Vertical degree of forms and some applications

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- What it looks like for spacetime with an observer field
- What it looks like for principal bundle with connection
- How it might help in computing exterior covariant derivative

Contents

Adapted basis in $L = L_1 \oplus L_2$

Let us call the subspaces L_1 and L_2 of $L_1 \oplus L_2 \equiv L$

$$\begin{aligned} L \supset L_1 &= \text{vertical} \\ L \supset L_2 &= \text{horizontal} \end{aligned}$$

Choose a basis $e_i \in L_1$ and $e_a \in L_2$. Then

$$\begin{aligned} e_\alpha \equiv (e_i, e_a) &= \text{adapted basis in } L = L_1 \oplus L_2 \\ e^\alpha \equiv (e^i, e^a) &= \text{adapted dual basis in } L^* = L_1^* \oplus L_2^* \end{aligned}$$

Duality conditions in detail

The overall duality conditions

$$\langle e^\alpha, e_\beta \rangle = \delta^\alpha_\beta \tag{1}$$

are equivalent to four particular cases:

$$\langle e^a, e_b \rangle = \delta^a_b \qquad \langle e^i, e_j \rangle = \delta^i_j \qquad \langle e^a, e_i \rangle = 0 \qquad \langle e^i, e_a \rangle = 0 \tag{2}$$

Then, clearly, we can write

$$v \text{ is vertical} \quad \Leftrightarrow \quad v = v^i e_i \;\Leftrightarrow\; \langle e^a, v \rangle = 0 \tag{3}$$

$$u \text{ is horizontal} \quad \Leftrightarrow \quad u = u^a e_a \;\Leftrightarrow\; \langle e^i, u \rangle = 0 \tag{4}$$

Forms in $L = L_1 \oplus L_2$

A general p-form in $L = L_1 \oplus L_2$ reads

$$\alpha = \frac{1}{p!}\alpha_{\alpha\dots\beta} e^\alpha \wedge \dots \wedge e^\beta \qquad (5)$$

(p pieces of basis 1-forms $e^\alpha, \dots, e^\beta$ are wedge-multiplied).
Now each e^α is either e^i or e^a. So, (5) is actually a sum of terms of the structure

$$\alpha = \hat{\alpha} + e^i \wedge \hat{\alpha}_i + e^i \wedge e^j \wedge \hat{\alpha}_{ij} + \dots \qquad (6)$$

where the hatted forms ($\hat{\alpha}$, $\hat{\alpha}_i$, $\hat{\alpha}_{ij}$, …) do not contain e^i (they only contain e^a).

Forms in $L = L_1 \oplus L_2$ (2)

Or, equivalently,

$$\alpha = \alpha_{(0)} + \alpha_{(1)} + \alpha_{(2)} + \dots \tag{7}$$

where

$$\alpha_{(0)} := \hat{\alpha} \tag{8}$$

$$\alpha_{(1)} := e^i \wedge \hat{\alpha}_i \tag{9}$$

$$\alpha_{(2)} := e^i \wedge e^j \wedge \hat{\alpha}_{ij} \tag{10}$$

$$\alpha_{(3)} := e^i \wedge e^j \wedge e^k \wedge \hat{\alpha}_{ijk} \tag{11}$$

$$\text{etc.} \tag{12}$$

Observation: Individual terms in the decomposition (7) are well defined (they do not depend on the choice of $e_i \in L_1$ or $e_a \in L_2$).

Forms in $L = L_1 \oplus L_2$ (3)

So, a general p-form in $L = L_1 \oplus L_2$ is a sum of terms characterized by another integer (in addition to the integer p).
We call it vertical degree (recall that p is the degree of α).
Namely,

$$\text{vertical degree of } \alpha_{(q)} \text{ is (by definition) } q \in \mathbb{Z} \tag{13}$$

A general p-form in $L = L_1 \oplus L_2$ is
(from the point of view of the vertical degree) inhomogeneous.
(It is a sum of homogeneous terms.)

Horizontal part of a form, horizontal form

The first term of the sum is of particular interest. We call it

$$\text{horizontal part of } \alpha \text{ is (by definition) } \alpha_{(0)} \tag{14}$$

Then we introduce the concept of

$$\text{horizontal form } \alpha : \ \text{such that } \alpha = \alpha_{(0)} \tag{15}$$

So, horizontal p-form is a particular homogeneous form:

$$\text{horizontal form } \alpha : \quad \text{such that its horizontal degree is } 0 \tag{16}$$

Some useful observations (1)

Observation 1:

$$\alpha \text{ is horizontal} \quad \Leftrightarrow \quad i_{e_j}\alpha = 0 \text{ (for all } j) \tag{17}$$

That is, α is killed by any vertical argument.
So, horizontal forms only can survive on horizontal arguments.

Observation 2 (more general):

$$i_{e_j}\alpha = 0 \quad \Leftrightarrow \quad \alpha = \alpha_{(0)} \tag{18}$$

$$i_{e_j}i_{e_k}\alpha = 0 \quad \Leftrightarrow \quad \alpha = \alpha_{(0)} + \alpha_{(1)} \tag{19}$$

$$i_{e_j}i_{e_k}i_{e_l}\alpha = 0 \quad \Leftrightarrow \quad \alpha = \alpha_{(0)} + \alpha_{(1)} + \alpha_{(2)} \tag{20}$$

$$\text{etc.} \tag{21}$$

Some useful observations (2)

Define

$$Q := j^k i_k \qquad i_k \equiv i_{e_k}, \quad j^k \alpha := e^k \wedge \alpha \tag{22}$$

Observation 3:

$$\boxed{Q\alpha_{(q)} = q\alpha_{(q)}} \tag{23}$$

Why?
Recall (see 5.8.11 in [1]), that

$$j^\beta i_\beta \alpha = p\alpha \qquad \text{for } \alpha \text{ a } p\text{-form} \tag{24}$$

(hint: $j^\alpha i_\beta$ is a derivation (of the algebra of forms) of degree 0).
For $j^k i_k$ similarly. (It works like number operator $a_k^+ a_k$:-)

Observer field V

Let (M, g) be a space-time (signature $+ - - -$) and
let V be a future-oriented vector field with $g(V, V) = 1$.
Integral curves of V may be regarded as world-lines of observers.
In each (world-)point,
- $V \equiv e_0$ defines (observer's, local) time direction
- $V^\perp$ defines (observer's, local) 3-space.

So, we can regard (in tangent space)

$$L_1 := \text{Span } V \qquad L_2 := V^\perp \equiv \text{the 3-space} \tag{25}$$

Then

$$e_i \leftrightarrow V \equiv e_0 \qquad e^i \leftrightarrow \tilde{V} \equiv g(V, \ \cdot \ ) \equiv e^0 \tag{26}$$

Observer field V and 1+3 decomposition

Since there is just a single $e^i \leftrightarrow \tilde{V}$, the general expression

$$\begin{aligned} \alpha &= \alpha_{(0)} + \alpha_{(1)} + \alpha_{(2)} + \dots && (27) \\ &= \hat{\alpha} + e^i \wedge \hat{\alpha}_i + e^i \wedge e^j \wedge \hat{\alpha}_{ij} + \dots && (28) \end{aligned}$$

reduces, here, to just two terms

$$\begin{aligned} \alpha &= \alpha_{(0)} + \alpha_{(1)} && (29) \\ &= \hat{r} + \tilde{V} \wedge \hat{s} && (30) \end{aligned}$$

A general form is given (w.r.t. V)
by a pair of spatial (= horizontal) forms $(\hat{s}, \hat{r})$.

Principal G-bundle - adapted frame and co-frame fields (1)

Let

- G be a semi-simple Lie group, $\mathcal{G}$ its Lie algebra
- E_i an orthonormal basis in $\mathcal{G}$ (w.r.t. Killing-Cartan K)
- $\pi : P \to M$ be a principal G-bundle over space-time $(M, \hat{g})$
- $\omega = \omega^i E_i$ be a connection form
- ξ_X, $X = X^i E_i$, be the fundamental field of $R_g : P \to P$
- ξ_{E_i} be the generators of $R_g : P \to P$
- w^h be the horizontal lift of a vector field w on M
- $\hat{e}_a$ be an orthonormal (w.r.t. $\hat{g}$) frame on (a part of) M

Principal G-bundle - adapted frame and co-frame fields (2)

Then

$$e_\alpha \equiv (e_i, e_a) \qquad e^\alpha \equiv (e^i, e^a) \qquad \langle e^\alpha, e_\beta \rangle = \delta^\alpha_\beta$$

where

$$e_i := \xi_{E_i} \qquad e_a := \hat{e}_a^h \tag{31}$$
$$e^i := \omega^i \qquad e^a := \pi^* \hat{e}^a \tag{32}$$

constitute a frame and the dual co-frame, respectively (on part of P).
Moreover, the frames happen to be ortho-normal w.r.t.

$$g := \pi^* \hat{g} + K(\omega, \omega) \tag{33}$$

Principal G-bundle - adapted frame and co-frame fields (3)

Now, in each tangent space $L \equiv T_p P$ in point $p \in P$, define

$$L_1 := \text{ver}\ T_p P \qquad L_2 := \text{hor}\ T_p P \tag{34}$$

in the sense of connection theory (!).
Then the decomposition (the essence of connection concept)

$$T_p P = \text{ver}\ T_p P \oplus \text{hor}\ T_p \tag{35}$$

exactly matches the decomposition studied above

$$L = L_1 \oplus L_2 \tag{36}$$

(including meaning of hor and ver).

Principal G-bundle - adapted frame and co-frame fields (4)

So, as an example,
- the curvature 2-forms $\Omega^i = \frac{1}{2}\Omega^i_{ab} e^a \wedge e^b$ are horizontal $(\equiv \alpha_{(0)})$
- the connection 1-forms ω^i have vertical degree equal 1 $(\equiv \alpha_{(1)})$
- the expression $\frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k$ has vertical degree equal 2 $(\equiv \alpha_{(2)})$
- and so on

Computation of $\Omega \equiv D\omega$

By definition, the curvature 2-form Ω
is the exterior covariant derivative

$$D := \text{hor} \circ d \tag{37}$$

of the connection form $\omega \equiv \omega^i E_i$.
So, we need to compute

$$\Omega^i := \text{hor}\,(d\omega^i) \equiv (de^i)_{(0)} \tag{38}$$

Computation of $\Omega \equiv D\omega$ (2)

Definition properties of connection form ω:

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega \qquad (39)$$

$$\langle \omega, \xi_X \rangle = X \qquad (40)$$

Their infinitesimal version

$$\mathcal{L}_{\xi_X} \omega = -\,\mathrm{ad}_X \omega \equiv -\,[X, \omega] \qquad (41)$$

$$i_{\xi_X} \omega = X \qquad (42)$$

is equivalent to

$$i_{\xi_X} d\omega = -\,[X, \omega] \qquad (43)$$

$$i_{\xi_X} \omega = X \qquad (44)$$

(use $\mathcal{L}_W = d i_W + i_W d$ and (42))

Computation of $\Omega \equiv D\omega$ (3)

Finally (for $X = E_j$, $\omega = \omega^i E_i$, $e^i = \omega^i$) we get

$$i_{e_j}(de^i) = -c^i_{jk} e^k \tag{45}$$

$$i_{e_j} e^i = \delta^i_j \tag{46}$$

Consequently,

$$\boxed{i_{e_k} i_{e_j}(de^i) = -c^i_{jk}} \qquad \boxed{i_{e_m} i_{e_k} i_{e_j}(de^i) = 0} \tag{47}$$

Computation of $\Omega \equiv D\omega$ (4)

From the second formula in (47) we can, first, deduce

$$de^i = (de^i)_{(0)} + (de^i)_{(1)} + (de^i)_{(2)} \tag{48}$$

From the first formula in (47), then, we have

$$(de^i)_{(2)} = -\frac{1}{2} c^i_{jk} e^j \wedge e^k \tag{49}$$

and, finally, from (45),

$$(de^i)_{(1)} = 0 \tag{50}$$

(no vertical degree one term is at the r.h.s of (45)).

Computation of $\Omega \equiv D\omega$ (5)

So, we learned that

$$de^i = (de^i)_{(0)} - \frac{1}{2} c^i_{jk} e^j \wedge e^k \tag{51}$$

or, already trivially,

$$(de^i)_{(0)} =: \mathrm{hor}\, de^i = de^i + \frac{1}{2} c^i_{jk} e^j \wedge e^k \tag{52}$$

So, remembering that $e^i = \omega^i$, we get well-known explicit formula

$$\boxed{\Omega^i := \mathrm{hor}\, d\omega^i = d\omega^i + \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k} \tag{53}$$

Bianchi identity: $D\Omega = 0$

In our language:

$$DDe^i = 0 \qquad \text{i.e.} \quad (dDe^i)_{(0)} = 0 \tag{54}$$

Why is it true?
Because actually

$$dDe^i = (dDe^i)_{(1)} + (dDe^i)_{(3)} \tag{55}$$

Indeed,

$$dDe^i = d(de^i + \frac{1}{2}c^i_{jk}e^j \wedge e^k) = c^i_{jk}(de^j) \wedge e^k \tag{56}$$

Combine with (68) $\Rightarrow$:-)

Computation of $D\alpha$ (1)

Definition properties of a horizontal p-form α of type ρ:

$$R_g^*\alpha = \rho(g^{-1})\alpha \qquad \text{(type } \rho) \tag{57}$$

$$i_{\xi_X}\alpha = 0 \qquad \text{(horizontal)} \tag{58}$$

Their infinitesimal version

$$\mathcal{L}_{\xi_X}\alpha = -\rho'(X)\alpha \tag{59}$$

$$i_{\xi_X}\alpha = 0 \tag{60}$$

is equivalent to

$$i_{\xi_X}(d\alpha) = -\rho'(X)\alpha \tag{61}$$

$$i_{\xi_X}\alpha = 0 \tag{62}$$

Computation of $D\alpha$ (2)

Finally (for $X = E_j$) we get

$$i_{e_j}(d\alpha) = -\rho'(E_j)\alpha \tag{63}$$
$$i_{e_j}\alpha = 0 \tag{64}$$

Consequently,

$$i_{e_k} i_{e_j}(d\alpha) = 0 \tag{65}$$

From (65) we, first, deduce

$$d\alpha = (d\alpha)_{(0)} + (d\alpha)_{(1)} \tag{66}$$

and from (63) we see that

$$(d\alpha)_{(1)} = -e^i \wedge \rho'(E_i)\alpha \tag{67}$$

Computation of $D\alpha$ (3)

Altogether, we learned that

$$d\alpha = (d\alpha)_{(0)} - e^i \wedge \rho'(E_i)\alpha \tag{68}$$

or, trivially,

$$(d\alpha)_{(0)} =: \text{hor}\, d\alpha = d\alpha + e^i \wedge \rho'(E_i)\alpha \tag{69}$$

So, remembering that $e^i = \omega^i$, we get well-known explicit formula

$$\boxed{D\alpha := \text{hor}\, d\alpha = d\alpha + \omega^i \wedge \rho'(E_i)\alpha \equiv d\alpha + \rho'(\omega)\dot{\wedge}\alpha} \tag{70}$$

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