Galilean and Carrollian invariant Hodge star operators

Marián Fecko

Department of Theoretical Physics
Comenius University in Bratislava
fecko@fmph.uniba.sk

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- Why (usual) $*$ does not exist in Galilean/Carrollian spacetimes
- How to construct (useful) analogs of $*$ there
- How the analogs work in Galilean/Carrollian electrodynamics

Contents

Standard Hodge star operator on (M, g, o)

Hodge star (duality) operator $*$ is a well-known linear map on forms

$$* : \Omega^p(M, g, o) \to \Omega^{n-p}(M, g, o) \qquad n = \dim M \qquad (1)$$

given in components as follows:

$$\boxed{(*\alpha)_{a \dots b} := \frac{1}{p!} \alpha_{c \dots d} \, \omega^{c \dots d}{}_{a \dots b}} \qquad (2)$$

where

$$\omega^{c \dots d}{}_{a \dots b} := g^{ce} \dots g^{df} \omega_{e \dots f a \dots b} \qquad (3)$$

Standard Hodge star operator on (M, g, o) (2)

So what is used for construction of the (standard) Hodge star is

- the metric volume form $\omega \equiv \omega_{g,o} \;\leftrightarrow\; \omega_{e \dots fa \dots b}$
- the cometric $g^{-1} \;\leftrightarrow\; g^{ab}$ (for raising of indices).

In terms of these objects, we can also write the operator in component-free way

$$\boxed{*_g \alpha \;\sim\; C \dots C(g^{-1} \otimes \dots \otimes g^{-1} \otimes \omega_g \otimes \alpha)} \tag{4}$$

where C denotes *contraction*.

Standard Hodge star operator on (M, g, o) (3)

Now, consider a diffeomorphism

$$f : M \to M \tag{5}$$

It is a folklore knowledge that (any) pull-back f^*

1. *commutes* with contractions
2. *preserves* tensor product

$$f^* C = C f^* \qquad f^*(a \otimes b) = f^* a \otimes f^* b \tag{6}$$

A bit less folklore knowledge says (still easy, see [2]), that $g \mapsto g^{-1}$ and $g \mapsto \omega_g$ are natural constructions, i.e.

$$f^* g^{-1} = (f^* g)^{-1} \qquad f^* \omega_g = \omega_{f^* g} \tag{7}$$

Standard Hodge star operator on (M, g, o) (4)

But then we can immediately see from (4) that the (standard) Hodge star is natural w.r.t. diffeomorphisms, i.e.

$$\boxed{f^*(*_g \alpha) = *_{f^*g}(f^*\alpha)} \tag{8}$$

And from this directly follows a key observation:

$$\boxed{f^*g = g \qquad \Rightarrow \qquad f^* *_g = *_g f^*} \tag{9}$$

In words: Hodge star operator is invariant w.r.t. isometries of g.

Standard Hodge star operator on (M, g, o) (5)

Recall that exterior derivative is invariant w.r.t. any diffeomorphism:

$$\boxed{f^*\, d = d\, f^*} \tag{10}$$

Then any combination of d and $*_g$ results in

- differential operator on forms, which is
- invariant w.r.t. isometries of g!

Standard Hodge star operator on (M, g, o) (6)

Notable general examples:

$$d \quad \text{itself} \qquad \text{differential} \tag{11}$$

$$\delta_g \sim *_g^{-1} d *_g \qquad \text{codifferential ("generalized divergence")} \tag{12}$$

$$\Delta_g \sim (\delta_g d + d\delta_g) \qquad \text{Laplace - de Rham operator} \tag{13}$$

Standard Hodge star operator on (M, g, o) (7)

Notable particular examples of notable general examples:

1. Euclidean space E^3: All relevant operators in vector calculus

$$\nabla \leftrightarrow \mathrm{grad} \qquad \nabla\times \leftrightarrow \mathrm{curl} \qquad \nabla\cdot \leftrightarrow \mathrm{div} \qquad \Delta \leftrightarrow \text{Laplace} \tag{14}$$

are translation and rotation invariant (isometries of E^3).

2. Minkowski space $E^{1,3}$: The popular operators

$$d \qquad d*_\eta \qquad \delta_\eta \qquad \Box_\eta \leftrightarrow \text{D'Alembert (wave) operator} \tag{15}$$

are Poincaré invariant (isometries of $E^{1,3}$).

Summing up (1)

Let us recapitulate:

- Often isometries of some g are physically important
- Like translations and rotations in common physics in E^3
- Or Poincaré transformations in special relativity

Then, if tensor fields are used as a mathematical tool,

- operators behaving naturally become of particular interest
- since they become invariant w.r.t. isometries
- i.e. invariant w.r.t. physically important transformations

Summing up (2)

In particular, if differential forms are used,

- exterior derivative d is invariant (w.r.t. any diffeomorphisms)
- Hodge star $*_g$ behaves naturally
- so the Hodge star $*_g$ becomes invariant w.r.t. isometries
- so invariant w.r.t. rotations and translations in E^3
- so invariant w.r.t. Poincaré transformations in special relativity

Galilean spacetime

Galilean spacetime - proper arena for non-relativistic physics.
Formally - an appropriate limit of Minkowski spacetime.
Galilean coordinates $(t, x, y, z) \equiv (x^0, x^1, x^2, x^3) \equiv (x^0, x^i) \leftrightarrow x^\mu$.

What does not change w.r.t. Minkowski spacetime:

- translations $x^\mu \mapsto x^\mu + k^\mu$ (incl. time transl.)
- spatial rotations $(t, \mathbf{r}) \mapsto (t, R\mathbf{r})$, $R^T R = \mathbb{I}$

What does change w.r.t. Minkowski spacetime:
- boosts

$$t' = t \qquad (\text{"universal time"}) \tag{16}$$

$$\mathbf{r}' = \mathbf{r} + \mathbf{v}t \tag{17}$$

Galilean generators

Galilean transformations close, w.r.t. compositions,
to Galilean group.
From explicit formulas one easily derives its 10 generators,
i.e. vector fields on Galilean spacetime,
whose flows are 1-parameter subgroups of the group.

They read:

$$\partial_t \ , \ \partial_i \quad = \quad \text{time and space translations} \tag{18}$$

$$\epsilon_{ijk} x^j \partial_k \quad = \quad \text{spatial rotations} \tag{19}$$

$$t\partial_i \quad = \quad \text{Galilean boosts} \tag{20}$$

They close, w.r.t. commutators, to Galilean Lie algebra.

Carrollian spacetime

Carrollian spacetime - another limit of Minkowski spacetime (J.M. Lévy Leblond, 1965).
Carrollian coordinates $(t, x, y, z) \equiv (x^0, x^1, x^2, x^3) \equiv (x^0, x^i) \leftrightarrow x^\mu$.

Still the same as for Minkowski and Galilean spacetime:

- translations $x^\mu \mapsto x^\mu + k^\mu$ (incl. time transl.)
- spatial rotations $(t, \mathbf{r}) \mapsto (t, R\mathbf{r})$, $R^T R = \mathbb{I}$

What does change w.r.t. Minkowski as well as Galilean spacetimes:
- boosts

$$t' = t + \mathbf{v} \cdot \mathbf{r} \tag{21}$$

$$\mathbf{r}' = \mathbf{r} \tag{22}$$

Carrollian generators

Carrollian transformations close, w.r.t. compositions,
to Carrollian group.
From explicit formulas one derives again its 10 generators.

They read:

$$\partial_t \ , \ \partial_i \quad = \quad \text{time and space translations} \tag{23}$$

$$\epsilon_{ijk} x^j \partial_k \quad = \quad \text{spatial rotations} \tag{24}$$

$$x^i \partial_t \quad = \quad \text{Carrollian boosts} \tag{25}$$

They close, w.r.t. commutators, to Carrollian Lie algebra.

Poincaré, Galilean and Carrollian generators

We can display all 10 generators of the 3 Lie algebras (Poincaré, Galilean and Carrollian):

Common:

$$\partial_t \ , \ \partial_i \quad = \quad \text{time and space translations} \tag{26}$$

$$\epsilon_{ijk} x^j \partial_k \quad = \quad \text{spatial rotations} \tag{27}$$

Specific:

$$x^i \partial_t + t\partial_i \quad = \quad \text{Lorentzian boosts} \tag{28}$$

$$t\partial_i \quad = \quad \text{Galilean boosts} \tag{29}$$

$$x^i \partial_t \quad = \quad \text{Carrollian boosts} \tag{30}$$

They all close, w.r.t. commutators, to corresponding Lie algebras.

Invariant metric tensor - Minkowski space

The great insight of Minkowski (1908) is that
there exists a metric tensor on spacetime
whose isometries coincide with
crucial special relativistic transformations:

$$f : x \mapsto \Lambda x + a \qquad f^* g = g \qquad g = \eta_{\mu\nu} dx^\mu \otimes dx^\nu \tag{31}$$

(So g is invariant w.r.t. physically crucial transformations.)

The natural questions then arise:
Is this also the case for Galilean (Carrollian) transformations?

Invariant metric tensors - Galilean and Carrollian spacetimes

Short and clear answer is: NO.

In slightly more words:

1. There is NO interesting metric tensor on Galilean spacetime, i.e. such which is invariant w.r.t. Galilean transformations.

2. There is NO interesting metric tensor on Carrollian spacetime, i.e. such which is invariant w.r.t. Carrollian transformations.

Neither Galilean nor Carrollian transformations are isometries.

Invariant Hodge stars - Galilean and Carrollian spacetimes

Direct consequence:

1. There is NO interesting Hodge star on Galilean spacetime,
i.e. such which is invariant w.r.t. Galilean transformations.

2. There is NO interesting Hodge star on Carrollian spacetime,
i.e. such which is invariant w.r.t. Carrollian transformations.

This is a bad news for those who plan to use differential forms
on Galilean and Carrollian spacetimes,
since without Hodge star
the number of interesting operators on forms is too limited.

Galilean and Carrollian invariant analogs of Hodge star

The good news is that one can easily construct
analogs of Hodge star operators,
which may become almost as useful
as the full fledged Hodge star is.

One way to achieve this is simply
substituting invariant metric tensor with other invariant tensors,
which are available on Galilean or Carrollian spacetimes.

And those "other invariant tensors" are well known for a long time!

(Another way to achieve this - see ([3]).)

Needed Galilean invariant tensors

Two Galilean invariant tensors are needed:

1. Volume form

$$\omega = dt \wedge dx \wedge dy \wedge dz \tag{32}$$

2. $\binom{2}{0}$-type (degenerate!) tensor

$$h = h^{\mu\nu}\partial_\mu \otimes \partial_\nu = \delta^{ij}\partial_i \otimes \partial_j \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix} \tag{33}$$

One easily checks their Galilean invariance:

$$\mathcal{L}_\xi \omega = 0 \qquad \mathcal{L}_\xi h = 0 \tag{34}$$

for $\xi =$ any of the 10 Galilean generators.

Needed Carrollian invariant tensors

Two Carrollian invariant tensors are needed:

1. $\binom{4}{0}$-type "volume form"

$$\tilde{\omega} = \partial_t \wedge \partial_x \wedge \partial_y \wedge \partial_z \tag{35}$$

2. $\binom{0}{2}$-type (degenerate!) tensor

$$\tilde{h} = \tilde{h}_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{ij} dx^i \otimes dx^j \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \tag{36}$$

One easily checks their Carrollian invariance:

$$\mathcal{L}_\xi \tilde{\omega} = 0 \qquad \mathcal{L}_\xi \tilde{h} = 0 \tag{37}$$

for $\xi =$ any of the 10 Carrollian generators.

Galilean and Carrollian invariant (analogs of) Hodge $*$

With the help of the above mentioned tensors,
one can add to the original Hodge star $*_g$ from (4)
two analogs of the latter, $*_{\omega,h}$ and $*_{\tilde{\omega},\tilde{h}}$:

$$*_g\,\alpha \sim C \dots C(g^{-1} \otimes \dots \otimes g^{-1} \otimes \omega_g \otimes \alpha) \qquad (38)$$

$$*_{\omega,h}\,\alpha \sim C \dots C(h \otimes \dots \otimes h \otimes \omega \otimes \alpha) \qquad (39)$$

$$*_{\tilde{\omega},\tilde{h}}\,\alpha \sim C \dots C(\tilde{h} \otimes \dots \otimes \tilde{h} \otimes \tilde{\omega} \otimes \alpha) \qquad (40)$$

Then, directly from the construction,
- $*_g$ is Poincaré invariant and
- $*_{\omega,h}$ is Galilean invariant and
- $*_{\tilde{\omega},\tilde{h}}$ is Carrollian invariant.

Galilean and Carrollian invariant (analogs of) Hodge $*$ (2)

Another way to communicate the same idea
is to remind the expression

$$(*\alpha)_{a\dots b} := \frac{1}{p!}\alpha_{c\dots d}\ \omega^{c\dots d}{}_{a\dots b} \tag{41}$$

and display detailed explicit formulas
giving the crucial mixed tensor:

$$\omega^{c\dots d}{}_{a\dots b} = (g^{-1})^{ce}\dots(g^{-1})^{df}(\omega_g)_{e\dots fa\dots b} \qquad \text{(standard)} \tag{42}$$
$$= h^{ce}\dots h^{df}\omega_{e\dots fa\dots b} \qquad \text{(Galilean)} \tag{43}$$
$$= \tilde{h}_{ae}\dots\tilde{h}_{bf}\tilde{\omega}^{c\dots de\dots f} \qquad \text{(Carrollian)} \tag{44}$$

Action of the three Hodge star operators on forms

In all three spacetimes, any p-form α, $p = 0, 1, 2, 3, 4$, may be uniquely decomposed as follows:

$$\alpha = dt \wedge \hat{s} + \hat{r} \tag{45}$$

where the two hatted forms $(\hat{s}, \hat{r})$ are spatial (no dt present). Explicit computation of the three Hodge stars leads to

$$* \, (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \qquad \textit{Minkowski} \text{ Hodge star} \tag{46}$$

$$* \, (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{*}\hat{r} \qquad \textit{Galilean} \text{ Hodge star} \tag{47}$$

$$* \, (dt \wedge \hat{s} + \hat{r}) = \hat{*}\hat{\eta}\hat{s} \qquad \textit{Carrollian} \text{ Hodge star} \tag{48}$$

Here $\hat{*}$ stands for standard Euclidean Hodge star and $\hat{\eta}$ produces just ± 1.

Action of the three Hodge star operators on F

Important example:
For 2-form of electromagnetic field F

$$F = dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S} \tag{49}$$

we get

$$*_M (dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = dt \wedge (-\mathbf{B}) \cdot d\mathbf{r} - \mathbf{E} \cdot d\mathbf{S} \tag{50}$$

$$*_G (dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = dt \wedge (-\mathbf{B}) \cdot d\mathbf{r} \tag{51}$$

$$*_C (dt \wedge \mathbf{E} \cdot d\mathbf{r} - \mathbf{B} \cdot d\mathbf{S}) = -\mathbf{E} \cdot d\mathbf{S} \tag{52}$$

Action of the three Hodge star operators on $(\mathbf{E}, \mathbf{B})$

So, effectively, in terms of electric and magnetic fields $(\mathbf{E}, \mathbf{B})$, this reads

$$*_M \; : \; (\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{E}) \tag{53}$$
$$*_G \; : \; (\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{0}) \tag{54}$$
$$*_C \; : \; (\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{0}, \mathbf{E}) \tag{55}$$

The $*_M$ is known to be a duality
(when applied twice, we essentially come back to the original).

It is clear from (54) and (55) that we can
no longer speak of Galilean and Carrollian Hodge duality!

(Free) Maxwell equations on Minkowski spacetime (R^4, η, o)

$$\boxed{d *_\eta F = 0 \qquad dF = 0} \qquad \text{Maxwell equations on } (R^4, \eta, o) \tag{56}$$

Here isometries are just Poincaré transformations

$$f : R^4 \to R^4 \qquad x \mapsto \Lambda x + a \qquad \Lambda^T \eta \Lambda = \eta \tag{57}$$

Then

$$d *_\eta (f^* F) = 0 \qquad d(f^* F) = 0 \tag{58}$$

Equations (56) are invariant w.r.t. Poincaré transformations.

(Free) Maxwell equations on (M, g, o)

$$\boxed{d *_g F = 0 \qquad dF = 0} \qquad \text{Maxwell equations on } (M, g, o) \tag{59}$$

Let

$$f : M \to M \qquad f^* g = g \qquad \text{isometry of } (M, g, o) \tag{60}$$

Then (just apply f^* on (59))

$$d *_g (f^* F) = 0 \qquad d(f^* F) = 0 \tag{61}$$

Equations (59) are invariant w.r.t. isometries of (M, g, o)

Maxwell equations - $(\mathbf{E}, \mathbf{B})$ language

It is clear, that the equations

$$d * F = 0 \qquad dF = 0 \tag{62}$$

represent, just because of properties of d and $*$,

- a system of 1-st order partial differential equations
- for the fields **E** and **B**
- which is general isometry, Poincaré, Galilean or Carrollian invariant
- depending on what $*$ is actually there ($*_g$, $*_M$, $*_G$ or $*_C$)

We can call all of them Maxwell equations
in the corresponding versions of electrodynamics.

Maxwell equations - $(\mathbf{E}, \mathbf{B})$ language (2)

Explicitly, we get the following lists:

Minkowski	Galilean	Carrollian	(63)
$\operatorname{div} \mathbf{E} = 0$		$\operatorname{div} \mathbf{E} = 0$	(64)
$\operatorname{curl} \mathbf{B} - \partial_t \mathbf{E} = 0$	$\operatorname{curl} \mathbf{B} = 0$	$\partial_t \mathbf{E} = 0$	(65)
$\operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0$	$\operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0$	$\operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0$	(66)
$\operatorname{div} \mathbf{B} = 0$	$\operatorname{div} \mathbf{B} = 0$	$\operatorname{div} \mathbf{B} = 0$	(67)

This coincides
(except for mysterious missing of the Gauss law in Galilean case :-(
with standard references, see
any textbook on electrodynamics (for Minkowski case)
Le Bellac and Levy-Leblond 1973 (for Galilean case)
Duval, Gibbons, Horvathy, Zhang 2014 (for Carrollian case).

Damned nice place to contemplate

on
Galilean and Carrollian spacetimes;
in particular on how analogs of
Hodge star operator
might be introduced on them.

Near saddleback Prielom, August 25, 2022 (The School's free day :-)

For Further Reading (1)

M. Fecko.
Differential geometry and Lie groups for physicists.
Cambridge University Press 2006 (paperback 2011)

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Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time.
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For Further Reading (3)

D. Hansen
Beyond Lorentzian Physics.
PhD Thesis, ETH Zurich, 267 pp. (2021)

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Beyond Lorentzian Physics.
PhD Thesis, ETH Zurich, 267 pp. (2021)

R. Grassie
Beyond Lorentzian Symmetry.
PhD Thesis, University of Edinburgh, 204 pp. (2021)

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