MATHEMATICAL PHYSICS
Syllabus of an undergraduate course at FMFI Comenius University in Bratislava
Assoc.Prof. Marián Fecko, PhD.
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The aim and the **target group** of the course:
The goal of the course (4h of lectures plus 2h of recitetions per week) is to provide students with the basics of analysis on manifolds (differential geometry) and Lie groups and their representations. It focuses on **aspects** and illustrative examples that are **relevant to physicists**. Its level and prerequisites are set to be understandable to any **motivated** interested person, who has completed basic mathematical courses (mainly **linear algebra** and calculus). Since all physicists take these courses, here, the course is not intended only for theoretical physicists. Its ambition is to teach students to actually work with the material (calculate).

Week 1: Linear algebra of tensors (multilinear algebra)

The dual space and the dual basis
Tensor of type $\binom{r}{s}$, its components and their transformation under change of basis
Tensor operations (linear combinations, tensor product, contractions, (anti)symmetrizations)
Metric tensor, raising and lowering indices

Week 2: Basic matrix Lie groups

Symmetries (automorphisms) of various structures and groups (plus examples)
Matrix Lie groups - $GL(n,\mathbb{R})$, $O(n)$, $SO(n)$, $SL(n)$, $U(n)$, $SU(n)$
Affine group $GA(n,\mathbb{R})$ and its embedding into matrix group $GL(n+1,\mathbb{R})$
Euclidean and Poincaré group
Direct product of groups

Week 3: The concept of a manifold

What is a manifold, mapping of manifolds, description of smooth surfaces in $\mathbb{R}^n$
A manifold defined implicitly (via constrains) and parametrically
A curve and a function on M and their coordinate presentations
The concept of a Lie group (combination of the concepts of a group and a manifold)

Week 4: Lie algebras

Lie algebra (as such, with no relation to Lie groups), examples
The Lie algebra $\mathcal{G}$ of a Lie group G
Matrix Lie algebras $gl(n,\mathbb{R})$, $o(n)$, $so(n)$, $sl(n)$ $u(n)$, $su(n)$ (derivation from groups, dimensions, bases)
Direct sum of Lie algebras
One-parameter subgroups and exponential mapping $\mathcal{G} \to G$, explicitly for $SO(2), SO(3), U(1), SU(2)$
Derived homomorphism $f' : \mathcal{G} \to \mathcal{H}$ of Lie algebras to a homomorphism $f : G \to H$ of Lie groups

Week 5: Representations of groups and Lie algebras

Representation ρ of a group G in V, derived representation ρ' of the Lie algebra $\mathcal{G}$ in V
Conjugation, Ad-representation of G and ad-representation of $\mathcal{G}$
Irreducible, reducible and completely reducible representations
Intertwining operator for two representations and the Schur's lemmas
Invariant scalar product, Killing-Cartan form and its Ad-invariance, semi-simple algebra
Tensor product $\rho_1 \otimes \rho_2$ and sum $\rho_1 \oplus \rho_2$ of representations of groups

Week 6: Vector fields on manifolds

The tangent space, vectors and vector fields, various equivalent definitions
The coordinate basis ∂_i in the tangent space
Vector field on M and its decomposition w.r.t. a coordinate basis
Transformation of components under change of coordinates

Integral curves of a vector field, equations for finding them, local flow of a field V

Week 7: Tensor fields on manifolds

The gradient df as a covector field, coordinate basis dx^i for covectors dual to ∂_i for vectors
Tensor field of type $\binom{r}{s}$, its components and their transformation under change of coordinates
Frame and coframe field, frame components of tensor fields

Week 8: Metric tensor on a manifold (Riemannian manifolds)

Orthogonal coordinates and orthonormal frame and co-frame fields
Metric tensor and kinetic energy in first-year mechanics
The length functional for a curve, geodesics as the shortest paths
The gradient ∇f as a vector field

Week 9: Mapping of tensors induced by mapping of manifolds

Push-forward f_* of a mapping f, its properties and coordinate expression
Pull-back f^* of a mapping f, its properties and coordinate expression
Induced metric tensor
Kinetic and potential energy in Lagrangian (= second-year) mechanics

Week 10: Lie derivative

Lie transport, Lie derivative and its component expression
General properties of Lie derivative
Exponent of Lie derivative
Commutator $[V, W]$ as a measure of non-closure of parallelogram from integral curves
Holonomic and non-holonomic (co)frame fields

Week 11: Killing equations

Preservation of lengths of curves and isometries of a Riemannian manifolds
Killing equations as conditions for infinitesimal isometries, their component expression
Killing vectors for E^n, translations and rotations (Euclidean group)
Killing vectors for $E^{r,s}$, translations, rotations and hyperbolic rotations (Poincaré group)
Tensor fields (e.g. metric tensor) with prescribed symmetry
Preservation of angles and conformal transformations, conformal Killing vectors

Week 12: G-space M, representation on functions on it

Action of a group on a set (right and left), orbit, stabilizer, G-space, homogeneous space; examples
Representation of G in the space of functions on G-space and its derived representation
Regular representation (in the space of functions on G)
Fundamental fields (generators) of an action of G on a manifold M, their properties
Fundamental fields for rotations and translations in E^3 and their relation to operators of angular momentum and momentum in quantum mechanics

Week 13: Homogeneous spaces and coverings

Homogeneous spacer G/H and its relation to "general" homogeneous spaces
The sphere S^2 (and in general S^n) as a homogeneous space (which G and H in G/H, ...)
Relation of Lie algebras and representations of groups, if one group covers another
Covering $\pi : SU(2) \to SO(3)$, topology of rotation group, non-equivalent loops
Normal subgroup and G/H as a group; homomorphism theorem

Reference :

M.Fecko: Differential geometry and Lie groups for physicists, Cambridge University Press, 2006