SYLLABUS
for optional post-graduate course
MATHEMATICAL METHODS OF THEORETICAL PHYSICS
winter term 2025/2026
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Some geometry around hydrodynamics

1. Integral invariants (what Poincaré and what Cartan)
2. Hydrodynamic equations in terms of differential forms
3. Kelvin's theorem on circulation and behavior of vortex lines: Helmholtz quickly and easily

Vector analysis in E^2

1. Two canonical ways of identifying vector fields with 1-forms
2. Two new (in comparison with E^3) differential operations
3. What results from $\nabla \cdot \mathbf{v} = 0$, here ("stream function" in 2d-hydrodynamics)

Energy-momentum tensor

1. Its appearance via variation of action w.r.t. metric tensor and its property $T^{\mu\nu}{}_{;\nu} = 0$
2. Closedness of $(n-1)$-form (usually 3-form) $*T(\xi,\ \cdot\ ) \equiv \xi_\mu T^{\mu\nu} d\Sigma_\nu$ (for Killing vector ξ)
3. Conservation law in terms of the 3-form
4. Interpretation of components $T^{\mu\nu}$ (from there, based on various Killings)
5. An effective algorithm for computation of T_{ab} from a term of the form $\langle\alpha, \alpha\rangle$ in action
6. Vanishing of its trace ($T^a_a = 0$) and conformal invariance of the theory

Nöther's theorem

1. How symmetry of action leads to a closed $(n-1)$-form
2. How energy-momentum tensor and the (already known) 3-form $*T(\xi,\ \cdot\ )$ appears here
3. How theory of symmetries of Hamiltonian systems (already known as well) appears here

Some geometry around Kaluza-Klein theory

1. Metric tensor g on P from metric tensor h on M and G-connection on P
2. Linear (RLC) connection on (P, g)
3. Geodesics on (P, g) and how we regard them on M (as a motion of charged particle)
4. Scalar curvature on (P, g) and how we regard it on M (both gauge fields *and* gravitation)

Some geometry around Einstein-Cartan formulation of gravitation

1. Hilbert's action as a functional of g and its presentation via curvature forms and co-frame field
2. Cartan's action as a functional of e^a and ω_{ab} (as independent variables)
3. Derivation of equations of motion via its variation, appearance of torsion (when and why)

Some geometry around Cartan's formulation of the Newtonian gravity

1. Newtonian space-time according to Cartan (1923) - derivation of appropriate connection
2. Cartan connection and (secondary school) free fall, parabolas as geodesics

Cohomologies of Lie algebras (Chevalley-Eilenberg)

1. Complex associated to $(\mathcal{G}, \rho, V)$
2. deRham complex as a particular case
3. $\rho = \mathrm{ad}$ a $\rho = 0$ as particular cases

Some geometry around Lie algebroids

1. Basic concepts (anchor on points and on sections, how a function gets out from commutator, ...)
2. Examples (TM and $\mathcal{G}$ as LA, LA for int.distribution, action LA for $\mathcal{G}$ on M, Poisson LA for (T^*M, π))
3. d_A and cohomologies of LA
4. Schouten-Nijenhuis bracket and Gerstenhaber bracket

Quantum dynamics as a ("classical") Hamiltonian system

1. Rewriting of the Schrödinger equation (on $\mathbb{C}^n$) as Hamiltonian equations with $H(z) = \langle z|\hat{H}|z\rangle \equiv \bar{E}$
2. Kähler structure (J, ω, g) on $\mathbb{C}^n$
3. New element: Length $||\zeta_f|| \equiv \sqrt{g(\zeta_f, \zeta_f)}$ (magnitude of the speed) of a Hamiltonian field ζ_f
4. Transition to the manifold of rays $\mathbb{C}P^n$: Here $||\zeta_f|| = \Delta E$ holds (so stationary states do not move)
5. Time-energy uncertainty relations $\Delta E \Delta t \geq 1$ (from this perspective)

A bit of supergeometry

1. Supermanifolds and supermaps (even and odd elements ...)
2. Bijection $A \to \{B \to C\}$ versus $(A \times B) \to C$
3. Bijection $\mathbb{R}^{0|1} \to M$ versus ΠTM ("points of" ΠTM as maps)
4. Differential forms on M as functions on ΠTM
5. Canonical vector fields Deg and Q on ΠTM and their role in the language of forms on M
6. Mapping $V \mapsto \theta \cdot V$ (action of functions on $\mathbb{R}^{0|1}$ on vector fields on ΠTM)
7. Lifts $V \mapsto \tilde{V}$ and $V \mapsto V^{\uparrow}$, identity $V^{\uparrow}Q + QV^{\uparrow} = \tilde{V}$ versus Cartan's identity $i_V d + di_V = \mathcal{L}_V$
8. Mappings $\mathbb{R}^{0|n} \to M$ as a supermanifold and differential worms

References:

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4. M.Dubois-Violette, J.Madore: Conservation Laws and Integrability Conditions for Gravitational and Yang-Mills Field Equations, Commun. Math. Phys. 108, (1987) 213-223
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6. P.Bóna: Extended Quantum Mechanics, Acta Physica Slovaca 50, No.1, 1 - 198 (2000) (arXiv: math-ph/9909022)
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