Saint-Venant's compatibility condition and Einstein tensor

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- How Einstein tensor in 3D is related to Saint-Venant's operator
- Why, then, Saint-Venant's compatibility condition is true

Contents

Reminder - gradient vector fields in 3D

There is a well-known query in vector analysis:
Under what circumstances a given vector field v
happens to be a gradient field?

$$\text{when} \qquad v \stackrel{?}{=} \nabla\psi \qquad \text{i.e.} \qquad v_i \stackrel{?}{=} \partial_i\psi \tag{1}$$

And there is also a well-known (local) answer:

$$\boxed{v = \nabla\psi \qquad \Leftrightarrow \qquad \operatorname{curl} v = 0} \tag{2}$$

Reminder - gradient vector fields in 3D (2)

So, there is a differential operator on vector fields

$$\boxed{\text{curl} \equiv \nabla \times \qquad (\text{curl}\, v)_i = \epsilon_{ijk} \partial_j v_k} \tag{3}$$

with a key property that, when applied on v, the result

- being zero guaranties existence of potential ψ
- being non-zero obstructs existence of potential ψ.

Reminder - gradient vector fields in 3D (3)

There are two implications involved in a single equivalence (2).
It turns out that one of them is easy part
whereas the other hard part of the statement.

Easy part:

$$\begin{aligned} (\mathrm{curl}\,\nabla\psi)_i &= \epsilon_{ijk}\partial_j(\nabla\psi)_k \\ &= \epsilon_{ijk}\partial_j\partial_k\psi \\ &= \epsilon_{i[jk]}\partial_{(j}\partial_{k)}\psi \\ &= 0 \end{aligned}$$

Hard part: Poincaré lemma for differential forms (see [Fecko 2006]).

Reminder - strain tensor in linear elasticity

In linear elasticity, for a given displacement (vector) field u, we compute corresponding

strain tensor $$\boxed{\epsilon_{ij} := \partial_i u_j + \partial_j u_i} \qquad (4)$$

Its geometrical (and therefrom physical) meaning:

$$\epsilon_{ij} = 0 \qquad \Leftrightarrow \qquad \text{no deformation caused by } u \qquad (5)$$

Reminder - strain tensor in linear elasticity (2)

Notice that strain tensor is always symmetric second rank tensor

$$\boxed{\epsilon_{ij} = \epsilon_{ji}} \tag{6}$$

Therefore a natural query arises:

Under what circumstances
a given symmetric second-rank tensor h
happens to be a strain tensor for some displacement field u?

$$\text{when} \qquad h_{ij} \stackrel{?}{=} \partial_i u_j + \partial_j u_i \tag{7}$$

Adhémar Jean Claude Barré de Saint-Venant (1797 - 1886)

The answer to the query was found, in 1864, by French mechanician and mathematician Barré de Saint-Venant.

(The figure is from Wikipedia.)

Operator $\mathrm{curl}\,\mathrm{curl}$

In order to formulate Saint-Venant's result,
we need, first, to define
(2-nd order differential) operator
$\mathrm{curl}\,\mathrm{curl}$ on 2-nd rank symmetric tensors
(rather than on vectors).

It is given by formula

$$\boxed{(\mathrm{curl}\,\mathrm{curl}\,h)_{ij} := \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c h_{bd}} \tag{8}$$

(So as if "usual" curl was applied separately on both indices.)
This operator is also called Saint-Venant operator.

Symmetry of the result

Notice that operator $\operatorname{curl}\operatorname{curl}$ indeed returns
symmetric output tensor for
symmetric input tensor:

$$\begin{aligned}
(\operatorname{curl}\operatorname{curl} h)_{ji} &= \epsilon_{jab}\epsilon_{icd}\partial_a\partial_c h_{bd} \\
&= \epsilon_{icd}\epsilon_{jab}\partial_a\partial_c h_{bd} \\
&= \epsilon_{iab}\epsilon_{jcd}\partial_c\partial_a h_{db} \\
&= \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c h_{bd} \\
&= (\operatorname{curl}\operatorname{curl} h)_{ij}
\end{aligned}$$

Saint-Venant's compatibility condition

Now we are already prepared to understand
what Saint-Venant's compatibility condition claims.

Namely, it is the following statement:

For symmetric tensor ($h_{ij} = h_{ji}$),

$$\boxed{(\mathrm{curl}\,\mathrm{curl}\, h)_{ij} = 0 \quad \Leftrightarrow \quad h_{ij} = \partial_i u_j + \partial_j u_i} \tag{9}$$

Easy part of Saint-Venant's compatibility condition

There are two implications involved in a single equivalence (9). As it was the case for gradient fields, one of them is easy part whereas the other hard part of the statement.

Easy part:

$$\begin{aligned}(\mathrm{curl}\,\mathrm{curl}\,\epsilon)_{ij} &= \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c(\partial_b u_d + \partial_d u_b) \\ &= \epsilon_{i[ab]}\epsilon_{jcd}\partial_{(a}\partial_c\partial_{b)} u_d + \epsilon_{iab}\epsilon_{j[cd]}\partial_{(a}\partial_c\partial_{d)} u_b \\ &= 0 + 0 = 0\end{aligned}$$

Hard part: The main topic of this lecture :-)

Summing up

Let us recapitulate:

For vector fields

- some of them happen to be gradient fields
- it is so iff apllication of curl on it vanishes

For symmetric 2-nd rank tensor fields

- some of them happen to be strain tensor fields
- it is so iff apllication of $\text{curl}\,\text{curl}$ on it vanishes

Reminder: Symmetries of Riemann curvature tensor R_{ijkl}

Complete Riemann curvature tensor R_{ijkl}
has, apriori, n^4 components.
However, because of its (index) symmetries

$$R_{abcd} \overset{1.}{=} -R_{bacd} \quad (10)$$

$$\overset{2.}{=} -R_{abdc} \quad (11)$$

$$\overset{3.}{=} +R_{cdab} \quad (12)$$

$$R_{a[bcd]} \overset{4.}{=} 0 \quad (13)$$

(for Levi-Civita connection)

$$\text{number of independent components} = \frac{n^2(n^2-1)}{12} \quad (14)$$

So, as an example, it reduces to just 1 for $n = 2$ and to 6 for $n = 3$.

Warm up: Riemann curvature tensor in 2D

In 2D (see 15.6.10 in [Fecko2006]), the symmetries lead to parametrization (check!)

$$R_{abcd} = \epsilon_{ab}\epsilon_{cd}\, b \tag{15}$$

Computation of scalar curvature R gives

$$R = 2b \tag{16}$$

and therefore, finally,

$$\boxed{R_{abcd} = \frac{1}{2}\epsilon_{ab}\epsilon_{cd}\, R} \tag{17}$$

(Btw., $b = R/2$ is just the celebrated Gauss curvature K.)

Riemann curvature tensor in 3D

Particular case of 3D in not treated at all in [Fecko2006]!
So, it is indeed a truly awkward book!!
Shame on it!!!

Fortunately, we fill the gap in our education, here!!!!

Symmetries lead to parametrization (check!)

$$R_{abcd} = \epsilon_{abi}\epsilon_{cdj}B_{ij} \tag{18}$$

in terms of a (yet unknown) symmetric tensor B_{ij}

$$B_{ij} = B_{ji} \tag{19}$$

Riemann curvature tensor in 3D (2)

Computation of Ricci tensor R_{ij} and scalar curvature R gives

$$R_{ij} = \delta_{ij} B - B_{ij} \tag{20}$$
$$R = 2B \tag{21}$$

where

$$B := B_{jj} \tag{22}$$

We use components w.r.t. an orthonormal frame, as we already did in 2D. So $g_{ij} = \delta_{ij}$ and raising/lowering of indices is trivial.

Riemann curvature tensor in 3D (3)

From (20) and (21) we get

$$B_{ij} = \frac{1}{2} R g_{ij} - R_{ij} \equiv -G_{ij} \tag{23}$$

where

$$G_{ij} := R_{ij} - \frac{1}{2} R g_{ij} \qquad \textit{Einstein tensor} \tag{24}$$

So, the parametrization (18) actually reads

$$\boxed{R_{abcd} = -\epsilon_{abi}\epsilon_{cdj} G_{ij}} \tag{25}$$

Riemann curvature tensor in 3D (4)

So, in 3D, we can reconstruct
complete Riemann tensor from just Einstein tensor alone!

Because of both-directions way $G_{ij} \leftrightarrow R_{ij}$

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \tag{26}$$

$$R_{ij} = G_{ij} + \frac{1}{2-n} G g_{ij} \tag{27}$$

(for $n \neq 2$), Ricci can be obtained from Einstein
and, consequently, we can reconstruct
complete Riemann tensor from just Ricci tensor as well!

Summing up

Let us recapitulate:

In 2D, one can reconstruct

- Riemann tensor R_{abcd} from just scalar curvature R alone
- so just a single independent component in R_{abcd}

In 3D, one can reconstruct

- Riemann tensor R_{abcd} from just Einstein tensor G_{ab} alone
- Riemann tensor R_{abcd} from just Ricci tensor R_{ab} alone
- so just six (3(3+1)/2) independent components in R_{abcd}

Einstein tensor for $g_{ij} = \delta_{ij} + \epsilon h_{ij}$

Consider variation of flat metric

$$\boxed{\delta_{ij} \mapsto g_{ij} = \delta_{ij} + \epsilon h_{ij}} \tag{28}$$

What the corresponding variation of Einstein tensor looks like?

For flat metric δ, complete Riemann tensor vanishes,
so, consequently, Einstein tensor vanishes.

Therefore Einstein tensor for $g = \delta + \epsilon h$ is necessarily of the form

$$G_{ij}[\delta + \epsilon h] = G_{ij}[\delta] + \epsilon(\dots)_{ij} = \epsilon(\dots)_{ij} \tag{29}$$

We want to compute $(\dots)_{ij}$ explicitly.

Einstein tensor for $g_{ij} = \delta_{ij} + \epsilon h_{ij}$ (2)

Recall standard general formulas for computing ${R^i}_{jkl}$ for given g_{ij}:

$$\Gamma_{ijk}[g] = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) \tag{30}$$

$$\Gamma^i_{jk}[g] = g^{il}\Gamma_{ljk}[g] \tag{31}$$

$${R^i}_{jkl}[\Gamma] = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl}{\Gamma^i}_{mk} - \Gamma^m_{jk}{\Gamma^i}_{ml} \tag{32}$$

Here, we have, to first order in ϵ,

$$g_{ij} = \delta_{ij} + \epsilon h_{ij} \qquad \text{so that} \qquad g^{ij} = \delta_{ij} - \epsilon h_{ij} \tag{33}$$

Einstein tensor for $g_{ij} = \delta_{ij} + \epsilon h_{ij}$ (3)

Then, in our case (up to first order), step by step,

$$\Gamma^i_{jk}[g] = \Gamma_{ijk}[g] = \epsilon \Gamma_{ijk}[h] \tag{34}$$

$$R^i{}_{jkl}[g] \equiv R^i{}_{jkl}[\Gamma[g]] = \Gamma^i_{jl,k}[g] - \Gamma^i_{jk,l}[g] \tag{35}$$

$$\equiv 2\Gamma^i_{j[l,k]}[g] \equiv 2\epsilon \Gamma_{ij[l,k]}[h] \tag{36}$$

Using (30), we get the full Riemann tensor in the form

$$\boxed{R^i{}_{jkl}[\delta + \epsilon h] = R_{ijkl}[\delta + \epsilon h] = \epsilon(h_{i[l,k]j} - h_{j[l,k]i})} \tag{37}$$

Einstein tensor for $g_{ij} = \delta_{ij} + \epsilon h_{ij}$ (4)

Then, by contractions, we get Ricci tensor and scalar curvature as follows:

$$R_{ij}[\delta + \epsilon h] = \frac{1}{2}\epsilon(h_{ik,kj} + h_{jk,ki} - h_{kk,ij} - h_{ij,kk}) \quad (38)$$

$$R[\delta + \epsilon h] = \epsilon(h_{jk,jk} - h_{jj,kk}) \quad (39)$$

This means that Einstein tensor

$$G_{ij}[\delta + \epsilon h] \equiv R_{ij}[\delta + \epsilon h] + \frac{1}{2}R[\delta + \epsilon h](\delta_{ij} + \epsilon h_{ij}) \quad (40)$$

becomes a sum of six terms containing expressions of structure $h_{ij,kl}$ with various combinations of indices.

Einstein tensor for $g_{ij} = \delta_{ij} + \epsilon h_{ij}$ in 3D

One can check that the six terms
may be compactly written as follows:

$$G_{ij}[\delta + \epsilon h] = \epsilon \frac{1}{2} 3! \delta_a^{[b} \delta_i^k \delta_j^{l]} h_{jl,ik} \tag{41}$$

Because of the well-known identity (in 3D)

$$3! \delta_{aij}^{bkl} \equiv 3! \delta_a^{[b} \delta_i^k \delta_j^{l]} = \epsilon_{aij} \epsilon^{bkl} \tag{42}$$

this can be rewritten as

$$G_{ij}[\delta + \epsilon h] = \epsilon \frac{1}{2} \epsilon_{aij} \epsilon_{bkl} h_{jl,ik} \tag{43}$$

Saint-Venant operator on the scene again!

But notice that at the r.h.s. of (43) we can identify nothing but the good old Saint-Venant operator $\mathrm{curl}\,\mathrm{curl}$ from (8)!

$$\boxed{(\mathrm{curl}\,\mathrm{curl}\,h)_{ij} := \epsilon_{aij}\epsilon_{bkl}h_{jl,ik}} \tag{44}$$

This leads us to the most important formula of the presentation.

It is so important that it deserves a separate frame :-)

Saint-Venant operator and Einstein tensor

The most important formula of the presentation:

in 3D $$\boxed{G_{ab}(\delta + \epsilon h) = \epsilon \frac{1}{2}(\mathrm{curl}\,\mathrm{curl}\,h)_{ab}} \tag{45}$$

It reveals that (in 3D) the Saint-Venant operator is (by definition)

- the first variation w.r.t. metric tensor
- of Einstein tensor
- evaluated at flat (= Euclidean) metric.

Why the formula (45) is important for us

The formula (45) is important for us
because we can use

standard knowledge from "general relativity"
(mainly properties of flat space :-)

to prove easily the hard part
of the Saint-Venant compatibility condition!

Hard part of the Saint-Venant's compatibility condition

So we want to understand
why the hard part
of the Saint-Venant's compatibility condition,
i.e. the statement that,
for symmetric tensor ($h_{ij} = h_{ji}$),

$$\boxed{(\mathrm{curl}\,\mathrm{curl}\, h)_{ij} = 0 \quad \overset{?}{\Rightarrow} \quad h_{ij} = \partial_i u_j + \partial_j u_i} \tag{46}$$

holds.

Hard part of the Saint-Venant's compatibility condition (2)

Well, assume that, in 3D,
a symmetric tensor ($h_{ij} = h_{ji}$) satisfies

$$(\mathrm{curl}\,\mathrm{curl}\,h)_{ij} = 0 \tag{47}$$

Then
1. From (45)

$$G_{ab}(\delta + \epsilon h) = \epsilon \frac{1}{2} (\mathrm{curl}\,\mathrm{curl}\,h)_{ab} \tag{48}$$

we deduce that

$$\boxed{G_{ab}(\delta + \epsilon h) = 0} \tag{49}$$

Hard part of the Saint-Venant's compatibility condition (2)

2. From (25)

$$R_{abcd} = -\epsilon_{abi}\epsilon_{cdj} G_{ij} \tag{50}$$

we can deduce, then, that
complete Riemann tensor (for $g = \delta + \epsilon h$) vanishes

$$\boxed{R_{abcd}(\delta + \epsilon h) = 0} \tag{51}$$

3. Of course, the same is true for "unperturbed" Euclidean space

$$\boxed{R_{abcd}(\delta) = 0} \tag{52}$$

Hard part of the Saint-Venant's compatibility condition (3)

4. Folklore wisdom from "general relativity" says, that
if
- connection is metric
- connection is symmetric (torsion-free)
- Riemann tensor vanishes, $R_{abcd}(\Gamma(g)) = 0$

then
there exist adapted coordinates, say $z_a \leftrightarrow g$,
in which the metric is Euclidean

$$g = \delta_{ab} dz_a \otimes dz_b \tag{53}$$

Hard part of the Saint-Venant's compatibility condition (4)

5. Then **2**. and **3**. say, that

$$\delta \leftrightarrow x_a \qquad \delta + \epsilon h \leftrightarrow y_a \tag{54}$$

i.e.

$$\delta = \delta_{ab} dx_a \otimes dx_b \qquad \delta + \epsilon h = \delta_{ab} dy_a \otimes dy_b \tag{55}$$

(coordinates x_a are adapted to δ, y_a are adapted to $\delta + \epsilon h$).

Hard part of the Saint-Venant's compatibility condition (5)

6. Metric tensors δ and $\delta + \epsilon h$ are close to one another.
Therefore also corresponding adapted coordinates
are necessarily close to one another:

$$\boxed{y_a = x_a + \epsilon u_a(x) \qquad \text{for some functions } u_a(x)} \tag{56}$$

and, consequently,

$$dy_a = dx_a + \epsilon u_{a,c} dx_c \equiv (\delta_{ac} + \epsilon u_{a,c}) dx_c \tag{57}$$

Hard part of the Saint-Venant's compatibility condition (6)

7. Then

$$\begin{aligned}\delta + \epsilon h &= \delta_{ab} dy_a \otimes dy_b \\ &= \delta_{ab}(\delta_{ac} + \epsilon u_{a,c}) dx_c \otimes (\delta_{bd} + \epsilon u_{b,d}) dx_d \\ &= \delta_{ab} dx_a \otimes dx_b + \epsilon(u_{a,b} + u_{b,a}) dx_a \otimes dx_b \\ &= \delta + \epsilon(u_{a,b} + u_{b,a}) dx_a \otimes dx_b\end{aligned}$$

and therefore, finally,

$$h = (u_{a,b} + u_{b,a}) dx_a \otimes dx_b \qquad \text{i.e.} \qquad \boxed{h_{ab} = u_{a,b} + u_{b,a}} \quad (58)$$

This is exactly what Saint-Venant's compatibility condition says.

Damned nice place to contemplate

on Saint-Venant's compatibility condition;
in particular on how it is related to
Einstein tensor.

From Kôprovský štít, August 25, 2021 (a trip within The School :-)

From the trip

Kôprovský štít, ("Dill" Peak), August 25, 2021

On the peak, with Tomáš (left) and Dominik (middle). Pretty nice trip, indeed.

For Further Reading (1)

M. Eastwood.
Ricci curvature and the mechanics of solids.
Austral. Math. Soc. Gaz. 37 238–241 (2010)

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