_0}^{x, t} L(x(t), \dot{x}(t))dt \qquad (76)$$

computed
- along the real trajectory (= obeying Lagrange equations)
- from a reference point x_0 at t_0
- to the point x at t

Action as a function of coordinates (2)

Then (see, e.g., Landau-Lifshitz, Mechanics, $ 43), it holds

$$\frac{\partial S(x,t)}{\partial x^a} = p_a \qquad \frac{\partial S(x,t)}{\partial t} = -H \tag{77}$$

Here

$$p_a = p_a(x,t) \qquad H = H(x,t) \tag{78}$$

are the values of the canonical momentum p_a and the energy H at the endpoint of the real trajectory.

What changes for $L(x, \dot{x}, \ddot{x})$?

Now, we have

$$S(x, v, t) = \int_{x_0, v_0, t_0}^{x, v, t} L(x(t), \dot{x}(t), \ddot{x}(t)) dt \qquad (79)$$

(here $v = \dot{x}$) computed
- along the real trajectory (= obeying Lagrange equations)
- from a reference point (x_0, v_0) at t_0
- to the point (x, v) at t

What changes for $L(x, \dot{x}, \ddot{x})$? (2)

Then, it holds

$$\frac{\partial S(x,v,t)}{\partial x^a} = p_{1a} \qquad \frac{\partial S(x,v,t)}{\partial v^a} = p_{2a} \qquad \frac{\partial S(x,v,t)}{\partial t} = -H \tag{80}$$

so that

$$\delta S = p_{1a}\delta x^a + p_{2a}\delta v^a - H\delta t \tag{81}$$

Here

$$\delta S := S(x+\delta x, v+\delta v, t+\delta t) - S(x,v,t) \tag{82}$$

Why the topic is relevant today

Pais, Uhlenbeck [3] (1950):

It is investigated whether suitable generalizations of the field equations of current field theories to equations of higher order may be of help in eliminating the divergent features of the present theory.
It turns out to be difficult, if feasible, to reconcile in this way the requirements of convergence, of positive definiteness of the free field energy, and of a strictly causal behavior of the state vector of a physical system. ...
A procedure for deriving a Hamiltonian corresponding to (13) was given long ago by Ostrogradski ...

Why the topic is relevant today (2)

Woodard [1] (2006):

I begin by reviewing a powerful no-go theorem which pervades and constrains fundamental theory so completely that most people assume its consequence without thinking. This is the theorem of Ostrogradski ...

Why the topic is relevant today (3)

Woodard [2] (2009):

... the renormalization of perturbative quantum general relativity requires that the equations of motion be changed to include terms with up to four derivatives. ... it is also subject to a virulent instability that is totally inconsistent with the observed reality of a universe which is 13.7 billion years old.

Why the topic is relevant today (4)

Woodard [3] (2015):

The resulting instability imposes by far the most powerful restriction on fundamental, interacting, continuum Lagrangian field theories.

For Further Reading (1)

M.V. Ostrogradsky.
Memoires sur les equations differentielles relatives au probleme des isoperimetres.
Mem. Acad. St. Petersburg 6 (4), 385 (1850)

For Further Reading (1)

- M.V. Ostrogradsky.
  Memoires sur les equations differentielles relatives au probleme des isoperimetres.
  Mem. Acad. St. Petersburg 6 (4), 385 (1850)
- E.T. Whittaker.
  A treatise on the analytical dynamics of particles and rigid bodies, p.266.
  Cambridge University Press, Cambridge, UK (1960), 4th edition

For Further Reading (1)

- M.V. Ostrogradsky.
  Memoires sur les equations differentielles relatives au probleme des isoperimetres.
  Mem. Acad. St. Petersburg 6 (4), 385 (1850)
- E.T. Whittaker.
  A treatise on the analytical dynamics of particles and rigid bodies, p.266.
  Cambridge University Press, Cambridge, UK (1960), 4th edition
- A. Pais and G.E. Uhlenbeck
  On Field Theories with Non-Localized Action.
  Phys. Rev. 79 145-165 (1950).

For Further Reading (1)

- M.V. Ostrogradsky.
  Memoires sur les equations differentielles relatives au probleme des isoperimetres.
  Mem. Acad. St. Petersburg 6 (4), 385 (1850)
- E.T. Whittaker.
  A treatise on the analytical dynamics of particles and rigid bodies, p.266.
  Cambridge University Press, Cambridge, UK (1960), 4th edition
- A. Pais and G.E. Uhlenbeck
  On Field Theories with Non-Localized Action.
  Phys. Rev. 79 145-165 (1950).

For Further Reading (2)

R.P. Woodard
Avoiding Dark Energy with 1/R Modifications of Gravity.
arXiv:astro-ph/0601672v2 (6 Feb 2006)

For Further Reading (2)

R.P. Woodard
Avoiding Dark Energy with 1/R Modifications of Gravity.
arXiv:astro-ph/0601672v2 (6 Feb 2006)

R.P. Woodard
How Far Are We from the Quantum Theory of Gravity?.
arXiv (gr-qc): 0907.4238v1. (24 Jul 2009)

For Further Reading (2)

- R.P. Woodard
  Avoiding Dark Energy with 1/R Modifications of Gravity.
  arXiv:astro-ph/0601672v2 (6 Feb 2006)
- R.P. Woodard
  How Far Are We from the Quantum Theory of Gravity?.
  arXiv (gr-qc): 0907.4238v1. (24 Jul 2009)
- R.P. Woodard
  The Theorem of Ostrogradsky.
  arxiv (hep-th): 1506.02210v1. (7 June 2015)

For Further Reading (2)

- R.P. Woodard
  Avoiding Dark Energy with 1/R Modifications of Gravity.
  arXiv:astro-ph/0601672v2 (6 Feb 2006)
- R.P. Woodard
  How Far Are We from the Quantum Theory of Gravity?.
  arXiv (gr-qc): 0907.4238v1. (24 Jul 2009)
- R.P. Woodard
  The Theorem of Ostrogradsky.
  arxiv (hep-th): 1506.02210v1. (7 June 2015)