Hamiltonian systems with degenerate Poisson tensor

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- What are Casimir functions
- What canonical coordinates look like
- What common features Poisson and sub-Riemannian geometries do share

Contents

Usual Hamilton's equations

Start from standard Hamilton's equations

$$\boxed{\dot{x}^a = \frac{\partial H}{\partial p_a}} \qquad \boxed{\dot{p}_a = -\frac{\partial H}{\partial x^a}} \qquad a = 1, \dots, n \tag{1}$$

In phase space, introduce coordinates

$$z^\alpha \equiv (x^a, p_a) \qquad \alpha = 1, \dots, 2n \tag{2}$$

and rewrite the equations as

$$\underbrace{(\dot{x}, \dot{p})}_{\dot{z}} = \underbrace{(\partial_x H, \partial_p H)}_{dH} \underbrace{\begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}}_{\mathcal{P}} \qquad \text{i.e.} \qquad \boxed{\dot{z}^\alpha = (dH)_\beta \mathcal{P}^{\beta\alpha}} \tag{3}$$

Usual Hamilton's equations (2)

Then, for for the curve $\gamma(t) \leftrightarrow z^\alpha(t)$, we have

$$\dot{\gamma} \equiv \dot{z}^\alpha \partial_\alpha = \mathcal{P}(dH, dx^\alpha)\partial_\alpha \equiv \mathcal{P}(dH, \ \cdot \ ) \tag{4}$$

So we get a coordinate-free version of Hamilton's equations

$$\boxed{\dot{\gamma} = \zeta_H} \qquad \boxed{\zeta_H := \mathcal{P}(dH, \ \cdot \ )} \tag{5}$$

where

$$\mathcal{P} \equiv \text{ Poisson tensor} \qquad \zeta_H \equiv \text{ Hamiltonian field} \tag{6}$$

Usual Hamilton's equations (3)

We see, that Poisson tensor is a bi-vector

(skew-symmetric $\binom{2}{0}$-type tensor)

and, since the matrix $\mathcal{P}$ in (3) is invertible,
the tensor $\mathcal{P}$ is non-degenerate:

$$\boxed{\mathcal{P}(\alpha, \ \cdot\ ) = 0 \qquad \Rightarrow \qquad \alpha = 0} \tag{7}$$

Usual Hamilton's equations (4)

This enables one to define
a non-degenerate skew-symmetric $\binom{0}{2}$-type tensor ω
(i.e. a 2-form)
via (minus of) the inverse matrix,

$$\omega_{\alpha\rho}\mathcal{P}^{\rho\beta} := -\delta_\alpha^\beta \tag{8}$$

In original coordinates (x^a, p_a) one can check that

$$\omega = \cdots = dp_a \wedge dq^a \qquad \Rightarrow \qquad \boxed{d\omega = 0} \quad \text{(it is closed)} \tag{9}$$

Usual Hamilton's equations (5)

From (3) we then see that

$$\dot{z}^{\alpha}\omega_{\alpha\rho} = (dH)_{\beta}\mathcal{P}^{\beta\alpha}\omega_{\alpha\rho} = -(dH)_{\rho} \tag{10}$$

So, we get another coordinate-free version of Hamilton's equations

$$\boxed{\dot{\gamma} = \zeta_H} \qquad \boxed{i_{\zeta_H}\omega := -dH} \tag{11}$$

$$\zeta_H = \text{ Hamiltonian field} \qquad \omega = \text{ symplectic form} \tag{12}$$

(Recall that, by definition, we call symplectic form any 2-form which is, in addition, closed and non-degenerate.)

Usual Hamilton's equations (6) - summing up

Let us recapitulate: Usual Hamilton's equations

- may be expressed in geometrical (coordinate-free) way
- as equations for integral curves of Hamiltonian field ζ_H

In order to tell what Hamiltonian field ζ_H is, we can choose

- either Poisson language (express it in terms of $\mathcal{P}$, see (5))
- or symplectic language (express it in terms of ω, see (11))

From practical side, symplectic language smoothly wins!
Why? Because

- there is exterior calculus available for differential forms, but
- nothing comparably powerful for bi-vectors.

Poisson bracket for non-degenerate case

Still in context of "usual" Hamilton's equations,
Poisson bracket is introduced as

$$\{f,g\} \equiv \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} - \frac{\partial g}{\partial p_a}\frac{\partial f}{\partial q^a} \tag{13}$$

In the two languages, its coordinate-free expression reads as follows:

Poisson $\qquad \{f,g\} = \mathcal{P}(df, dg) \qquad (14)$

symplectic $\qquad \{f,g\} = \omega(\zeta_f, \zeta_g) \qquad i_{\zeta_f}\omega := -df \qquad (15)$

Poisson bracket for no-degenerate case (2)

In either of the two languages, one easily checks that the following well-known identities hold

$$\{f,g\} = -\{g,f\} \qquad \textit{skew-symmetry} \tag{16}$$

$$\{f,g+\lambda h\} = \{f,g\} + \lambda\{f,h\} \qquad \textit{linearity} \tag{17}$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \qquad \textit{derivation} \tag{18}$$

$$\{\{f,g\},h\} + \text{ cycl.} = 0 \qquad \textit{Jacobi identity} \tag{19}$$

Poisson bracket given by properties alone

It proves to be a very fruitful idea to study
Poisson bracket in general, simply given by its properties

$$\{f,g\} = -\{g,f\} \quad \text{skew-symmetry} \tag{20}$$
$$\{f,g+\lambda h\} = \{f,g\} + \lambda\{f,h\} \quad \text{linearity} \tag{21}$$
$$\{f,gh\} = \{f,g\}h + g\{f,h\} \quad \text{derivation} \tag{22}$$
$$\{\{f,g\},h\} + \text{ cycl.} = 0 \quad \text{Jacobi identity} \tag{23}$$

with no assumption about how it is constructed.
Surprisingly, an interesting new possibility emerges in this way,
namely the one with degenerate Poisson tensor (see below).

How to "solve" axioms (20)-(23)

First, combination of (21) and (22) reveals that Poisson bracket acts as a vector field on $\mathcal{F}(M)$

$$\{f,\ \cdot\ \} = \zeta_f \qquad \{f,g\} = \zeta_f g \qquad \zeta_f \in \mathfrak{X}(M) \tag{24}$$

Then, with the help of (20) we obtain

$$\{f,g\} = \langle dg, \zeta_f \rangle = -\langle df, \zeta_g \rangle \tag{25}$$

This says that the function $\{f,g\}$ depends linearly on two covectors, df and dg.

How to "solve" axioms (20)-(23) - cont.

Therefore, a $\binom{2}{0}$-type tensor, let us call it Poisson tensor $\mathcal{P}$, is behind, defined by

$$\boxed{\{f, g\} = \mathcal{P}(df, dg)} \tag{26}$$

Because of (20), $\mathcal{P}$ is actually a bi-vector (skew-symmetry holds).

So, the first three properties, (20)-(22)

- are "solved" by the construction (26),
- i.e. they just reveal the existence of the bi-vector $\mathcal{P}$.

How to "solve" axioms (20)-(23) - cont.

And what about the last property (Jacobi identity) of the Poisson bracket?

First notice that the vector field ζ_f may be written as

$$\boxed{\zeta_f = \{f, \cdot\,\} = \mathcal{P}(df, \cdot\,)} \qquad \text{Hamiltonian field} \tag{27}$$

Then, Jacobi identity is translated to the following property of Poisson tensor $\mathcal{P}$:

$$\forall f : \quad \boxed{\mathcal{L}_{\zeta_f}\mathcal{P} = 0} \qquad \Leftrightarrow \qquad \text{Jacobi identity} \tag{28}$$

i.e. to the statement that Poisson tensor is Lie-invariant w.r.t. to any Hamiltonian field.

A proof of (28)

▼ Indeed, for any f, g, h,
formal computation of Lie derivative reads

$$\mathcal{L}_{\zeta_f}(\mathcal{P}(dg, dh)) = (\mathcal{L}_{\zeta_f}\mathcal{P})(dg, dh) + \mathcal{P}(\mathcal{L}_{\zeta_f} dg, dh) + \mathcal{P}(dg, \mathcal{L}_{\zeta_f} dh)$$

On the other hand,
Jacobi identity may be re-written, step by step, as

$$\begin{aligned} \{f, \{g, h\}\} &= \{\{f, g\}, h\} + \{g, \{f, h\}\} \\ \zeta_f\{g, h\} &= \{\zeta_f g, h\} + \{g, \zeta_f h\} \\ \mathcal{L}_{\zeta_f}(\mathcal{P}(dg, dh)) &= \mathcal{P}(\mathcal{L}_{\zeta_f} dg, dh) + \mathcal{P}(dg, \mathcal{L}_{\zeta_f} dh) \end{aligned}$$

Therefore, for any g, h

$$(\mathcal{L}_{\zeta_f}\mathcal{P})(dg, dh) = 0 \qquad \text{i.e.} \qquad \boxed{\mathcal{L}_{\zeta_f}\mathcal{P} = 0} \ \forall f$$

▲

Possibly degenerate $\mathcal{P}$?

Notice that our "solution" (26) of axioms of Poisson bracket says nothing about (non-)degeneracy of the Poisson tensor $\mathcal{P}$.

Originally, in (5),
the tensor resulted directly from Hamilton's equations
and in this case it was non-degenerate
(there is invertible matrix in (3)).

Here, in (26),
it emerges from general properties of Poisson bracket
and there is no sign for necessity of non-degeneracy.

It seems like as if non-degeneracy is not needed to fulfil all axioms.

Possibly degenerate $\mathcal{P}$? (2)

And indeed, consider the following highly trivial example:

- three-dimensional space $\mathbb{R}^3$ with coordinates (q, p, y)
- endowed with the following formula for Poisson bracket

$$\{f, g\} := \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q} \tag{29}$$

$$= \underbrace{(\partial_q f, \partial_p f, \partial_y f)}_{df} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathcal{P}} \underbrace{\begin{pmatrix} \partial_q g \\ \partial_p g \\ \partial_y g \end{pmatrix}}_{dg} \tag{30}$$

Possibly degenerate $\mathcal{P}$? (3)

One can easily check that

- it works (= satisfies everything needed to fulfil the axioms),
- the corresponding Poisson tensor is degenerate.

(There is no skew-symmetric odd-dimensional regular matrix.
So any Poisson bracket in $\mathbb{R}^3$ is necessarily degenerate).

So, degenerate Poisson tensors (and, consequently, brackets)

- really do exist,
- might be of interest somewhere (they indeed are),
- are to be studied in more detail.

How Hamiltonian distribution emerges

Any $\binom{2}{0}$-type tensor t may be regarded as a linear mapping "covector to vector"

$$t: \quad \alpha \mapsto t(\alpha, \;\cdot\;) \equiv v \tag{31}$$

and defines as many as two subspaces,

- the kernel (a subspace "in covectors"),
- the image (a subspace "in vectors").

For us

- the latter case is relevant, now,
- with $t = \mathcal{P}$.

How Hamiltonian distribution emerges (2)

So, on $(M, \mathcal{P})$, let us consider a distribution $\mathcal{D}_{\mathcal{P}}$
given as the image space of the Poisson tensor

$$\mathcal{D}_{\mathcal{P}} := \mathrm{Im}\,\mathcal{P} \equiv \{v \in \mathfrak{X}(M) \mid \exists \alpha, v = \mathcal{P}(\alpha, \ \cdot\ )\} \qquad (32)$$

Notice that the distribution becomes technically trivial
if the Poisson tensor is non-degenerate (i.e. isomorphism).

(Each vector is in the distribution, then.)

So, in order to speak of something (technically) worth attention,
the Poisson tensor $\mathcal{P}$ is to be degenerate.

How Hamiltonian distribution emerges (3)

Now, realize that

- any Hamiltonian field, $\zeta_f = \mathcal{P}(df, \ \cdot \ )$, is in $\mathcal{D}_\mathcal{P}$,
- so that Hamiltonian fields provide a subset of $\mathcal{D}_\mathcal{P}$,
- for a coordinate basis dx^α, $\alpha = 1, \dots, \dim M$, we have

$$\mathcal{D}_\mathcal{P} := \mathrm{Im}\, \mathcal{P} := \mathrm{Span}\, \{\mathcal{P}(dx^\alpha, \ \cdot \ ) = \mathrm{Span}\, \{\zeta_{x^\alpha}\} \qquad (33)$$

- so that (point-wise)

$$\boxed{\mathcal{D}_\mathcal{P} = \mathcal{D}_{\text{ham}} \equiv \mathrm{Span}\, \{\text{Hamiltonian fields}\}} \qquad (34)$$

So the distribution $\mathcal{D}_\mathcal{P}$ (given as $\mathrm{Im}\, \mathcal{P}$) coincides with the distribution given by (all) Hamiltonian fields.

Integrability of the Hamiltonian distribution

Therefore,

$$\boxed{\mathcal{D}_{\mathcal{P}} \quad \text{is integrable}} \qquad \text{(see the next page)} \tag{35}$$

So, our Poisson manifold $(M, \mathcal{P})$ is foliated by leaves (= integral sub-manifolds of the distribution).

Since time evolution of a Hamiltonian system $(M, \mathcal{P}, H)$ consists of motion of a point along Hamiltonian field ζ_H, the point spends whole life on a single leaf (fixed by initial conditions).

A proof of integrability of $\mathcal{D}_{\mathcal{P}}$ (1)

For any pair of Hamiltonian fields it holds

$$[\zeta_f, \zeta_g] = \zeta_{\{f,g\}} \tag{36}$$

- Indeed,

$$\begin{aligned}
[\zeta_f, \zeta_g] &= \mathcal{L}_{\zeta_f}(\mathcal{P}(dg, \ \cdot\ )) \\
&= (\mathcal{L}_{\zeta_f}\mathcal{P})(dg, \ \cdot\ ) + \mathcal{P}(\mathcal{L}_{\zeta_f} dg, \ \cdot\ ) \\
&= 0 + \mathcal{P}(d(\zeta_f g), \ \cdot\ ) \\
&= \mathcal{P}(d\{f,g\}, \ \cdot\ ) \\
&= \zeta_{\{f,g\}}
\end{aligned}$$

▲

A proof of integrability of $\mathcal{D}_{\mathcal{P}}$ (2)

In particular,

$$[\zeta_{x^\alpha}, \zeta_{x^\beta}] = \zeta_{\{x^\alpha, x^\beta\}} = \zeta_{\mathcal{P}^{\alpha\beta}} = \mathcal{P}(d\mathcal{P}^{\alpha\beta}, \ \cdot \ ) = \mathcal{P}^{\alpha\beta}{}_{,\rho}\zeta_{x^\rho} \qquad (37)$$

Then, for $U, V \in \mathcal{D}_{\mathcal{P}}$,

$$[U, V] = [f_\alpha \zeta_{x^\alpha}, g_\beta \zeta_{x^\beta}] = \dots = h_\alpha \zeta_{x^\alpha} \equiv W \in \mathcal{D}_{\mathcal{P}} \qquad (38)$$

So, due to Frobenius integrability criterion, $\mathcal{D}_{\mathcal{P}}$ is integrable.

Convenient description of $\mathcal{D}_{\mathcal{P}}$ on $(M, \mathcal{P})$

Fix, locally, an adapted frame (e_a, e_i), i.e. let $e_a \in \mathcal{D}_{\mathcal{P}}$.
(Well, this needs constant rank of $\mathcal{P}$ within the patch.
This is not always the case.)
So,

- e_a-part is directed along leaves,
- therefore it may serve (when restricted) as a frame on a particular leaf;
- e_i-part is transversal to leaves.

Write

$$\mathcal{P} = \mathcal{P}^{ab} e_a \otimes e_b + \mathcal{P}^{ai} e_a \otimes e_i + \mathcal{P}^{ia} e_i \otimes e_a + \mathcal{P}^{ij} e_i \otimes e_j \qquad (39)$$

Convenient description of $\mathcal{D}_\mathcal{P}$ on $(M, \mathcal{P})$ (2)

Or, as a matrix,

$$\mathcal{P} \leftrightarrow \begin{pmatrix} \mathcal{P}^{ab} & \mathcal{P}^{ai} \\ \mathcal{P}^{ia} & \mathcal{P}^{ij} \end{pmatrix} \tag{40}$$

Then, for general

$$\alpha = \alpha_a e^a + \alpha_i e^i \tag{41}$$

we get

$$\mathcal{P}(\alpha, \ . \ ) = \cdots = (\alpha_b \mathcal{P}^{ba} + \alpha_i \mathcal{P}^{ia}) e_a + (\alpha_a \mathcal{P}^{ai} + \alpha_j \mathcal{P}^{ji}) e_i \tag{42}$$

Since, however, $\mathcal{D}_\mathcal{P} \equiv \mathrm{Im}\, \mathcal{P} = \text{Span } \{e_a\}$, we can deduce

$$\boxed{\mathcal{P}^{ai} = 0 = \mathcal{P}^{ji}} \tag{43}$$

Convenient description of $\mathcal{D}_\mathcal{P}$ on $(M, \mathcal{P})$ (3)

Thus, so far,

$$\mathcal{P} \leftrightarrow \begin{pmatrix} \mathcal{P}^{ab} & 0 \\ \mathcal{P}^{ia} & 0 \end{pmatrix} \tag{44}$$

Now, the tensor $\mathcal{P}$ is to be skew-symmetric,
so we actually have

$$\boxed{\mathcal{P} \leftrightarrow \begin{pmatrix} \mathcal{P}^{ab} & 0 \\ 0 & 0 \end{pmatrix}} \qquad \mathcal{P}^{ab} = -\mathcal{P}^{ba} \tag{45}$$

The non-zero block $\mathcal{P}^{ab}$ is already non-degenerate (why?)
and, therefore, it is even-dimensional (why?). Dimensionally,

$$\boxed{n \equiv \dim M = 2m + k} \qquad a = 1, \dots, 2m \ , \ i = 1, \dots, k \tag{46}$$

Convenient description of $\mathcal{D}_\mathcal{P}$ on $(M, \mathcal{P})$ (4)

In (any) adapted frame (e_a, e_i), things look pretty simple:

$$\mathcal{P}: \quad e^a \mapsto \mathcal{P}(e^a, \cdot\,) = \mathcal{P}^{ab} e_b \neq 0 \qquad (47)$$

$$e^i \mapsto \mathcal{P}(e^i, \cdot\,) = 0 \qquad (48)$$

$$\mathcal{P} = \mathcal{P}^{ab} e_a \otimes e_b \qquad (49)$$

$$\{f, g\} = \mathcal{P}(df, dg) = (e_a f)\mathcal{P}^{ab}(e_b g) \qquad (50)$$

$$\zeta_f = \mathcal{P}(df, \cdot\,) = (e_a f)\mathcal{P}^{ab} e_b \qquad (51)$$

Notice that no e_i appear in either $\mathcal{P}$, ζ_f or $\{f, g\}$.
So functions $f, g, \dots$ are only differentiated along leaves.

Symplectic form ω on leaves $\mathcal{S} \subset (M, \mathcal{P})$

Consider formula

$$\omega(\zeta_f, \zeta_g) := \mathcal{P}(df, dg) \equiv \{f, g\} \qquad \zeta_f \equiv \mathcal{P}(df, \,\cdot\,) \tag{52}$$

In non-degenerate case it describes two equivalent ways of expressing the same Poisson bracket on $(M, \mathcal{P})$, see (14) and (15).

In degenerate case, on the other hand, it turns out that it may be regarded as a definition of symplectic form ω living (only) on leaves of $\mathcal{D}_{\mathcal{P}}$.

Symplectic form ω on leaves $\mathcal{S} \subset (M, \mathcal{P})$ (2)

Indeed, $\omega(\zeta_f, \zeta_g) := \dots$

- defines a $\binom{0}{2}$-type tensor field (only) on leaves of $\mathcal{D}_{\mathcal{P}}$,
- the tensor field is clearly skew-symmetric,
- so a 2-form ω is defined on (each) leaf $\mathcal{S} \subset (M, \mathcal{P})$

What is not yet clear is

- whether ω is closed and
- whether ω is non-degenerate.

Actually both statements do hold (see the next page).
Hence the standard nomenclature - symplectic leaves.

Closedness of ω

Closedness of ω:

$$
\begin{aligned}
(d\omega)(\zeta_f,\zeta_g,\zeta_h) &= \zeta_f(\omega(\zeta_g,\zeta_h)) + \cdots - \omega([\zeta_f,\zeta_g],\zeta_h) + \ldots \\
&= \{f,\{g,h\}\} + \cdots - \{\{f,g\},h\} + \ldots \\
&= \ldots (\text{six terms}) \\
&= 2(\{f,\{g,h\}\} + \text{cycl.}) \\
&= 0 \qquad (\text{Jacobi identity})
\end{aligned}
$$

(In the first line, Cartan formulas for computing d were used.)

So,

$$\omega \text{ is closed}: \qquad \boxed{d\omega = 0} \tag{53}$$

Components ω_{ab} of ω

Let us compute components of ω w.r.t. adapted frame (e_a, e_i).
Using (50), (51) and (52) we get

$$\omega((e_a f)\mathcal{P}^{ac}e_c, (e_b g)\mathcal{P}^{bd}e_d) := (e_a f)\mathcal{P}^{ab}(e_b g) \qquad (54)$$

or

$$(e_a f)(-\mathcal{P}^{ac}\omega_{cd}\mathcal{P}^{db})(e_b g) = (e_a f)\mathcal{P}^{ab}(e_b g) \qquad (55)$$

This says, since f, g are arbitrary, that

$$-\mathcal{P}^{ac}\omega_{cd}\mathcal{P}^{db} = \mathcal{P}^{ab} \qquad (56)$$

or, since $\mathcal{P}^{ac}$ is regular, that

$$\mathcal{P}^{ac}\omega_{cb} = -\delta^a_b \quad \text{i.e.} \quad \boxed{\omega_{ab} = -(P^{-1})_{ab}} \qquad (57)$$

Non-degeneracy of ω

The formulas

$$\mathcal{P}^{ac}\omega_{cb} = -\delta^a_b \quad \text{i.e.} \quad \boxed{\omega_{ab} = -(P^{-1})_{ab}} \tag{58}$$

show, as a by-product, that the matrix ω_{ab} of the form ω,
being the inverse to the regular matrix $\mathcal{P}^{ac}$,
is regular as well.
But this is just a way to say
that the form ω itself is non-degenerate:

$$\omega(v, \,\cdot\,) = 0 \qquad \Rightarrow \qquad v = 0 \tag{59}$$

(Compare with (7).)

Casimir functions - definition

Functions which Poisson commute with all functions
play important role and deserve special name:

$$g \text{ is Casimir function : } \qquad \{f, g\} = 0 \quad \forall f \tag{60}$$

It is useful to understand this property
also from a different point of view.

Casimir functions - properties

Recall (see (50),(51)) that the following expressions hold:

$$\{f, g\} = (e_a f)\mathcal{P}^{ab}(e_b g) \qquad (61)$$

$$\zeta_f = (e_a f)\mathcal{P}^{ab} e_b \qquad (62)$$

From them one can easily see that

$$\zeta_f = 0 \quad \Leftrightarrow \quad e_a f = 0 \quad \Leftrightarrow \quad f = \text{const. on leaves} \qquad (63)$$

$$\{f, g\} = 0 \ \forall g \quad \Leftrightarrow \quad e_a f = 0 \quad \Leftrightarrow \quad f = \text{const. on leaves} \qquad (64)$$

Casimir functions - various characterizations

So we can equivalently characterize Casimir functions by any of the following properties:
They

- Poisson commute with all functions
- are constant on each symplectic leaf
- are conserved quantities for any Hamiltonian on $(M, \mathcal{P}, H)$
- generate vanishing Hamiltonian fields
- generate (as Hamiltonians) trivial Hamiltonian dynamics

Canonical coordinates on $(M, \mathcal{P})$

We already learned that adapted frame (e_a, e_i) helps to treat the "$\mathcal{D}_{\mathcal{P}}$-distribution stuff" of $(M, \mathcal{P})$.

Now, since the distribution is integrable,
we can choose (due to Frobenius theorem)
the (still adapted) frame to be coordinate (holonomic)

$$(e_a, e_i) = (\partial_a, \partial_i) \ \leftrightarrow \ (x^a, y^i) \quad \text{(local coordinates)} \tag{65}$$

Here

- x^a, $a = 1, \dots, 2m$, may be used as coordinates on leaves
- y^i, $i = 1, \dots, k$ label the leaves themselves
- Casimir functions depend on y^i alone

Canonical coordinates on $(M, \mathcal{P})$ (2)

Finally, since there is (canonical) symplectic structure on each leaf, we can further optimize coordinates x^a, $a = 1, \dots, 2m$ and come to canonical Darboux-type coordinates

$$x^a = (q^\mu, p_\mu) \qquad \mu = 1, \dots, m \tag{66}$$

So, altogether, we can use, on $(M, \mathcal{P})$, local canonical coordinates

$$(q^\mu, p_\mu, y^i) \qquad \mu = 1, \dots, m; \;\; i = 1, \dots, k \tag{67}$$

Here, again,

- (q^μ, p_μ), $\mu = 1, \dots, m$ may be used as coordinates on leaves
- y^i, $i = 1, \dots, k$ label the leaves themselves
- Casimir functions depend on y^i alone

Canonical coordinates on $(M, \mathcal{P})$ (3)

In canonical coordinates (q^μ, p_μ, y^i) important quantities acquire their "usual look":

$$\{f, g\} = \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial q^\mu} - \frac{\partial g}{\partial p_\mu}\frac{\partial f}{\partial q^\mu} \tag{68}$$

$$\zeta_f = \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial g}{\partial p_\mu}\frac{\partial}{\partial q^\mu} \tag{69}$$

Bear in mind, however, that the functions f, g live on $(M, \mathcal{P})$, so they, in general, also depend on Casimir part of the coordinates

$$f(q, p, y)\ ,\ \ g(q, p, y) \tag{70}$$

There is, nevertheless, no derivative w.r.t. y^i present in (68)-(69).

Canonical coordinate version of Hamilton equations

From (69) we see that Hamilton equations,
being the equations for integral curves
of the Hamiltonian field ζ_H,
look as follows:

$$\boxed{\dot{q}^\mu = \frac{\partial H}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu} \qquad \dot{y}^i = 0} \tag{71}$$

where

$$H(q, p, y) \qquad \mu = 1, \dots, m \ , \ \ i = 1, \dots, k \tag{72}$$

Common description of a distribution $\mathcal{D}$ on M

Both

- sub-Riemannian geometry (MF, Stará Lesná 2009, [3])
- Poisson geometry (MF, Stará Lesná 2020 :-)

use a $\binom{2}{0}$-type degenerate tensor field t
for description of a distribution $\mathcal{D}$
in terms of the image space of the tensor:

- $\mathcal{D} = \text{Im } t$
  where t is regarded as a linear map

$$t_x : T_x^* M \to T_x M \qquad \alpha \mapsto t(\alpha, \ . \ ) \qquad \alpha_\mu \mapsto \alpha_\nu t^{\nu\mu}$$

Common description of a distribution $\mathcal{D}$ on M (2)

In both cases, we proceed as follows: Fix, locally,

- an adapted frame (e_a, e_i),
  i.e. such that $e_a \in \mathcal{D}$
- and (e^a, e^i) - the dual coframe.

This means that

- e_a-part is directed "along" the distribution $\mathcal{D}$,
- e_i-part is "transversal" to the distribution $\mathcal{D}$.

This choice simplifies the description of $\mathcal{D}$ in terms of t a lot!

Common description of a distribution $\mathcal{D}$ on M (3)

Namely, write

$$t = t^{ab} e_a \otimes e_b + t^{ai} e_a \otimes e_i + t^{ia} e_i \otimes e_a + t^{ij} e_i \otimes e_j$$

i.e.

$$t \leftrightarrow \begin{pmatrix} t^{ab} & t^{ai} \\ t^{ia} & t^{ij} \end{pmatrix}$$

Then, for $\alpha = \alpha_a e^a + \alpha_i e^i$,

$$t(\alpha, \ . \ ) = \cdots = (\alpha_b t^{ba} + \alpha_i t^{ia}) e_a + (\alpha_a t^{ai} + \alpha_j t^{ji}) e_i$$

Since $\mathcal{D} = \text{Span } \{e_a\}$, we get $t^{ai} = 0 = t^{ji}$.

Two important special cases

Thus, so far,

$$t \leftrightarrow \begin{pmatrix} t^{ab} & 0 \\ t^{ia} & 0 \end{pmatrix} \tag{73}$$

Now, in two important special cases
we automatically, free of charge, get rid of t^{ia} as well:
Namely, when the tensor t is

- symmetric - co-metric h in sub-Riemannian geometry,
- skew-symmetric - Poisson tensor $\mathcal{P}$ in Poisson geometry.

Two important special cases (2)

We then actually have

$$h \leftrightarrow \begin{pmatrix} h^{ab} & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{P} \leftrightarrow \begin{pmatrix} \mathcal{P}^{ab} & 0 \\ 0 & 0 \end{pmatrix} \tag{74}$$

Or, equivalently,

$$h = h^{ab} e_a \otimes e_b \qquad h^{ab} = h^{ba} \tag{75}$$

$$\mathcal{P} = \mathcal{P}^{ab} e_a \otimes e_b \qquad \mathcal{P}^{ab} = -\mathcal{P}^{ba} \tag{76}$$

Induced structures in the distribution $\mathcal{D}$

Now both $\binom{2}{0}$-type tensors, h and $\mathcal{P}$,
canonically induce $\binom{0}{2}$-type tensors, g and ω.

However, only in the distribution $\mathcal{D}$!

The corresponding formulas share the same pattern:

$$g(h(\alpha,\ \cdot\ ), h(\beta,\ \cdot\ ) := h(\alpha,\beta) \qquad g_{ac}h^{cb} = \delta^b_a \tag{77}$$

$$\omega(\mathcal{P}(\alpha,\ \cdot\ ), \mathcal{P}(\beta,\ \cdot\ ) := \mathcal{P}(\alpha,\beta) \qquad \omega_{ac}\mathcal{P}^{cb} = -\delta^b_a \tag{78}$$

Induced structures in the distribution $\mathcal{D}$ (2)

So we can sum up comparison yet presented:
On the whole manifold M we have

- either $\mathcal{P}$ - the Poisson tensor (in Poisson geometry),
- or h - the co-metric (in sub-Riemannian geometry).

Only in the distribution we have, then,

- ω - the symplectic form (in Poisson geometry),
- g - the metric (in sub-Riemannian geometry).

Notice that a single (degenerate) tensor (h or $\mathcal{P}$, respectively), carries full information about
both $\mathcal{D}$ and g or ω, respectively.

Important difference between Poisson and sub-Riemannian

Perhaps the most important difference is that the distribution $\mathcal{D}$ is

- integrable in Poisson geometry
- non-integrable in sub-Riemannian geometry.

So, there are

- symplectic leaves in Poisson geometry, whereas
- no metric leaves in sub-Riemannian geometry.

Technically, integrability of "Poisson" distribution originates in Jacobi identity fulfilled by $\mathcal{P}$ (see (36)).
There is no counterpart assumed to hold by h.

Damned nice place to contemplate,

of course,
on Hamiltonian
systems;
especially
on those
with
degenerate Poisson
tensor.

Jahňací štít (Lamb Peak), August 26, 2020 (within The School :-)

For Further Reading (1)

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