HAWKINGOV JAV

- HAWKING: PARTICLE CREATION BY BLACK HOLES
  Commun. math. Phys. 43, 199-220 (1975)
- JACOBSON: arXiv:gr-qc/0308048
- MUKHANOV, WINITZKI: INTRODUCTION TO QUANTUM FIELDS IN CLASSICAL BACKGROUNDS

KVANTOVANIE:

HARM. OSCILÁTOR $\longrightarrow$ QFT

OSCILÁTOR S $\omega(t)$ $\longrightarrow$ QFT NA ZAKRIVENOM POZADÍ

OSCILÁTOR: $\boxed{m=1,\ \hbar=1}$

$$S=\int dt\left(\frac{1}{2}\dot{q}^2-\frac{1}{2}\omega(t)^2q^2\right)\longrightarrow \ddot{q}+\omega(t)^2q=0$$

$$H=\frac{p^2}{2}+\frac{1}{2}\omega(t)^2q^2\longrightarrow \dot{q}=p \qquad \dot{p}=-\omega(t)^2q$$

ZVYŠOVACIE/ZNIŽOVACIE OPERÁTORY

$$\hat{a}=\sqrt{\frac{\omega(t)}{2}}\left(\hat{q}+\frac{i}{\omega(t)}\hat{p}\right) \qquad \hat{q}=\frac{1}{\sqrt{2\omega(t)}}\left(\hat{a}+\hat{a}^+\right) \qquad [\hat{q},\hat{p}]=i$$

$$\hat{a}^+=\sqrt{\frac{\omega(t)}{2}}\left(\hat{q}-\frac{i}{\omega(t)}\hat{p}\right) \qquad \hat{p}=i\sqrt{\frac{\omega(t)}{2}}\left(\hat{a}^+-\hat{a}\right) \qquad [\hat{a},\hat{a}^+]=1$$

$$\hat{H}=\omega(t)\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)=\omega(t)\left[\frac{\omega(t)}{2}\left(\hat{q}^2+\frac{i}{\omega(t)}[\hat{q},\hat{p}]+\frac{\hat{p}^2}{\omega(t)^2}\right)+\frac{1}{2}\right]=$$

$$=\frac{1}{2}\omega(t)^2\hat{q}^2+\frac{\hat{p}^2}{2}$$

$$\hat{H}=\omega(t)\left(\hat{N}+\frac{1}{2}\right) \qquad \hat{N}=\hat{a}^+\hat{a}\leftarrow \text{OPERÁTOR EN. HLADINY}$$

BÁZA V HILBERTOVOM PRIESTORE $|n\rangle$ $\qquad n=0,1,2,3,\dots$

$$\hat{a}|0\rangle=0 \qquad \hat{a}^+|n\rangle=\sqrt{n+1}\,|n+1\rangle$$

$$|n\rangle=\frac{1}{\sqrt{n!}}\left(\hat{a}^+\right)^n|0\rangle \qquad \langle n|m\rangle=\delta_{nm}$$

PROBLÉM S PREMENNÝM ω

$$\dot{\hat{a}} = \frac{1}{2}\frac{\dot\omega}{\sqrt{\omega}}\frac{1}{\sqrt{2}}\left(\hat q + \frac{i}{\omega}\hat p\right) + \sqrt{\frac{\omega}{2}}\left(\dot{\hat q} - \frac{i\dot\omega}{\omega^2}\hat p + \frac{i}{\omega}\dot{\hat p}\right) =$$

$$= \frac{1}{2}\frac{\dot\omega}{\omega}\sqrt{\frac{\omega}{2}}\left(\hat q + \frac{i}{\omega}\hat p + 2\frac{\omega}{\dot\omega}\sqrt{\frac{2}{\omega}}\sqrt{\frac{\omega}{2}}\frac{-i\dot\omega}{\omega^2}\hat p\right) +$$

$$+\sqrt{\frac{\omega}{2}}\left(\hat p + \frac{i}{\omega}(-\omega^2 q)\right) =$$

$$= \frac{1}{2}\frac{\dot\omega}{\omega}\sqrt{\frac{\omega}{2}}\left(\hat q - \frac{i}{\omega}\hat p\right) - i\omega\sqrt{\frac{\omega}{2}}\left(\frac{i}{\omega}\hat p + \hat q\right) = -i\omega\hat a + \frac{1}{2}\frac{\dot\omega}{\omega}\hat a^+$$

$\Rightarrow \dot{\hat a}|0\rangle = \frac{1}{2}\frac{\dot\omega}{\omega}|1\rangle$ TO JE V SPORE S:

$$\hat a|0\rangle = 0 \Rightarrow \frac{d}{dt}\hat a|0\rangle = \frac{d}{dt}0 = 0$$

TENTO PROBLÉM VYRIEŠIME NESKÔR ZAVEDENÍM BOGOLIUBOVOVEJ TRANSFORMÁCIE

AK $\dot\omega = 0$ TAK $\dot{\hat a} = -i\omega\hat a \rightarrow \hat a = \hat a(0)e^{-i\omega t}$
ODTERAZ KEĎ BUDEME PÍSAŤ $\hat a$ A $\hat a^+$ TAK BUDEME MAŤ NA MYSLI $\hat a(0)$ A $\hat a^+(0)$ POTOM

$$\hat q = \frac{1}{\sqrt{2\omega}}\left(\hat a e^{-i\omega t} + \hat a^+ e^{i\omega t}\right)$$

$\hat a$ A $\hat a^+$ SÚ INT. KONŠT. POVÝŠENÉ NA OPERÁTORY

AK $\dot\omega \neq 0$ TAK

$$\hat q = v(t)\hat a + v^*(t)\hat a^+$$

KDE $v(t)$ A $v^*(t)$ SÚ NEZÁVISLÉ RIEŠENIA $\ddot q + \omega(t)^2 q = 0$

HERMITOVSKOSŤ $\hat q \longleftrightarrow v^*$ JE KOMPL. ZDRUŽ. K U v

NEZÁVISLOSŤ v A $v^* \longleftrightarrow$ NENULOVOSŤ WRONSKIÁNU

$$W(v, v^*) = \begin{vmatrix} \dot v & v \\ \dot v^* & v^* \end{vmatrix} = \dot v v^* - v\dot v^* \neq 0$$

$$\dot W = \ddot v v^* + \dot v \dot v^* - \dot v\dot v^* - v\ddot v^* \overset{P.R.}{=} -\omega^2 v v^* - (-\omega^2 v^*)v = 0$$

$W^* = -W \Rightarrow W$ – RÝDZO IMAGINÁRNE

$$i = [\hat{q}, \hat{p}] = [v\hat{a} + v^*\hat{a}^+, \dot{v}\hat{a} + \dot{v}^*\hat{a}^+] =$$

$$= v\dot{v}^* \underbrace{[\hat{a}, \hat{a}^+]}_{1} + v^*\dot{v}\underbrace{[\hat{a}^+, \hat{a}]}_{-1} = -W \to W(v, v^*) = -i$$

ZAVEĎME ~~SKALÁRNY~~ SÚČIN FUNKCIÍ (t):

$$\langle f, g\rangle = i W(f, g^*) = -i(f\dot{g}^* - \dot{f}g^*) \overset{\text{OZN.}}{=} -i f \overleftrightarrow{\partial_t} g^*$$

PRE MÓDY v A v^*:

$$\langle v, v\rangle = i W(v, v^*) = 1$$

$$\langle v^*, v^*\rangle = i W(v^*, v) = -i W(v, v^*) = -1$$

$$\langle v, v^*\rangle = i W(v, v) = 0$$

VŠEOBECNÉ VLASTNOSTI SÚČINU

$$\langle \alpha f, \beta g\rangle = \alpha\beta^*\langle f, g\rangle, \quad \langle f, g\rangle^* = \langle g, f\rangle, \quad \langle f, f\rangle = -\langle f^*, f^*\rangle$$

$$\ddot{f} + \omega^2 f = 0 \ \& \ \ddot{g} + \omega^2 g = 0 \Rightarrow \frac{d}{dt}\langle f, g\rangle = 0$$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2 = \frac{1}{2}(\dot{v}\hat{a} + \dot{v}^*\hat{a}^+)^2 + \frac{1}{2}\omega^2(v\hat{a} + v^*\hat{a}^+)^2 =$$

$$= \frac{1}{2}\left[\dot{v}^2\hat{a}^2 + \dot{v}^{*2}\hat{a}^{+2} + |\dot{v}|^2\underbrace{(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})}\right] +$$

$$+ \frac{1}{2}\omega^2\left[v^2\hat{a}^2 + v^{*2}\hat{a}^{+2} + |v|^2\overbrace{(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})}^{2\hat{a}^+\hat{a} + \hat{1}}\right] =$$

$$= \frac{1}{2}(\dot{v}^2 + \omega^2 v^2)\hat{a}^2 + \frac{1}{2}(\dot{v}^{*2} + \omega^2 v^{*2})\hat{a}^{+2} +$$

$$+ (|\dot{v}|^2 + \omega^2|v|^2)\hat{a}^+\hat{a} + \frac{1}{2}(|\dot{v}|^2 + \omega^2|v|^2)\hat{1}$$

$$\hat{H}|0\rangle = \frac{1}{2}(\dot{v}^{*2} + \omega^2 v^{*2})\underbrace{\hat{a}^{+2}|0\rangle}_{\hat{a}^+\sqrt{1}|1\rangle = \sqrt{1}\sqrt{2}|2\rangle = \sqrt{2}|2\rangle} + \frac{1}{2}(|\dot{v}|^2 + \omega^2|v|^2)|0\rangle$$

ABY $|0\rangle$ BOL VLASTNÝM STAVOM $\hat{H}$ MUSÍ PLATIŤ

$$\dot{v}^{*2} + \omega^2 v^{*2} = 0 \longrightarrow \dot{v}^* = \pm i\omega v^* \;/\; \dot{v} = \mp i\omega v$$

$$1 = \langle v, v\rangle = -i v \overleftrightarrow{\partial_t} v^* = -i(v\dot{v}^* - \dot{v}v^*) =$$

$$= -i\left(v(\pm i)\omega v^* - (\mp i)\omega v v^*\right) = \pm 2\omega |v|^2$$

$\Rightarrow$ PLATÍ HORNÉ ZNAMIENKO A $v = \frac{1}{\sqrt{2\omega}} e^{i\alpha}$

MÁME TU VŠAK SPOR LEBO

$$\ddot{v} + \omega^2 v = 0 \;\;\&\;\; \dot{v} = -i\omega v \Rightarrow \ddot{v} = -i\dot{\omega}v - i\omega(-i\omega v) \rightarrow$$

$$\rightarrow \ddot{v} + \omega^2 v = -i\dot{\omega}v \Rightarrow \dot{\omega} = 0$$ ČO NIE JE PRAVDA

NIE JE MOŽNÉ ZAVIESŤ BÁZU $|m\rangle$, $m = 0, 1, 2, 3, \ldots$ V HILBERTOVOM PRIESTORE TAK, ABY V ĽUBOVOĽNOM ČASE STAVY $|m\rangle$ ZODPOVEDALI m-TÝM ENERGETICKÝM HLADINÁM

VIEME VŠAK ZAVIESŤ BÁZU $|m\rangle$ TAK ABY $|m\rangle$ ZODPOVEDALO E_m VO FIXOVANOM ČASE t_0 ROBÍ SA TO NASLEDOVNE:

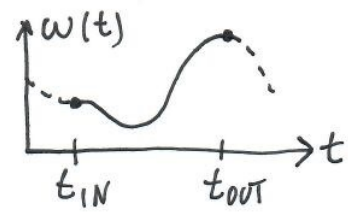
$v(t)$: $\ddot{v} + \omega(t)^2 v = 0$ & POČ. PODM. $v(t_0) = \dfrac{e^{i\alpha}}{\sqrt{2\omega(t_0)}}$

$$\dot{v}(t_0) = -i\omega(t_0)v(t_0) = -i\sqrt{\frac{\omega(t_0)}{2}}\, e^{i\alpha}$$

POTOM $\hat{q} = v(t)\hat{a}_{t_0} + v^*(t)\hat{a}^+_{t_0}$

V KAŽDOM ČASE MÁME INÉ $\hat{a}$ A $\hat{a}^+$ A INÚ BÁZU $|m_{t_0}\rangle$

TRANSFORMÁCIA MEDZI BÁZAMI V DVOCH ČASOCH t_{IN} A t_{OUT} SA NAZÝVA BOGOLIUBOVOVA TRANSFORMÁCIA



$|m_{IN}\rangle \ldots \hat{a}_{IN}, \hat{a}^+_{IN} \ldots v_{IN}, v^*_{IN}$

$|m_{OUT}\rangle \ldots \hat{a}_{OUT}, \hat{a}^+_{OUT} \ldots v_{OUT}, v^*_{OUT}$

$$\boxed{v_{IN}(t) = \alpha v_{OUT}(t) + \beta v^*_{OUT}(t)}$$

α, β – BOGOLIUBOVOVE KOEFICIENTY

$$\hat{q}(t) = \begin{cases} v_{IN}(t)\hat{a}_{IN} + v^*_{IN}(t)\hat{a}^+_{IN} \\ v_{OUT}(t)\hat{a}_{OUT} + v^*_{OUT}(t)\hat{a}^+_{OUT} \end{cases}$$

$$\hat{q} = v_{IN}\hat{a}_{IN} + v_{IN}^*\hat{a}^+_{IN} = (\alpha v_{OUT} + \beta v_{OUT}^*)\hat{a}_{IN} + (\alpha^* v_{OUT}^* + \beta^* v_{OUT})\hat{a}^+_{IN} =$$

$$= v_{OUT}(\alpha\hat{a}_{IN} + \beta^*\hat{a}^+_{IN}) + v_{OUT}^*(\beta\hat{a}_{IN} + \alpha^*\hat{a}^+_{IN})$$

$$\boxed{\hat{a}_{OUT} = \alpha\hat{a}_{IN} + \beta^*\hat{a}^+_{IN}}$$

INVERZNÉ TRANSFORMÁCIE

$$\boxed{v_{OUT} = \alpha^* v_{IN} - \beta v_{IN}^*}$$

$$\boxed{\hat{a}_{IN} = \alpha^*\hat{a}_{OUT} - \beta^*\hat{a}^+_{OUT}}$$

OVERENIE:

- $\alpha^*(\alpha v_{OUT} + \beta v_{OUT}^*) - \beta(\alpha^* v_{OUT}^* + \beta^* v_{OUT}) = (|\alpha|^2 - |\beta|^2) v_{OUT}$
- $\alpha^*(\alpha\hat{a}_{IN} + \beta^*\hat{a}^+_{IN}) - \beta^*(\alpha^*\hat{a}^+_{IN} + \beta\hat{a}_{IN}) = (|\alpha|^2 - |\beta|^2)\hat{a}_{IN}$

NORMOVANIE:

$$[\hat{a}_{OUT}, \hat{a}^+_{OUT}] = [\alpha\hat{a}_{IN} + \beta^*\hat{a}^+_{IN}, \beta\hat{a}_{IN} + \alpha^*\hat{a}^+_{IN}] = (|\alpha|^2 - |\beta|^2)[\hat{a}_{IN}, \hat{a}^+_{IN}]$$

$$\rightarrow \quad \boxed{|\alpha|^2 - |\beta|^2 = 1}$$

POČÍTANIE BOGOLIUBOVOVÝCH KOEFICIENTOV

$$\langle v_{IN}, v_{OUT}\rangle = -i\, v_{IN}\overleftrightarrow{\partial_t} v_{OUT}^* = -i(v_{IN}\dot{v}_{OUT}^* - \dot{v}_{IN} v_{OUT}^*) =$$

$$= -i\, v_{IN}(\alpha^*\dot{v}_{IN} - \beta\dot{v}_{IN}^*)^* + i\,\dot{v}_{IN}(\alpha^* v_{IN} - \beta v_{IN}^*)^* =$$

$$= \alpha(-i\, v_{IN}\dot{v}_{IN}^* + i\,\dot{v}_{IN} v_{IN}^*) + \beta(i\, v_{IN}\dot{v}_{IN} - i\,\dot{v}_{IN} v_{IN}) =$$

$$= \alpha(-i)\, v_{IN}\overleftrightarrow{\partial_t} v_{IN}^* = \alpha\underbrace{\langle v_{IN}, v_{IN}\rangle}_{1} \rightarrow \boxed{\alpha = \langle v_{IN}, v_{OUT}\rangle}$$

$$\langle v_{IN}, v_{OUT}^*\rangle = -i\, v_{IN}\overleftrightarrow{\partial_t} v_{OUT} = -i(v_{IN}\dot{v}_{OUT} - \dot{v}_{IN} v_{OUT}) =$$

$$= -i\, v_{IN}(\alpha^*\dot{v}_{IN} - \beta\dot{v}_{IN}^*) + i\,\dot{v}_{IN}(\alpha^* v_{IN} - \beta v_{IN}^*) =$$

$$= \alpha^*(-i\, v_{IN}\dot{v}_{IN} + i\,\dot{v}_{IN} v_{IN}) + \beta(i\, v_{IN}\dot{v}_{IN}^* - i\,\dot{v}_{IN} v_{IN}^*) =$$

$$= \beta\, i\, v_{IN}\overleftrightarrow{\partial_t} v_{IN}^* = -\beta\underbrace{\langle v_{IN}, v_{IN}\rangle}_{1} \rightarrow \boxed{\beta = -\langle v_{IN}, v_{OUT}^*\rangle}$$

NEZÁVISLOSŤ BOGOLIUBOVOVÝCH KOEFICIENTOV OD ČASU

$$\dot{\alpha} = -i\left(v_{IN}\dot{v}^*_{OUT} - \dot{v}_{IN}v^*_{OUT}\right)^{\bullet} =$$

$$= -i\left(\dot{v}_{IN}\dot{v}^*_{OUT} + v_{IN}\underbrace{\ddot{v}^*_{OUT}}_{-\omega^2 v^*_{OUT}} - \underbrace{\ddot{v}_{IN}}_{-\omega^2 v_{IN}}v^*_{OUT} - \dot{v}_{IN}\dot{v}^*_{OUT}\right) = 0$$

POH. ROVN.: $-\omega^2 v^*_{OUT}$ $\quad -\omega^2 v_{IN}$

$$\dot{\beta} = (-1)(-i)\left(v_{IN}\dot{v}_{OUT} - \dot{v}_{IN}v_{OUT}\right)^{\bullet} =$$

$$= i\left(\dot{v}_{IN}\dot{v}_{OUT} + v_{IN}\underbrace{\ddot{v}_{OUT}}_{-\omega^2 v_{OUT}} - \underbrace{\ddot{v}_{IN}}_{-\omega^2 v_{IN}}v_{OUT} - \dot{v}_{IN}\dot{v}_{OUT}\right) = 0$$

POH. ROVN.: $-\omega^2 v_{OUT}$ $\quad -\omega^2 v_{IN}$

EXCITOVANIE STAVU V ČASE t_{OUT} KTORÝ BOL V ČASE t_{IN} NA ZÁKLADNEJ ENERGETICKEJ HLADINE

$$\langle 0_{IN}|\hat{N}_{OUT}|0_{IN}\rangle = \langle 0_{IN}|\hat{a}^+_{OUT}\hat{a}_{OUT}|0_{IN}\rangle =$$

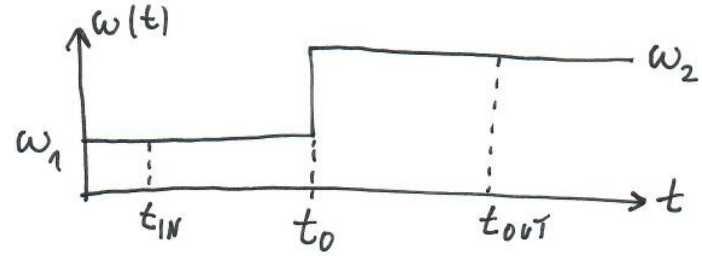
$$= \langle 0_{IN}|\left(\alpha^*\hat{a}^+_{IN} + \beta\hat{a}_{IN}\right)\left(\alpha\hat{a}_{IN} + \beta^*\hat{a}^+_{IN}\right)|0_{IN}\rangle =$$

$$= \langle 0_{IN}|\left(|\alpha|^2\hat{a}^+_{IN}\hat{a}_{IN} + \alpha^*\beta^*\hat{a}^{+2}_{IN} + \alpha\beta\hat{a}^2_{IN} + |\beta|^2\underbrace{\hat{a}_{IN}\hat{a}^+_{IN}}_{2\hat{a}^+_{IN}\hat{a}_{IN} + \hat{1}}\right)|0_{IN}\rangle =$$

$$= |\beta|^2$$

$$\boxed{\langle 0_{IN}|\hat{N}_{OUT}|0_{IN}\rangle = |\beta|^2 = \left|\langle v_{IN}, v^*_{OUT}\rangle\right|^2}$$

PRÍKLAD



PRUDKÝ SKOK $\omega(t)$ Z ω_1 NA ω_2 V ČASE t_0

POČ. PODM.
$$v_{IN}(t_0) = \frac{e^{i\alpha_1}}{\sqrt{2\omega_1}} \qquad v_{OUT}(t_0) = \frac{e^{i\alpha_2}}{\sqrt{2\omega_2}}$$
$$\dot{v}_{IN}(t_0) = -i\sqrt{\frac{\omega_1}{2}}\,e^{i\alpha_1} \qquad \dot{v}_{OUT}(t_0) = -i\sqrt{\frac{\omega_2}{2}}\,e^{i\alpha_2}$$

POH. ROVN.
$$\ddot{v}_{IN} + \omega_1 v_{IN} = 0 \qquad \ddot{v}_{OUT} + \omega_2 v_{OUT} = 0$$

RIEŠENIA
$$v_{IN}(t) = \begin{cases} \dfrac{e^{i\alpha_1}}{\sqrt{2\omega_1}}\,e^{-i\omega_1(t-t_0)} & \ldots\ t<t_0 \\ \dfrac{1}{2}\dfrac{e^{i\alpha_1}}{\sqrt{2\omega_1}}\left[\left(1-\dfrac{\omega_1}{\omega_2}\right)e^{i\omega_2(t-t_0)} + \left(1+\dfrac{\omega_1}{\omega_2}\right)e^{-i\omega_2(t-t_0)}\right. & \ldots\ t>t_0 \end{cases}$$

$$v_{OUT}(t) = \begin{cases} \dfrac{1}{2}\dfrac{e^{i\alpha_2}}{\sqrt{2\omega_2}}\left[\left(1-\dfrac{\omega_2}{\omega_1}\right)e^{i\omega_1(t-t_0)} + \left(1+\dfrac{\omega_2}{\omega_1}\right)e^{-i\omega_1(t-t_0)}\right. & \ldots\ t<t_0 \\ \dfrac{e^{i\alpha_2}}{\sqrt{2\omega_2}}\,e^{-i\omega_2(t-t_0)} & \ldots\ t>t_0 \end{cases}$$

BOGOLIUBOVOVE KOEFICIENTY
$$\alpha = -i\, v_{IN} \overleftrightarrow{\partial_t} v^*_{OUT} = \frac{1}{2}\left(\sqrt{\frac{\omega_2}{\omega_1}} + \sqrt{\frac{\omega_1}{\omega_2}}\right)e^{i(\alpha_1-\alpha_2)}$$
$$\beta = i\, v_{IN} \overleftrightarrow{\partial_t} v_{OUT} = \frac{1}{2}\left(\sqrt{\frac{\omega_2}{\omega_1}} - \sqrt{\frac{\omega_1}{\omega_2}}\right)e^{i(\alpha_1+\alpha_2)}$$

EXCITOVANIE
$$\langle 0_{IN}|\hat{N}_{OUT}|0_{IN}\rangle = |\beta|^2 = \frac{1}{4}\left(\sqrt{\frac{\omega_2}{\omega_1}} - \sqrt{\frac{\omega_1}{\omega_2}}\right)^2 = \frac{(\omega_2-\omega_1)^2}{4\omega_1\omega_2}$$

QFT NA ZAKRIVENOM POZADÍ

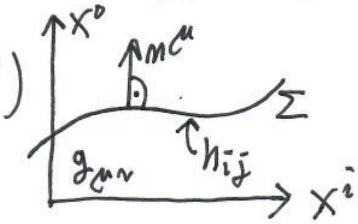
$$(+) \rightsquigarrow S = \int dt \left(\tfrac{1}{2} \dot{q}^2 - \tfrac{1}{2} \omega(t)^2 q^2 \right) \rightsquigarrow \langle q_1, q_2 \rangle = -i\, q_1 \overleftrightarrow{\partial_t} q_2^*$$

$$\downarrow$$

$$(+, \underbrace{-, \dots, -}_{m-1}) \rightsquigarrow S = \int d^m x \sqrt{|g|} \left(\tfrac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \tfrac{1}{2} m^2 \phi^2 - \tfrac{1}{2} \xi R \phi^2 \right)$$

$$\rightsquigarrow \langle \phi_1, \phi_2 \rangle = -i \int_{\Sigma} \underbrace{d^{m-1}x \sqrt{|h|}\, n^{\mu}}_{\text{OZN. } d\Sigma^{\mu} \equiv d\Sigma\, n^{\mu}} \left(\phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \right)$$

PRIESTOROVOPODOBNÁ NADPLOCHA $\longrightarrow \Sigma$

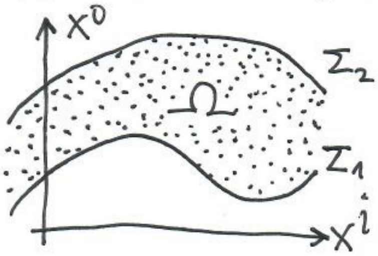


POHYBOVÁ ROVNICA (KLEIN GORDON)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \sqrt{|g|} \left(-m^2 \phi - \xi R \phi \right) - \underbrace{\partial_\mu \left(\sqrt{|g|}\, g^{\mu\nu} \phi_{,\nu} \right)}_{\sqrt{|g|}\, \underbrace{\nabla_\mu \nabla^\mu}_{\Box_g} \phi}$$

$$\hookrightarrow \boxed{\left(\Box_g + m^2 + \xi R \right) \phi = 0}$$

NEZÁVISLOSŤ (SKALÁRNEHO) SÚČINU OD VÝBERU NADPLOCHY Σ



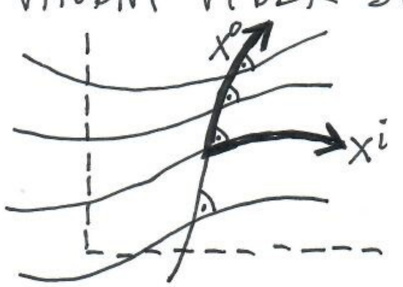
$$\langle \phi_1, \phi_2 \rangle_{\Sigma_1} - \langle \phi_1, \phi_2 \rangle_{\Sigma_2} = -i \int_{\partial\Omega} d\Sigma^{\mu} \left(\phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \right) =$$

$$= -i \int_{\Omega} d^m x\, \underbrace{\partial^{\mu} \left(\sqrt{|g|} \left(\phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \right) \right)}_{\sqrt{|g|}\, \nabla^{\mu} \left(\phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \right)} = 0$$

$$\nabla^{\mu} \left(\phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \right) = \nabla^{\mu} \left(\phi_1 \phi^*_{2,\mu} - \phi_{1,\mu} \phi_2^* \right) = \phi_1{}^{,\mu} \phi^*_{2,\mu} + \phi_1 \Box_g \phi_2^* -$$

$$- \left(\Box_g \phi_1 \right) \phi_2^* - \phi_{1,\mu} \phi_2^{*,\mu} = \begin{bmatrix} \text{POH.} \\ \text{ROVN.} \end{bmatrix} = \phi_1 \left(-m^2 - \xi R \right) \phi_2^* - \left(-m^2 - \xi R \right) \phi_1 \phi_2^* = 0$$

VHODNÝ VÝBER SÚRADNÍC



$$ds^2 = dt^2 - h_{ij}\, dx^i dx^j$$

$$\Sigma = \{ t = \text{KONŠT.} \}$$

$$n^{\mu} = \delta^{\mu}_0 = (1, 0, \dots, 0)$$

$$\sqrt{|g|} = \sqrt{|h|}$$

$$\partial^0 = \partial_0$$

NEZÁVISLOSŤ (SKALÁRNEHO) SÚČINU OD ČASU

$$\partial_0 \langle \phi_1 | \phi_2 \rangle = -i \int_V d^{n-1}x \, \partial_0 \left[\sqrt{|h|} \left(\phi_1 \overleftrightarrow{\partial_0} \phi_2^* \right) \right] =$$

$$= -i \int_V d^{n-1}x \, \underbrace{\partial_\mu \left[\sqrt{|h|} \left(\phi_1 \overleftrightarrow{\partial^\mu} \phi_2^* \right) \right]}_{\sqrt{|g|}\, \nabla_\mu \left(\phi_1 \overleftrightarrow{\partial^\mu} \phi_2^* \right) = 0} + i \underbrace{\int_V d^{n-1}x \, \partial_i \left[\sqrt{|h|} \left(\phi_1 \overleftrightarrow{\partial^i} \phi_2^* \right) \right]}_{\int_{\partial V} dS_i (\dots)^i = 0}$$

POH. ROVN. OKRAJ

KANONICKÁ HYBNOSŤ

$$S = \int dx^0 L \,, \quad L = \int d^{n-1}x \, \mathcal{L} \,, \quad \mathcal{L} = \sqrt{|g|} \left(\tfrac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \dots \right)$$

$$\Pi = \frac{\delta L}{\delta(\partial_0 \phi)} = \underbrace{\sqrt{|g|}}_{\sqrt{|h|}} \underbrace{g^{0\mu} \phi_{,\mu}}_{g^{00} \phi_{,0}} = \sqrt{|h|} \underbrace{n^\mu \phi_{,\mu}}_{\phi_{,0}}$$

KVANTOVANIE:

① VŠEOBECNÉ RIEŠENIE $\phi(x) = \sum_k \left(a_k f_k(x) + a_k^+ f_k^*(x) \right)$

KDE $a_k = \langle \phi, f_k \rangle$, $a_k^+ = -\langle \phi, f_k^* \rangle$ A PRE f_k A f_k^*:

$$\left(\Box_g + m^2 + \xi R \right) f_k = 0 \,, \quad \left(\Box_g + m^2 + \xi R \right) f_k^* = 0$$

$$\langle f_k, f_{k'} \rangle = \delta_{kk'} \,, \quad \langle f_k^*, f_{k'}^* \rangle = -\delta_{kk'} \,, \quad \langle f_k, f_{k'}^* \rangle = 0$$

② $(\ ) \to \widehat{(\ )}$: $\hat{\phi}(x) = \sum_k \left(f_k(x) \hat{a}_k + f_k^*(x) \hat{a}_k^+ \right)$

$\hat{a}_k$ A $\hat{a}_k^+$ GENERUJÚ FOCKOV PRIESTOR A $[\hat{a}_k, \hat{a}_q^+] = \delta_{kq}$

$$[\hat{a}_k, \hat{a}_q] = 0 = [\hat{a}_k^+, \hat{a}_q^+]$$

MUSÍ TO BYŤ V SÚLADE S KANONICKÝMI KOM. VZŤAHMI

$$[\hat{\phi}(x), \hat{\Pi}(y)]_{x,y \in \Sigma} = [\hat{\phi}(x^0, \vec{x}), \hat{\Pi}(x^0, \vec{y})] = i\, \delta(\vec{x} - \vec{y})$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 = [\hat{\Pi}(x), \hat{\Pi}(y)]$$

NA OVERENIE KOMUTAČNÝCH VZŤAHO TREBA POUŽIŤ

$$\sum_k \left(f_k(x)\, f_k^*(y) - f_k^*(x)\, f_k(y)\right) = 0$$

$$-i\sqrt{|h|}\, n^\mu(y) \sum_k \left(f_k(x)\, \partial_\mu f_k^*(y) - f_k^*(x)\, \partial_\mu f_k(y)\right) = \delta(x-y)$$

ČO SA DÁ SKONTROLOVAŤ NASLEDOVNE

$$\phi(x) = \sum_k \left(f_k(x) \langle \phi, f_k \rangle - f_k^* \langle \phi, f_k^* \rangle\right) =$$

$$= \sum_k \Big[-i\, f_k(x) \int d\Sigma_{(y)}\, n^\mu(y) \left(\phi(y)\, \partial_\mu f_k^*(y) - f_k^*(y)\, \partial_\mu \phi(y)\right) +$$

$$+ i\, f_k^*(x) \int d\Sigma_{(y)}\, n^\mu(y) \left(\phi(y)\, \partial_\mu f_k(y) - f_k(y)\, \partial_\mu \phi(y)\right) \Big] =$$

$$= i \int \sqrt{|h|}\, d^{n-1}y \Big[-n^\mu(y)\, \phi(y) \sum_k \left(f_k(x)\, \partial_\mu f_k^*(y) - f_k^*(x)\, \partial_\mu f_k(y)\right) +$$

$$+ n^\mu(y) \left(\partial_\mu \phi(y)\right) \sum_k \left(f_k(x)\, f_k^*(y) - f_k^*(x)\, f_k(y)\right) \Big] \stackrel{!}{=}$$

$$\stackrel{!}{=} \int d^{n-1}y\;\; \phi(y)\, \delta(x-y)$$

POTOM

$$\hat{\phi}(x) = \sum_k \left[f_k(x)\, \hat{a}_k + f_k^*(x)\, \hat{a}_k^+ \right]$$

$$\hat{\pi}(y) = \sqrt{|h|}\, n^\mu(y) \left(\partial_\mu \hat{\phi}(y)\right) = \sqrt{|h|}\, n^\mu(y) \sum_q \left[\left(\partial_\mu f_q(y)\right) \hat{a}_q + h.c. \right]$$

$$[\hat{\phi}(x), \hat{\pi}(y)] = \sqrt{|h|}\, n^\mu(y) \sum_k \sum_q \Big[f_k(x) \left(\partial_\mu f_q^*(y)\right) \underbrace{[\hat{a}_k, \hat{a}_q^+]}_{\delta_{kq}} +$$

$$+ f_k^*(x) \left(\partial_\mu f_q(y)\right) \underbrace{[\hat{a}_k^+, \hat{a}_q]}_{-\delta_{kq}} \Big] =$$

$$= \sqrt{|h|}\, n^\mu(y) \sum_k \left(f_k(x)\, \partial_\mu f_k^*(y) - f_k^*(x)\, \partial_\mu f_k(y) \right) = i\, \delta(x-y)$$

OSTATNÉ KOMUTAČNÉ VZŤAHY SÚ NULOVÉ

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \sum_k \sum_q \Big[f_k(x) f_q^*(y) \underbrace{\left[\hat{a}_k, \hat{a}_q^+\right]}_{\delta_{kq}} + f_k^*(x) f_q(y) \underbrace{\left[\hat{a}_k^+, \hat{a}_q\right]}_{-\delta_{kq}} \Big] =$$

$$= \sum_k \Big(f_k(x) f_k^*(y) - f_k^*(x) f_k(y) \Big) = 0$$

$$\left[\hat{\pi}(x), \hat{\pi}(y)\right] = \sqrt{|h(x)|}\sqrt{|h(y)|}\, n^\mu(x) n^\nu(y) \cdot$$

$$\cdot \sum_k \sum_q \Big[(\partial_\mu f_k(x))(\partial_\nu f_q^*(y)) \underbrace{\left[\hat{a}_k, \hat{a}_q^+\right]}_{\delta_{kq}} + (\partial_\mu f_k^*(x))(\partial_\nu f_q(y)) \underbrace{\left[\hat{a}_k^+, \hat{a}_q\right]}_{-\delta_{kq}} \Big] =$$

$$= \sqrt{|h(x)|}\sqrt{|h(y)|}\, \underbrace{n^\mu(x) n^\nu(y)}_{\text{SYM.}} \sum_k \underbrace{\Big[(\partial_\mu f_k(x))(\partial_\nu f_k^*(y)) - (\partial_\mu f_k^*(x))(\partial_\nu f_k(y)) \Big]}_{\text{ANTISYM.}} = 0$$

POUŽÍVALI SME $\sum_k$ ALE TO MÔŽE ZNAMENAŤ RÔZNE VECI, NAPRÍKLAD $\int \frac{d^3k}{(2\pi)^3}$ ALEBO $\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty d\omega$

$$\hat{\phi}(x) \overset{a)}{=} \int \frac{d^3k}{(2\pi)^3} \left(f_k(x) \hat{a}_{\vec{k}} + f_k^*(x) \hat{a}_{\vec{k}}^+ \right)$$

$$\overset{b)}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty d\omega \left(f_{\omega l m}(x) \hat{a}_{\omega l m} + f_{\omega l m}^* \hat{a}_{\omega l m}^+ \right)$$

$$\vdots$$

$$\overset{\text{OZN.}}{=} \sum_A \left(f_A(x) \hat{a}_A + f_A^*(x) \hat{a}_A^+ \right)$$

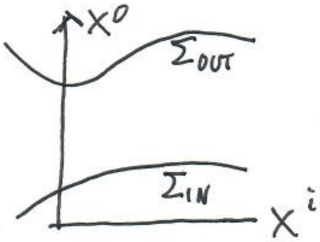
$$\langle f_A, f_B \rangle \overset{\text{OZN.}}{=} \delta_{AB} \overset{a)}{=} \delta^{(3)}(\vec{k} - \vec{k}') \overset{b)}{=} \delta_{ll'} \delta_{mm'} \delta(\omega' - \omega)$$

$$\langle f_A^*, f_B^* \rangle = -\delta_{AB} \quad , \langle f_A, f_B^* \rangle = 0$$

PRI OSCILÁTORE S PREMENNOU FREKVENCIOU SME VEDELI ZAVIESŤ ROZUMNÚ BÁZU V HILBERTOVOM PRIESTORE IBA V JEDNOM FIXOVANOM ČASE t_0

POTREBOVALI SME BOGOLIUBOVOVE TRANSFORMÁCIE

PRI POLIACH NA ZAKRIVENOM POZADÍ VIEME ZAVIESŤ ROZUMNÚ BÁZU VO FOCKOVOM PRIESTORE IBA NA JEDNEJ FIXOVANEJ NADPLOCHE Σ_0



TIEŽ POTREBUJEME BOGOLIUBOVOVE TRANSFORMÁCIE

$$f_A^{IN} = \sum_B \left(\alpha_{AB} f_B^{OUT} + \beta_{AB} f_B^{OUT*} \right)$$

$$\hat{a}_A^{IN} = \sum_B \left(\alpha_{AB}^* \hat{a}_B^{OUT} - \beta_{AB}^* \hat{a}_B^{OUT+} \right)$$

$$f_A^{OUT} = \sum_B \left(\alpha_{AB}^* f_B^{IN} - \beta_{AB} f_B^{IN*} \right)$$

$$\hat{a}_A^{OUT} = \sum_B \left(\alpha_{AB} \hat{a}_B^{IN} + \beta_{AB}^* \hat{a}_B^{IN+} \right)$$

SÚVIS MEDZI TÝMITO A PREDCHÁDZAJÚCIMI BOGOLIUBOVOVÝMI TRANSFORMÁCIAMI VIDNO NAJLEPŠIE PRI PRÍKLADE S ROZPÍNAJÚCIM SA VESMÍROM $g_{\mu\nu}$ = ROBERTSON – WALKER

NORMOVANIE BOGOLIUBOVOVÝCH KOEFICIENTOV

$$\sum_B \left(\alpha_{AB} \alpha_{CB}^* - \beta_{AB} \beta_{CB}^* \right) = \delta_{AC}$$

$$\sum_B \left(|\alpha_{AB}|^2 - |\beta_{AB}|^2 \right) = \cancel{\sum_C} \delta_{CC} \leftarrow \delta(0) \ldots \text{OBJEM}$$

KREOVANIE ČASTÍC

$$\langle 0_{IN} | \hat{a}_A^{OUT+} \hat{a}_B^{OUT} | 0_{IN} \rangle = \langle 0_{IN} | \sum_C \sum_D \left(\alpha_{AC}^* \hat{a}_C^{IN+} + \beta_{AC} \hat{a}_C^{IN} \right) \cdot$$

$$\cdot \left(\alpha_{BD} \hat{a}_D^{IN} + \beta_{BD}^* \hat{a}_D^{IN+} \right) | 0_{IN} \rangle = \langle 0_{IN} | \ldots \sum_C \sum_D \beta_{AC} \beta_{BD}^* \underbrace{\hat{a}_C^{IN} \hat{a}_D^{IN+}}_{2 \hat{a}_C^{IN+} \hat{a}_D^{IN} + \delta_{CD}} | 0_{IN} \rangle =$$

$$= \sum_C \beta_{AC} \beta_{BC}^*$$

$$\langle 0_{IN} | \hat{N}_A^{OUT} | 0_{IN} \rangle = \langle 0_{IN} | \hat{a}_A^{OUT+} \hat{a}_A^{OUT} | 0_{IN} \rangle = \sum_C |\beta_{AC}|^2 \supset \text{OBJEM}$$

KREUJE SA OBJEMOVÁ HUSTOTA POČTU ČASTÍC

S.W. Hawking: Particle Creation by Black Holes,
Commun.Math.Phys. 43 (1975) 199-220

2. Gravitational Collapse

It is now generally believed that, according to classical theory, a gravitational collapse will produce a black hole which will settle down rapidly to a stationary axisymmetric equilibrium state characterized by its mass, angular momentum and electric charge [7, 13]. The Kerr-Newman solution represent one such family of black hole equilibrium states and it seems unlikely that there are any others. It has therefore become a common practice to ignore the collapse phase and to represent a black hole simply by one of these solutions. Because these solutions are stationary there will not be any mixing of positive and negative frequencies and so one would not expect to obtain any particle creation. However there is a classical phenomenon called superradiance [14–17] in which waves incident in certain modes on a rotating or charged black hole are scattered with increased amplitude [see Section (3)]. On a particle description this amplification must correspond to an increase in the number of particles and therefore to stimulated emission of particles. One would therefore expect on general grounds that there would also be a steady rate of spontaneous emission in these superradiant modes which would tend to carry away the angular momentum or charge of the black hole [16]. To understand how the particle creation can arise from mixing of positive and negative frequencies, it is essential to consider not only the quasi-stationary final state of the black hole but also the time-dependent formation phase. One would hope that, in the spirit of the "no hair" theorems, the rate of emission would not depend on details of the collapse process except through the mass, angular momentum and charge of the resulting black hole. I shall show that this is indeed the case but that, in addition to the emission in the superradiant modes, there is a steady rate of emission in all modes at the rate one would expect if the black hole were an ordinary body with temperature $\kappa/2\pi$.

I shall consider first of all the simplest case of a non-rotating uncharged black hole. The final stationary state for such a black hole is represented by the Schwarzschild solution with metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.1}$$

As is now well known, the apparent singularities at $r=2M$ are fictitious, arising merely from a bad choice of coordinates. The global structure of the analytically extended Schwarzschild solution can be described in a simple manner by a Penrose diagram of the r-t plane (Fig. 1) [6, 13]. In this diagram null geodesics in the r-t plane are at $\pm 45°$ to the vertical. Each point of the diagram represents a 2-sphere of area $4\pi r^2$. A conformal transformation has been applied to bring infinity to a finite distance: infinity is represented by the two diagonal lines (really null surfaces) labelled $\mathscr{I}^+$ and $\mathscr{I}^-$, and the points I^+, I^-, and I^0. The two horizontal lines $r=0$ are curvature singularities and the two diagonal lines $r=2M$ (really null surfaces) are the future and past event horizons which divide the solution up into regions from which one cannot escape to $\mathscr{I}^+$ and $\mathscr{I}^-$. On the left of the diagram there is another infinity and asymptotically flat region.

Most of the Penrose diagram is not in fact relevant to a black hole formed by gravitational collapse since the metric is that of the Schwarzchild solution

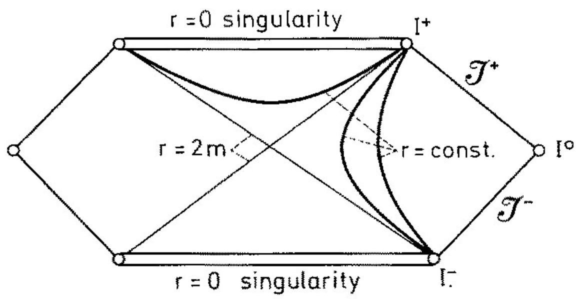


Fig. 1. The Penrose diagram for the analytically extended Schwarzschild solution

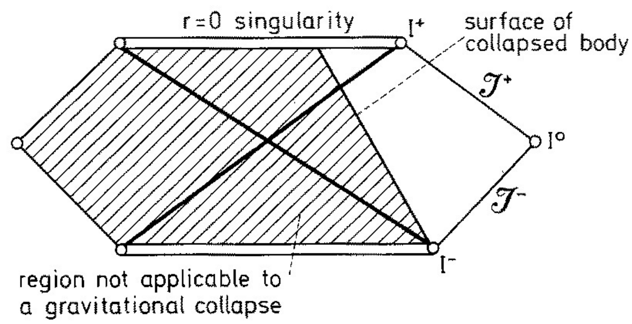


Fig. 2. Only the region of the Schwarzschild solution outside the collapsing body is relevant for a black hole formed by gravitational collapse. Inside the body the solution is completely different

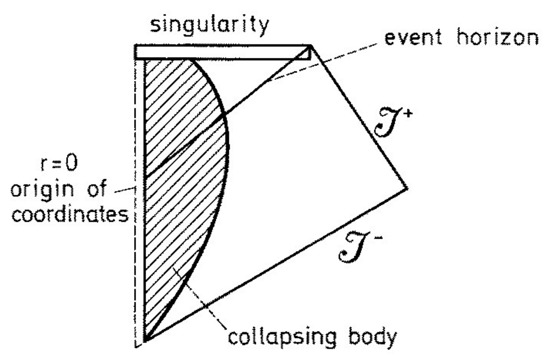


Fig. 3. The Penrose diagram of a spherically symmetric collapsing body producing a black hole. The vertical dotted line on the left represents the non-singular centre of the body

only in the region outside the collapsing matter and only in the asymptotic future. In the case of exactly spherical collapse, which I shall consider for simplicity, the metric is exactly the Schwarzchild metric everywhere outside the surface of the collapsing object which is represented by a timelike geodesic in the Penrose diagram (Fig. 2). Inside the object the metric is completely different, the past event horizon, the past $r=0$ singularity and the other asymptotically flat region do not exist and are replaced by a time-like curve representing the origin of polar coordinates. The appropriate Penrose diagram is shown in Fig. 3 where the conformal freedom has been used to make the origin of polar coordinates into a vertical line.

In this space-time consider (again for simplicity) a massless Hermitian scalar field operator ϕ obeying the wave equation

$$\phi_{;ab}g^{ab}=0\,. \tag{2.2}$$

(The results obtained would be the same if one used the conformally invariant wave equation:

$$\phi_{;ab}g^{ab}+\tfrac{1}{6}R\phi=0\,.)$$

The operator ϕ can be expressed as

$$\phi=\sum_i\{f_i\boldsymbol{a}_i+\bar{f}_i\boldsymbol{a}_i^\dagger\}\,. \tag{2.3}$$

The solutions $\{f_i\}$ of the wave equation $f_{i;ab}g^{ab}=0$ can be chosen so that on past null infinity $\mathscr{I}^-$ they form a complete family satisfying the orthonormality conditions (1.2) where the surface S is $\mathscr{I}^-$ and so that they contain only positive frequencies with respect to the canonical affine parameter on $\mathscr{I}^-$. (This last condition of positive frequency can be uniquely defined despite the existence of "supertranslations" in the Bondi-Metzner-Sachs asymptotic symmetry group [21, 22].) The operators $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$ have the natural interpretation as the annihilation and creation operators for ingoing particles i.e. for particles at past null infinity $\mathscr{I}^-$. Because massless fields are completely determined by their data on $\mathscr{I}^-$, the operator ϕ can be expressed in the form (2.3) everywhere. In the region outside the event horizon one can also determine massless fields by their data on the event horizon and on future null infinity $\mathscr{I}^+$. Thus one can also express ϕ in the form

$$\phi=\sum_i\{p_i\boldsymbol{b}_i+\bar{p}_i\boldsymbol{b}_i^\dagger+q_i\boldsymbol{c}_i+\bar{q}_i\boldsymbol{c}_i^\dagger\}\,. \tag{2.4}$$

Here the $\{p_i\}$ are solutions of the wave equation which are purely outgoing, i.e. they have zero Cauchy data on the event horizon and the $\{q_i\}$ are solutions which contain no outgoing component, i.e. they have zero Cauchy data on $\mathscr{I}^+$. The $\{p_i\}$ and $\{q_i\}$ are required to be complete families satisfying the orthonormality conditions (1.2) where the surface S is taken to be $\mathscr{I}^+$ and the event horizon respectively. In addition the $\{p_i\}$ are required to contain only positive frequencies with respect to the canonical affine parameter along the null geodesic generators of $\mathscr{I}^+$. With the positive frequency condition on $\{p_i\}$, the operators $\{\boldsymbol{b}_i\}$ and $\{\boldsymbol{b}_i^\dagger\}$ can be interpreted as the annihilation and creation operators for outgoing particles, i.e. for particles on $\mathscr{I}^+$. It is not clear whether one should impose some positive frequency condition on the $\{q_i\}$ and if so with respect to what. The choice of the $\{q_i\}$ does not affect the calculation of the emission of particles to $\mathscr{I}^+$. I shall return to the question in Section (4).

Because massless fields are completely determined by their data on $\mathscr{I}^-$ one can express $\{p_i\}$ and $\{q_i\}$ as linear combinations of the $\{f_i\}$ and $\{\bar{f}_i\}$:

$$p_i=\sum_j(\alpha_{ij}f_j+\beta_{ij}\bar{f}_j)\,, \tag{2.5}$$

$$q_i=\sum_j(\gamma_{ij}f_j+\eta_{ij}\bar{f}_j)\,. \tag{2.6}$$

These relations lead to corresponding relations between the operators

$$\boldsymbol{b}_i=\sum_j(\bar{\alpha}_{ij}\boldsymbol{a}_j-\bar{\beta}_{ij}\boldsymbol{a}_j^\dagger)\,, \tag{2.7}$$

$$\boldsymbol{c}_i=\sum_j(\bar{\gamma}_{ij}\boldsymbol{a}_j-\bar{\eta}_{ij}\boldsymbol{a}_j^\dagger)\,. \tag{2.8}$$

The initial vacuum state $|0\rangle$, the state containing no incoming particles, i.e. no particles on $\mathscr{I}^-$, is defined by

$$\boldsymbol{a}_i|0\rangle = 0 \quad \text{for all } i\,. \tag{2.9}$$

However, because the coefficients β_{ij} will not be zero in general, the initial vacuum state will not appear to be a vacuum state to an observer at $\mathscr{I}^+$. Instead he will find that the expectation value of the number operator for the ith outgoing mode is

$$\langle 0_-|\boldsymbol{b}_i^\dagger \boldsymbol{b}_i|0_-\rangle = \textstyle\sum_j |\beta_{ij}|^2\,. \tag{2.10}$$

Thus in order to determine the number of particles created by the gravitational field and emitted to infinity one simply has to calculate the coefficients β_{ij}. One would expect this calculation to be very messy and to depend on the detailed nature of the gravitational collapse. However, as I shall show, one can derive an asymptotic form for the β_{ij} which depends only on the surface gravity of the resulting black hole. There will be a certain finite amount of particle creation which depends on the details of the collapse. These particles will disperse and at late retarded times on $\mathscr{I}^+$ there will be a steady flux of particles determined by the asymptotic form of β_{ij}.

In order to calculate this asymptotic form it is more convenient to decompose the ingoing and outgoing solutions of the wave equation into their Fourier components with respect to advanced or retarded time and use the continuum normalization. The finite normalization solutions can then be recovered by adding Fourier components to form wave packets. Because the space-time is spherically symmetric, one can also decompose the incoming and outgoing solutions into spherical harmonics. Thus, in the region outside the collapsing body, one can write the incoming and outgoing solutions as

$$f_{\omega' lm} = (2\pi)^{-\frac{1}{2}} r^{-1} (\omega')^{-\frac{1}{2}} F_{\omega'}(r) e^{i\omega' v} Y_{lm}(\theta, \phi)\,, \tag{2.11}$$

$$p_{\omega lm} = (2\pi)^{-\frac{1}{2}} r^{-1} \omega^{-\frac{1}{2}} P_{\omega}(r) e^{i\omega u} Y_{lm}(\theta, \phi)\,, \tag{2.12}$$

where v and u are the usual advanced and retarded coordinates defined by

$$v = t + r + 2M \log\left|\frac{r}{2M} - 1\right|, \tag{2.13}$$

$$u = t - r - 2M \log\left|\frac{r}{2M} - 1\right|. \tag{2.14}$$

Each solution $p_{\omega lm}$ can be expressed as an integral with respect to ω' over solutions $f_{\omega' lm}$ and $\bar{f}_{\omega' lm}$ with the same values of l and $|m|$ (from now on I shall drop the suffices l, m):

$$p_\omega = \textstyle\int_0^\infty (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} \bar{f}_{\omega'}) d\omega'\,. \tag{2.15}$$

To calculate the coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$, consider a solution p_ω propagating backwards from $\mathscr{I}^+$ with zero Cauchy data on the event horizon. A part $p_\omega^{(1)}$ of the solution p_ω will be scattered by the static Schwarzchild field outside the collapsing body and will end up on $\mathscr{I}^-$ with the same frequency ω. This will give a $\delta(\omega' - \omega)$ term in $\alpha_{\omega\omega'}$. The remainder $p_\omega^{(2)}$ of p_ω will enter the collapsing body

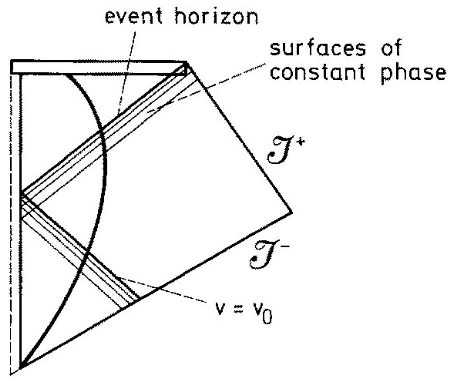


Fig. 4. The solution p_ω of the wave equation has an infinite number of cycles near the event horizon and near the surface $v=v_0$

where it will be partly scattered and partly reflected through the centre, eventually emerging to $\mathscr{I}^-$. It is this part $p_\omega^{(2)}$ which produces the interesting effects. Because the retarded time coordinate u goes to infinity on the event horizon, the surfaces of constant phase of the solution p_ω will pile up near the event horizon (Fig. 4). To an observer on the collapsing body the wave would seem to have a very large blue-shift. Because its effective frequency was very high, the wave would propagate by geometric optics through the centre of the body and out on $\mathscr{I}^-$. On $\mathscr{I}^-$ $p_\omega^{(2)}$ would have an infinite number of cycles just before the advanced time $v=v_0$ where v_0 is the latest time that a null geodesic could leave $\mathscr{I}^-$, pass through the centre of the body and escape to $\mathscr{I}^+$ before being trapped by the event horizon. One can estimate the form of $p_\omega^{(2)}$ on $\mathscr{I}^-$ near $v=v_0$ in the following way. Let x be a point on the event horizon outside the matter and let l^a be a null vector tangent to the horizon. Let n^a be the future-directed null vector at x which is directed radially inwards and normalized so that $l^a n_a = -1$. The vector $-\varepsilon n^a$ (ε small and positive) will connect the point x on the event horizon with a nearby null surface of constant retarded time u and therefore with a surface of constant phase of the solution $p_\omega^{(2)}$. If the vectors l^a and n^a are parallelly transported along the null geodesic γ through x which generates the horizon, the vector $-\varepsilon n^a$ will always connect the event horizon with the same surface of constant phase of $p_\omega^{(2)}$. To see what the relation between ε and the phase of $p_\omega^{(2)}$ is, imagine in Fig. 2 that the collapsing body did not exist but one analytically continued the empty space Schwarzchild solution back to cover the whole Penrose diagram. One could then transport the pair (l^a, n^a) back along to the point where future and past event horizons intersected. The vector $-\varepsilon n^a$ would then lie along the past event horizon. Let λ be the affine parameter along the past event horizon which is such that at the point of intersection of the two horizons, $\lambda=0$ and $\dfrac{dx^a}{d\lambda}=n^a$. The affine parameter λ is related to the retarded time u on the past horizon by

$$\lambda = -Ce^{-\kappa u} \tag{2.16}$$

where C is constant and κ is the surface gravity of the black hole defined by $K^a_{;b} K^b = -\kappa K^a$ on the horizon where K^a is the time translation Killing vector.

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$\left(\text{For a Schwarzchild black hole } \kappa = \frac{1}{4M}\right)$. It follows from this that the vector $-\varepsilon n^a$ connects the future event horizon with the surface of constant phase $-\frac{\omega}{\kappa}(\log\varepsilon - \log C)$ of the solution $p_\omega^{(2)}$. This result will also hold in the real space-time (including the collapsing body) in the region outside the body. Near the event horizon the solution $p_\omega^{(2)}$ will obey the geometric optics approximation as it passes through the body because its effective frequency will be very high. This means that if one extends the null geodesic γ back past the end-point of the event horizon and out onto $\mathscr{I}^-$ at $v = v_0$ and parallelly transports n^a along γ, the vector $-\varepsilon n^a$ will still connect γ to a surface of constant phase of the solution $p_\omega^{(2)}$. On $\mathscr{I}^-$ n^a will be parallel to the Killing vector K^a which is tangent to the null geodesic generators of $\mathscr{I}^-$:

$$n^a = DK^a\,.$$

Thus on $\mathscr{I}^-$ for $v_0 - v$ small and positive, the phase of the solution will be

$$-\frac{\omega}{\kappa}(\log(v_0 - v) - \log D - \log C)\,. \tag{2.17}$$

Thus on $\mathscr{I}^-$ $p_\omega^{(2)}$ will be zero for $v > v_0$ and for $v < v_0$

$$p_\omega^{(2)} \sim (2\pi)^{-\frac{1}{2}}\omega^{-\frac{1}{2}}r^{-1}P_\omega^- \exp\left(-i\frac{\omega}{\kappa}\left(\log\left(\frac{v_0 - v}{CD}\right)\right)\right) \tag{2.18}$$

where $P_\omega^- \equiv P_\omega(2M)$ is the value of the radial function for P_ω on the past event horizon in the analytically continued Schwarzchild solution. The expression (2.18) for $p_\omega^{(2)}$ is valid only for $v_0 - v$ small and positive. At earlier advanced times the amplitude will be different and the frequency measured with respect to v, will approach the original frequency ω.

By Fourier transforming $p_\omega^{(2)}$ one can evaluate its contributions to $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$. For large values of ω' these will be determined by the asymptotic form (2.18). Thus for large ω'

$$\alpha_{\omega\omega'}^{(2)} \approx (2\pi)^{-1}P_\omega^-(CD)^{\frac{i\omega}{\kappa}} \exp(i(\omega - \omega')v_0)\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \Gamma\left(1 - \frac{i\omega}{\kappa}\right)(-i\omega')^{-1+\frac{i\omega}{\kappa}}\,, \tag{2.19}$$

$$\beta_{\omega\omega'}^{(2)} \approx -i\alpha_{\omega(-\omega')}^{(2)}\,. \tag{2.20}$$

The solution $p_\omega^{(2)}$ is zero on $\mathscr{I}^-$ for large values of v. This means that its Fourier transform is analytic in the upper half ω' plane and that $p_\omega^{(2)}$ will be correctly represented by a Fourier integral in which the contour has been displaced into the upper half ω' plane. The Fourier transform of $p_\omega^{(2)}$ contains a factor $(-i\omega')^{-1+\frac{i\omega}{\kappa}}$ which has a logarithmic singularity at $\omega' = 0$. To obtain $\beta_{\omega\omega'}^{(2)}$ from $\alpha_{\omega\omega'}^{(2)}$ by (2.20) one has to analytically continue $\alpha_{\omega\ \omega'}^{(2)}$ anticlockwise round this singularity. This means that

$$|\alpha_{\omega\omega'}^{(2)}| = \exp\left(\frac{\pi\omega}{\kappa}\right)|\beta_{\omega\omega'}^{(2)}|\,. \tag{2.21}$$

Actually, the fact that $p_\omega^{(2)}$ is not given by (2.18) at early advanced times means that the singularity in $\alpha_{\omega\omega'}$ occurs at $\omega'=\omega$ and not at $\omega'=0$. However the relation (2.21) is still valid for large ω'.

The expectation value of the total number of created particles at $\mathscr{I}^+$ in the frequency range ω to $\omega+d\omega$ is $d\omega\int_0^\infty|\beta_{\omega\omega'}|^2 d\omega'$. Because $|\beta_{\omega\omega'}|$ goes like $(\omega')^{-\frac{1}{2}}$ at large ω' this integral diverges. This infinite total number of created particles corresponds to a finite steady rate of emission continuing for an infinite time as can be seen by building up a complete orthonormal family of wave packets from the Fourier components p_ω. Let

$$p_{jn}=\varepsilon^{-\frac{1}{2}}\int_{j\varepsilon}^{(j+1)\varepsilon} e^{-2\pi i n\varepsilon^{-1}\omega} p_\omega d\omega \tag{2.22}$$

where j and n are integers, $j\geqq 0$, $\varepsilon>0$. For ε small these wave packets will have frequency $j\varepsilon$ and will be peaked around retarded time $u=2\pi n\varepsilon^{-1}$ with width ε^{-1}. One can expand $\{p_{jn}\}$ in terms of the $\{f_\omega\}$

$$p_{jn}=\int_0^\infty(\alpha_{jn\omega'}f_{\omega'}+\beta_{jn\omega'}\bar{f}_{\omega'})d\omega' \tag{2.23}$$

where

$$\alpha_{jn\omega'}=\varepsilon^{-\frac{1}{2}}\int_{j\varepsilon}^{(j+1)\varepsilon}e^{-2\pi i n\varepsilon^{-1}\omega}\alpha_{\omega\omega'}d\omega \qquad \text{etc.} \tag{2.24}$$

For $j\gg\varepsilon$, $n\gg\varepsilon$

$$\begin{aligned}|\alpha_{jn\omega'}|&=\left|(2\pi)^{-1}P_\omega^-\omega^{-\frac{1}{2}}\Gamma\left(1-\frac{i\omega}{\kappa}\right)\varepsilon^{-\frac{1}{2}}(\omega')^{-\frac{1}{2}}\right.\\&\qquad\left.\cdot\int_{j\varepsilon}^{(j+1)\varepsilon}\exp i\omega''(-2\pi n\varepsilon^{-1}+\kappa^{-1}\log\omega')d\omega''\right|\\&=\left|\pi^{-1}P_\omega^-\omega^{-\frac{1}{2}}\Gamma\left(1-\frac{i\omega}{\kappa}\right)\varepsilon^{-\frac{1}{2}}(\omega')^{-\frac{1}{2}}z^{-1}\sin\tfrac{1}{2}\varepsilon z\right|\end{aligned} \tag{2.25}$$

where $\omega=j\varepsilon$ and $z=\kappa^{-1}\log\omega'-2\pi n\varepsilon^{-1}$. For wave-packets which reach $\mathscr{I}^+$ at late retarded times, i.e. those with large values of n, the main contribution to $\alpha_{jn\omega'}$ and $\beta_{jn\omega'}$ come from very high frequencies ω' of the order of $\exp(2\pi n\kappa\varepsilon^{-1})$. This means that these coefficients are governed only by the asymptotic forms (2.19, 2.20) for high ω' which are independent of the details of the collapse.

The expectation value of the number of particles created and emitted to infinity $\mathscr{I}^+$ in the wave-packet mode p_{jn} is

$$\int_0^\infty|\beta_{jn\omega'}|^2 d\omega'\,. \tag{2.26}$$

One can evaluate this as follows. Consider the wave-packet p_{jn} propagating backwards from $\mathscr{I}^+$. A fraction $1-\Gamma_{jn}$ of the wave-packet will be scattered by the static Schwarzchild field and a fraction Γ_{jn} will enter the collapsing body.

$$\Gamma_{jn}=\int_0^\infty(|\alpha_{jn\omega'}^{(2)}|^2-|\beta_{jn\omega'}^{(2)}|^2)d\omega' \tag{2.27}$$

where $\alpha_{jn\omega'}^{(2)}$ and $\beta_{jn\omega'}^{(2)}$, are calculated using (2.19, 2.20) from the part $p_{jn}^{(2)}$ of the wave-packet which enters the star. The minus sign in front of the second term on the right of (2.27) occurs because the negative frequency components of $p_{jn}^{(2)}$ make a negative contribution to the flux into the collapsing body. By (2.21)

$$|\alpha_{jn\omega'}^{(2)}|=\exp(\pi\omega\kappa^{-1})|\beta_{jn\omega'}^{(2)}|\,. \tag{2.28}$$

Thus the total number of particles created in the mode p_{jn} is

$$\Gamma_{jn}(\exp(2\pi\omega\kappa^{-1})-1)^{-1}. \tag{2.29}$$

But for wave-packets at late retarded times, the fraction Γ_{jn} which enters the collapsing body is almost the same as the fraction of the wave-packet that would have crossed the past event horizon had the collapsing body not been there but the exterior Schwarzchild solution had been analytically continued. Thus this factor Γ_{jn} is also the same as the fraction of a similar wave-packet coming from $\mathscr{I}^-$ which would have crossed the future event horizon and have been absorbed by the black hole. The relation between emission and absorption cross-section is therefore exactly that for a body with a temperature, in geometric units, of $\kappa/2\pi$.

Similar results hold for the electromagnetic and linearised gravitational fields. The fields produced on $\mathscr{I}^-$ by positive frequency waves from $\mathscr{I}^+$ have the same asymptotic form as (2.18) but with an extra blue shift factor in the amplitude. This extra factor cancels out in the definition of the scalar product so that the asymptotic forms of the coefficients α and β are the same as in the Eqs. (2.19) and (2.20). Thus one would expect the black hole also to radiate photons and gravitons thermally. For massless fermions such as neutrinos one again gets similar results except that the negative frequency components given by the coefficients β now make a positive contribution to the probability flux into the collapsing body. This means that the term $|\beta|^2$ in (2.27) now has the opposite sign. From this it follows that the number of particles emitted in any outgoing wave packet mode is $(\exp(2\pi\omega\kappa^{-1})+1)^{-1}$ times the fraction of that wave packet that would have been absorbed by the black hole had it been incident from $\mathscr{I}^-$. This is again exactly what one would expect for thermal emission of particles obeying Fermi-Dirac statistics.

Fields of non-zero rest mass do not reach $\mathscr{I}^-$ and $\mathscr{I}^+$. One therefore has to describe ingoing and outgoing states for these fields in terms of some concept such as the projective infinity of Eardley and Sachs [23] and Schmidt [24]. However, if the initial and final states are asymptotically Schwarzchild or Kerr solutions, one can describe the ingoing and outgoing states in a simple manner by separation of variables and one can define positive frequencies with respect to the time translation Killing vectors of these initial and final asymptotic space-times. In the asymptotic future there will be no bound states: any particle will either fall through the event horizon or escape to infinity. Thus the unbound outgoing states and the event horizon states together form a complete basis for solutions of the wave equation in the region outside the event horizon. In the asymptotic past there could be bound states if the body that collapses had had a bounded radius for an infinite time. However one could equally well assume that the body had collapsed from an infinite radius in which case there would be no bound states. The possible existence of bound states in the past does not affect the rate of particle emission in the asymptotic future which will again be that of a body with temperature $\kappa/2\pi$. The only difference from the zero rest mass case is that the frequency ω in the thermal factor $(\exp(2\pi\omega\kappa^{-1})\mp 1)^{-1}$ now includes the rest mass energy of the particle. Thus there will not be much emission of particles of rest mass m unless the temperature $\kappa/2\pi$ is greater than m.

BEZHMOTOVÉ POLE V SCHWARZSCHILDOVOM ČASOPRIESTORE

$$\left(\Box_g + m^2 + \xi R\right)\phi = 0 \longrightarrow \Box_g \phi = 0$$

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

$$\Box_g \phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu \phi\right) = \quad \left[(-g) = r^4 \sin^2\vartheta\right]$$

$$= \underbrace{g^{00}}_{1/1-\frac{2M}{r}}\partial_t^2\phi + \underbrace{g^{\varphi\varphi}}_{-1/r^2\sin^2\vartheta}\partial_\varphi^2\phi + \frac{1}{r^2\sin\vartheta}\partial_r\Big(r^2\sin\vartheta\, \underbrace{g^{rr}}_{-(1-\frac{2M}{r})}\partial_r\phi\Big) +$$

$$+ \frac{1}{r^2\sin\vartheta}\partial_\vartheta\Big(r^2\sin\vartheta\, \overbrace{g^{\vartheta\vartheta}}^{-1/r^2}\partial_\vartheta\phi\Big) =$$

$$= \left(1-\frac{2M}{r}\right)^{-1}\partial_t^2\phi - \frac{1}{r^2\sin^2\vartheta}\partial_\varphi^2\phi + \frac{1}{r^2}\left(2Mr - r^2\right)\partial_r^2\phi +$$

$$+ \frac{1}{r^2}\left(2M - 2r\right)\partial_r\phi - \frac{1}{r^2\sin\vartheta}\left(\cos\vartheta\,\partial_\vartheta\phi + \sin\vartheta\,\partial_\vartheta^2\phi\right) =$$

$$= \left(1-\frac{2M}{r}\right)^{-1}\partial_t^2\phi - \left(1-\frac{2M}{r}\right)\partial_r^2\phi + \left(-\frac{2}{r} + \frac{2M}{r^2}\right)\partial_r\phi -$$

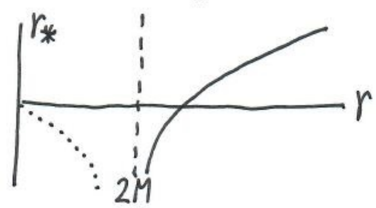
$$- \frac{1}{r^2}\partial_\vartheta^2\phi - \frac{1}{r^2}\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta\phi - \frac{1}{r^2\sin^2\vartheta}\partial_\varphi^2\phi$$

SUBSTITÚCIA $\phi = \frac{1}{r}\psi$

PRECHOD OD r KU KORYKTNAČEJ SÚRADNICI r_*:

$$ds^2 \stackrel{!}{=} \left(1-\frac{2M}{r}\right)\left(dt^2 - dr_*^2\right) + \ldots$$

$$r_*(r) = \int_0^r \left(1-\frac{2M}{r}\right)^{-1}dr = r + 2M\ln\left|\frac{r}{2M} - 1\right|$$



BUDEME UVAŽOVAŤ IBA OBLASŤ NAD HORIZONTOM $r > 2M$, $r_* \in (-\infty, \infty)$

$$\frac{d}{dr} = \frac{dr_*}{dr}\frac{d}{dr_*} = \left(1-\frac{2M}{r}\right)^{-1}\frac{d}{dr_*}$$

$$\frac{d^2}{dr^2} = -\left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\frac{d}{dr_*} + \left(1-\frac{2M}{r}\right)^{-2}\frac{d^2}{dr_*^2}$$

$$\partial_r \phi = -\frac{1}{r^2}\psi + \frac{1}{r}\partial_r \psi = -\frac{1}{r^2}\psi + \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\partial_{r_*}\psi$$

$$\partial_r^2 \phi = \frac{2}{r^3}\psi - \frac{1}{r^2}\partial_r\psi - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\partial_{r_*}\psi - \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\partial_{r_*}\psi +$$

$$+ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\partial_r\partial_{r_*}\psi =$$

$$= \frac{2}{r^3}\psi + \left[-\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1} - \frac{2M}{r^3}\left(1-\frac{2M}{r}\right)^{-2}\right]\partial_{r_*}\psi +$$

$$+ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{2M}{r}\right)^{-1}\partial_{r_*}^2\psi$$

$$-\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\left[\underbrace{2\left(1-\frac{2M}{r}\right) + \frac{2M}{r}}_{2-\frac{2M}{r}}\right]$$

$$\Box_g \phi = \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2\psi - \left(1-\frac{2M}{r}\right)\left[\frac{2}{r^3}\psi - \cancel{\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\left(2-\frac{2M}{r}\right)\partial_{r_*}\psi} +\right.$$

$$\left.+ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-2}\partial_{r_*}^2\psi\right] - \frac{1}{r}\left(2-\frac{2M}{r}\right)\left[-\frac{1}{r^2}\psi + \cancel{\frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\partial_{r_*}\psi}\right] -$$

$$-\frac{1}{r^3}\left(\underbrace{\partial_\vartheta^2\psi + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta\psi}_{\frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta\psi)} + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\psi\right) =$$

$$\Delta_s \equiv \frac{1}{\sin^2\vartheta}\partial_\varphi^2 + \frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta)$$

$$= \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2\psi - \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\partial_{r_*}^2\psi +$$

$$+ \underbrace{\left(-\frac{2}{r^3} + \frac{4M}{r^4} + \frac{2}{r^3} - \frac{2M}{r^4}\right)\psi - \frac{1}{r^3}\Delta_s\psi}_{\frac{1}{r}\left(\frac{2M}{r^3} - \frac{1}{r^2}\Delta_s\right)\psi} =$$

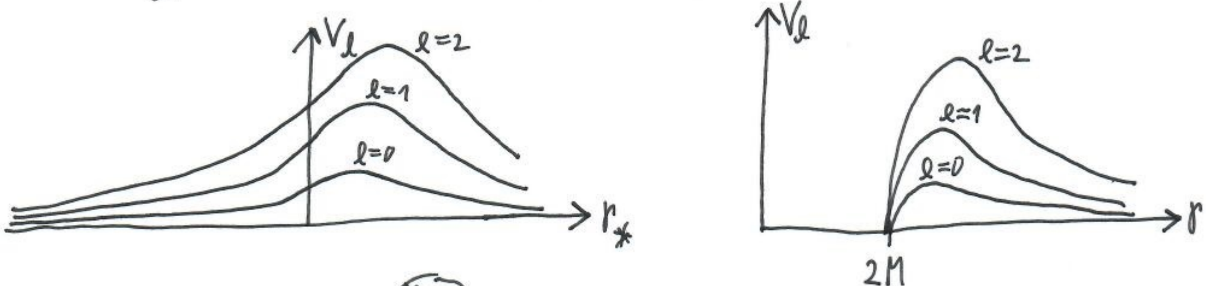
$$= \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\left[\left(\partial_t^2 - \partial_{r_*}^2\right)\psi + \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{1}{r^2}\Delta_s\right)\psi\right]$$

BUDEME UVAŽOVAŤ SFÉRICKÚ VLNU S FREKVENCIOU ω A MULTIPÓLOVÝM MOMENTOM l, m

$$\psi(t, r_*, \vartheta, \varphi) = e^{\pm i\omega t} R(r_*) Y_{lm}(\vartheta, \varphi)$$

$$\Box_g \phi = -\frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1} e^{\pm i\omega t}\, Y_{\ell m}(\vartheta,\varphi)\left[\partial^2_{r_*}R + \left(\omega^2 - V_\ell(r_*)\right)R\right]$$

$$V_\ell(r_*) = \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right) \leftarrow \Delta_S Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$$



SFÉRICKÁ VLNA  ŠÍRIACA K HORIZONTU SA MÔŽE ČIASTOČNE ODRAZIŤ OD POTENCIÁLOVEJ BARIÉRY

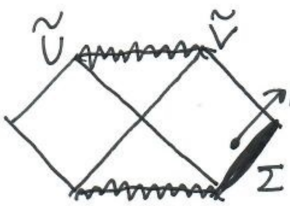
V LIMITE $r_* \to \pm\infty$: $\Box_g \phi = 0 \to \left(\partial^2_{r_*} + \omega^2\right) R_{\omega\ell} = 0$

$$\to R = \text{KONŠT}\; e^{\pm i\omega r_*}$$

MIMO LIMITY $r_* \to \pm\infty$ $\quad R(r_*) = R_{\omega\ell}(r_*)\, e^{\pm i\omega r_*}$

$\to$ TÁTO KONŠTANTA SA DÁ URČIŤ Z NORMOVACEJ PODMIENKY $\langle \phi_A, \phi_{A'}\rangle = \delta_{AA'}$ KDE SKALÁRNY SÚČIN JE DEFINOVANÝ NA NADPLOCHE KTORÁ SA DÁ PEKNE VYJADRIŤ V EDDINGTONOVÝCH–FINKELSTEINOVÝCH SÚRADNICIACH

	KRUSKAL–SZEKERES	PENROSE
$u = t - r_*(r)$	$U = -e^{-\frac{u}{4M}}$, $V = e^{\frac{v}{2M}}$	$\tilde{U} = \operatorname{arctg} U$
$v = t + r_*(r)$		$\tilde{V} = \operatorname{arctg} V$



$n^a \propto \delta^a_v$

$$\langle f, g\rangle = -i \lim_{r\to\infty} \int_{-\infty}^{\infty} dv \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi\; r^2 \sin\vartheta \left(f \overleftrightarrow{\partial_v} g^*\right)$$

SPRÁVNE NORMOVANIE VCHÁDZAJÚCEJ VLNY V LIMITE $r_* \to \infty$ JE

$$\phi_A(t, r_*, \vartheta, \varphi) = \frac{1}{\sqrt{4\pi\omega}}\,\frac{1}{r}\, Y_{\ell m}(\vartheta,\varphi)\, e^{-i\omega\underbrace{(t+r_*)}_{v}}$$

$A \equiv \omega\ell m$

$\phi = \frac{1}{r}\psi$

$t \nearrow \leftrightarrow r_* \searrow$

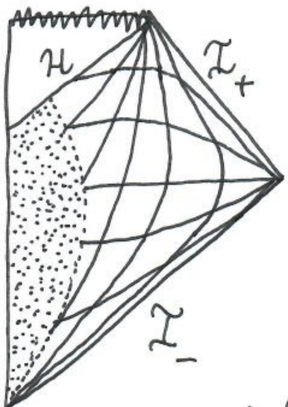
$$\delta_{AA'} \overset{!}{=} \langle \phi_{A} | \phi_{A'} \rangle = -i \lim_{r\to\infty} \int_{-\infty}^{\infty} dv \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \, r^2 \sin\vartheta \left(\phi_A \overset{\leftrightarrow}{\partial_v} \phi^*_{A'} \right) =$$

$$= -i \lim_{r\to\infty} \int_{-\infty}^{\infty} dv \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \, r^2 \sin\vartheta \, \frac{1}{4\pi\sqrt{\omega\omega'}} \frac{1}{r^2} Y_{lm}(\vartheta,\varphi) Y^*_{l'm'}(\vartheta,\varphi) \left(e^{-i\omega v} \overset{\leftrightarrow}{\partial_v} e^{i\omega' v} \right) =$$

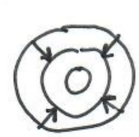
$$= \frac{-i}{4\pi\sqrt{\omega\omega'}} \underbrace{\int_{-\infty}^{\infty} dv \left(i\omega' - (-i)\omega \right) e^{i(\omega'-\omega)v}}_{i(\omega'+\omega)\, 2\pi\, \delta(\omega'-\omega)} \underbrace{\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sin\vartheta \, Y_{lm}(\vartheta,\varphi) Y^*_{l'm'}(\vartheta,\varphi)}_{\delta_{ll'}\delta_{mm'}} =$$

$$= \frac{(-i)\, i\, (\omega'+\omega)}{4\pi\sqrt{\omega\omega'}} 2\pi \, \delta_{ll'}\delta_{mm'}\delta(\omega'-\omega) = \delta_{ll'}\delta_{mm'}\delta(\omega'-\omega)$$

HAWKING UVAŽOVAL KOLABUJÚCU ČIERNU DIERU



MÓD VCHÁDZAJÚCEJ VLNY



$$f_A(t, r_*, \vartheta, \varphi) = \frac{1}{\sqrt{4\pi\omega}} \frac{1}{r} Y_{lm}(\vartheta,\varphi) F_{\omega l}(r_*) e^{-i\omega v}$$

MÓD VYCHÁDZAJÚCEJ VLNY

$$p_A(t, r_*, \vartheta, \varphi) = \frac{1}{\sqrt{4\pi\omega}} \frac{1}{r} Y_{lm}(\vartheta,\varphi) P_{\omega l}(r_*) e^{-i\omega u}$$

LIMITNÉ SPRÁVANIE: $\quad t\nearrow \leftrightarrow r_*\nearrow \to u = t - r_*$

$F_{\omega l}(r_* \to +\infty) = 1$ NA $\mathcal{I}_-$, $P_{\omega l}(r_* \to +\infty) = 1$ NA $\mathcal{I}_+$

$\mathcal{I}_- \ldots f_A \ldots \hat{a}^{IN}_A$

$\mathcal{I}_+ \ldots p_A \ldots \hat{a}^{OUT}_A$

BOGOLIUBOVOVA TRANSFORMÁCIA

SKALÁRNY SÚČIN PRE VÝPOČET BOGOLIUBOVOVÝCH KOEFICIENTOV BUDEME POČÍTAŤ NA $\mathcal{I}_-$ TAKŽE POTREBUJEME UVAŽOVAŤ VÝVOJ p_A DOZADU V ČASE Z $\mathcal{I}_+$ NA $\mathcal{I}_-$

VYCHÁDZAJÚCU VLNU ROZDELÍME NA DVE ČASTI

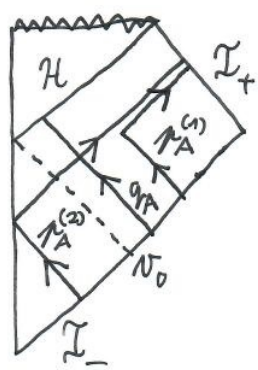
$$p_A = p_A^{(1)} + p_A^{(2)}$$

$p_A^{(1)}$ – VLNA SA ODRAZILA OD POTENCIÁLOVEJ BARIÉRY NAD HORIZONTOM

$p_A^{(2)}$ – VLNA KTORÁ PREŠLA CEZ HVIEZDU PREDTÝM NEŽ SKOLABOVALA, TÁTO ČASŤ BUDE MAŤ FAKTOR $(-1)^l$ NAVYŠE KVÔLI PREVRÁTENIU

$$Y_{lm}(-\vec{n}) = (-1)^l Y_{lm}(\vec{n})$$

V PENROSEOVOM DIAGRAME SA VLNOPLOCHY = PLOCHY KONŠTANTNEJ FÁZY PRE $p_A^{(1)}$ A $p_A^{(2)}$ ŠÍRIA PO TAKÝCHTO ČIARACH (NA CELEJ ČIARE JE KONŠTANTNÁ FÁZA)



CEZ N_0 PRECHÁDZA HRANIČNÁ VLNOPLOCHA KTORÁ PRELETÍ CEZ KOLABUJÚCU HVIEZDU A DO CENTRA DORAZÍ V OKAMIHU VZNIKU HORIZONTU UDALOSTÍ V CENTRE

VLNOPLOCHY q_A DORAZIA NESKÔR A VOJDÚ DO HORIZONTU KDE ZANIKNÚ A NEPRISPEJÚ KU KREOVANIU ČASTÍC NA $\mathcal{I}_+$

$$\hat{\phi}(x) = \begin{cases} \sum_A \left(f_A \hat{a}_A^{IN} + f_A^* \hat{a}_A^{IN+} \right) \\ \sum_A \left(p_A \hat{a}_A^{OUT} + p_A^* \hat{a}_A^{OUT+} + q_A \hat{a}_A^{HOR} + q_A^* \hat{a}_A^{HOR+} \right) \end{cases}$$

MÓDY q_A IBA DEFINUJÚ BÁZU VO FOCKOVOM PRIESTORE KTORÁ SA SLUŠNE SPRÁVA NA HORIZONTE $\mathcal{H}$ A NEPRISPIEVAJÚ KU SKALÁRNEMU SÚČINU PRE VÝPOČET BOGOLIUBOVOVÝCH KOEFICIENTOV PRE BOGOLIUBOVOVE TRANSFORMÁCIE MEDZI BÁZAMI DEFINOVANÝMI NA $\mathcal{I}_-$ A $\mathcal{I}_+$

KĽÚČOVOU VLASTNOSŤOU VLNOPLÔCH PRISLÚCHAJÚCICH $p_A^{(1)}$ JE TO, ŽE IDÚ CEZ OBLASTI, KDE JE IBA SCHWA-RZSCHILDOVA METRIKA. KVÔLI ČASOVEJ SYMETRII SCH-

-WARZSCHILDOVEJ METRIKY, TEDA $p_A^{(1)}$ BUDE MAŤ ROVNAKÝ TVAR NA $\mathcal{I}_-$ AJ $\mathcal{I}_+$, TAKŽE NA $\mathcal{I}_-$ JE $p_A^{(1)}$ ÚMERNÉ f_A:

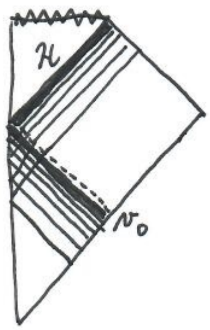
$$\langle p_A^{(1)}, f_B \rangle = \sqrt{\langle p_A^{(1)}, p_A^{(1)} \rangle}\, \delta_{AB} \propto \delta(\omega' - \omega)$$

$$\langle p_A^{(1)}, f_B^* \rangle = 0 \longrightarrow \beta_{AB}^{(1)} = 0$$

TEDA $p_A^{(1)}$ NEZMENIA FREKVENCIU A NEBUDÚ PRISPIEVAŤ K BOGOLIUBOVOVÝM KOEFICIENTOM β_{AB}

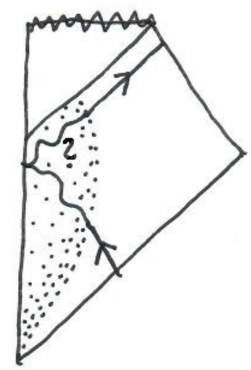
$$\beta_{AB} = -\langle p_A, f_B^* \rangle = -\langle p_A^{(2)}, f_B^* \rangle \equiv \beta_{AB}^{(2)}$$

VEČNÁ ČIERNA DIERA PRODUKUJE ROVNAKO VEĽA ČASTÍC AKO ICH POHLCUJE, TAKŽE ZOSTÁVA VEČNOU

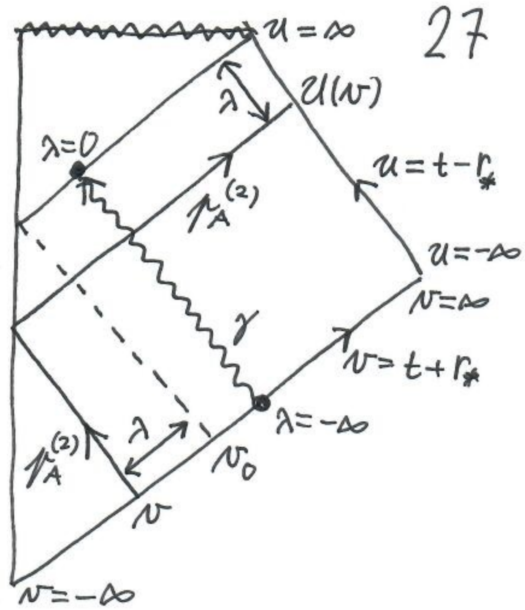


KVÔLI NEKONEČNÉMU MODRÉMU POSUNU NA HORIZONTE SA VLNOPLOCHY $p_A^{(2)}$ NAKOPIA TESNE NAD HORIZONTOM, TAKŽE NAJVÄČŠÍ PRÍSPEVOK BUDÚ MAŤ VLNOPLOCHY S v TESNE POD v_0

VĎAKA MODRÉMU POSUNU MOŽNO POUŽIŤ APROXIMÁCIU GEOMETRICKEJ OPTIKY PRI KTOREJ SA VLNOPLOCHY ŠÍRIA POZDĹŽ GEODETÍK



ZANEDBÁME JAVY PRI PRECHODE CEZ KOLABUJÚCU HVIEZDU, ALE TIE ZODPO-VEDAJÚ IBA KONEČNÉMU POČTU ČASTÍC VYKREOVANÝCH PRI KOLAPSE HVIEZDY (TO ZÁLEŽÍ AJ OD DETAILOV PRIEBEHU KOLAPSU) ALE TENTO KONEČNÝ POČET ČASTÍC SA ROZPTÝLI NA $\mathcal{I}_+$ A HLAVNÝ PRÍSPEVOK K HAWKINGOVMU ŽIARENIU BUDE OD NASLEDUJÚCICH VÝPOČTOV



VLNOPLOCHA $\mathcal{P}_A^{(2)}$ POČAS VCHÁDZANIA JE DANÁ PODMIENKOU $u = \text{KONŠT}$. NÁJSŤ VÝVOJ $\mathcal{P}_A^{(2)}$ DOZADU V ČASE ZNAMENÁ NÁJSŤ NÁJSŤ ZÁVISLOSŤ MEDZI u A v ČO UROBÍME V APROXIMÁCII GEOMETRICKEJ OPTIKY CEZ RADIÁLNE GEODETIKY. ZAVEDIEME SI POMOCNÝ VCHÁDZAJÚCI LÚČ γ CEZ KTORÝ ZADEFINUJEME GEODETICKÚ VZDIALENOSŤ VYCHÁDZAJÚCEHO $\mathcal{P}_A^{(2)}$ POZDĹŽ u NAD HORIZONTOM, ZÁROVEŇ JE TO GEODETICKÁ VZDIALENOSŤ VCHÁDZAJÚCEHO $\mathcal{P}_A^{(2)}$ POZDĹŽ v OD HRANIČNÉHO VCHÁDZAJÚCEHO LÚČA PRECHÁDZAJÚCEHO CEZ v_0. MÁME TEDA

$$v - v_0 = K_1 \lambda$$

A ZÁVISLOSŤ u OD λ NÁJDEME CEZ GEODETIKU POMOCNÉHO VCHÁDZAJÚCEHO LÚČA γ

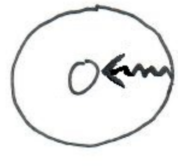
PRE RADIÁLNE NULOVÉ GEODETIKY:

$$\left(1-\frac{2M}{r}\right)\dot{t} = E = \text{KONŠT} \quad \& \quad \left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 = 0$$

$$\frac{E^2}{\left(1-\frac{2M}{r}\right)^2}$$

$$\frac{E - \dot{r}^2}{1-\frac{2M}{r}} = 0$$

$$\dot{r} = \pm E$$



$$r = K_0 - E\lambda'$$

$$\dot{t} = \frac{E}{1-\frac{2M}{K_0 - E\lambda'}} = \frac{E(K_0 - E\lambda')}{K_0 - E\lambda' - 2M} = \frac{E(K_0 - E\lambda' - 2M + 2M)}{K_0 - E\lambda' - 2M} =$$

$$= E + \frac{2ME}{K_0 - E\lambda' - 2M}$$

$$t = E\lambda' - 2M \ln|K_0 - 2M - E\lambda'|$$
$$r = K_0 - E\lambda'$$
$$r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right| = K_0 - E\lambda' + 2M \ln\left|\frac{K_0 - E\lambda' - 2M}{2M}\right| =$$
$$= K_0 \underbrace{- E\lambda' + 2M \ln|K_0 - E\lambda' - 2M|}_{-t} - 2M \ln(2M) =$$
$$= -t + K_0 - 2M \ln(2M)$$
$$u = t - r_* = 2t - K_0 + 2M \ln(2M) =$$
$$= 2E\lambda' - 4M \ln|K_0 - 2M - E\lambda'| - K_0 + 2M \ln(2M) =$$
$$= 2E\lambda' - 4M \ln \frac{e^{\frac{K_0}{2M}}|K - 2M - E\lambda|}{\sqrt{2M}} =$$
$$= 2E\lambda' - 4M \ln\left(\left|\lambda' + \frac{2M - K_0}{E}\right| \frac{E e^{\frac{K_0}{4M}}}{\sqrt{2M}}\right) =$$
$$= 2E\Big(\underbrace{\lambda' + \frac{2M - K_0}{E}}_{\lambda}\Big) - 4M \ln\Big(\Big|\underbrace{\lambda' + \frac{2M - K_0}{K}}_{\lambda}\Big| \underbrace{\frac{E e^{\frac{K_0}{4M}}}{\sqrt{2M}} e^{\frac{2(2M - K_0)}{4M}}}_{K_2^{-1} = \frac{E}{\sqrt{2M}} e^{1 - \frac{K_0}{4M}}}\Big)$$
$$\lambda = \lambda' + \frac{2M - K_0}{E}$$
$$K_2 = \frac{\sqrt{2M}}{E} e^{\frac{K_0}{4M} - 1} \qquad u = 2E\lambda - 4M \ln\left|\frac{\lambda}{K_2}\right|$$

PRIPOMEŇME $v - v_0 = K_1 \lambda \rightarrow \lambda = \dfrac{v - v_0}{K_1}$ & $v < v_0$

$$\Rightarrow u(v) = 2E \frac{v - v_0}{K_1} - 4M \ln \frac{|v - v_0|}{|K_1 K_2|} \approx \left[\lim_{v \to v_0^-}\right]$$
$$\approx -4M \ln \frac{v_0 - v}{K} \qquad \text{KDE } K = |K_1 K_2|$$

BUDEME POČÍTAŤ SKALÁRNE SÚČINY MÓDOV $p_A^{(2)}$ A f_A NA $\mathcal{I}_-$

$$f_A\big|_{\mathcal{I}_-}(t,r_*,\vartheta,\varphi) = \frac{1}{\sqrt{4\pi\omega}}\frac{1}{r}Y_{\ell m}(\vartheta,\varphi)\,e^{-i\omega v}$$

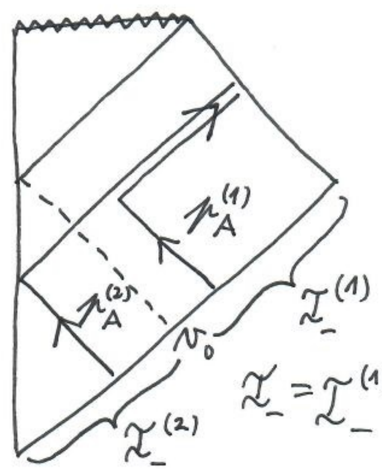
$$p_A^{(2)}\big|_{\mathcal{I}_-}(t,r_*,\vartheta,\varphi) \approx \frac{1}{\sqrt{4\pi\omega}}\frac{1}{r}(-1)^\ell Y_{\ell m}(\vartheta,\varphi)\,T_{\omega\ell}\,e^{-i\omega\left(-4M\ln\frac{v_0-v}{K}\right)}$$

$(-1)^\ell$ ... VLNA SA PRI PRECHODE CEZ POČIATOK PREVRÁTILA

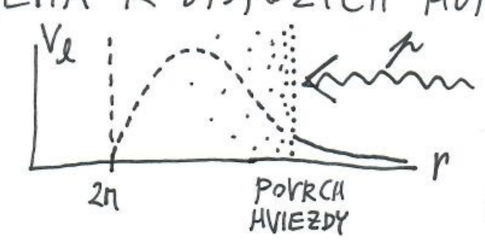
$T_{\omega\ell}$ ... ABSORPČNÝ FAKTOR ZA PRECHOD VLNY CEZ KOLABUJÚCU HVIEZDU

K ABSORPČNÉMU KOEFICIENTU:

$p_A^{(1)}$ A $p_A^{(2)}$ MAJÚ DISJUNKTNÉ SUPORTY NA $\mathcal{I}_-$ PRETOŽE VLNA PRICHÁDZAJÚCA Z $\mathcal{I}_-^{(1)}$ PRÍDE AŽ PO KOLAPSE A UŽ NEMÔŽE PREJSŤ CEZ KOLABU-JÚCU HVIEZDU A VLNA PRICHÁDZAJÚCA Z $\mathcal{I}_-^{(2)}$ NEMÁ K DISPOZÍCII HORIZONT A LEN ŤAŽKO SA ODRAZÍ



$\mathcal{I}_- = \mathcal{I}_-^{(1)} \cup \mathcal{I}_-^{(2)}$



Z TOHO VYPLÝVAJÚ DVE VECI:

① K $p_A^{(2)}\big|_{\mathcal{I}}$ TREBA PRIDAŤ HEAVISIDEOVU FUNKCIU $\vartheta(v_0 - v)$

② $\delta_{AB} = \langle p_A, p_B\rangle =$

$$= \underbrace{\langle p_A^{(1)}, p_B^{(1)}\rangle}_{(1-\Gamma_A)\delta_{AB}} + \underbrace{\langle p_A^{(2)}, p_B^{(2)}\rangle}_{\Gamma_A\delta_{AB}} + \underbrace{\langle p_A^{(1)}, p_B^{(2)}\rangle + \langle p_A^{(2)}, p_B^{(1)}\rangle}_{0}$$

Γ_A – ABSORPČNÝ KOEFICIENT $\in\langle 0,1\rangle$, $\Gamma_A = |T_{\omega\ell}|^2$

$$p_A^{(2)}\Big|_{\mathcal{I}_-}(t,r_*,\vartheta,\varphi) \approx \frac{1}{\sqrt{4\pi\omega}}\frac{1}{r}(-1)^l Y_{lm}(\vartheta,\varphi)\, T_{\omega l}\left(\frac{v_0-v}{K}\right)^{i\omega 4M}\vartheta(v_0-v)$$

$$\alpha_{AA'} = \langle p_A^{(2)}, f_{A'}\rangle = -i\int_{-\infty}^{\infty}dv\int_0^{\pi}d\vartheta\int_0^{2\pi}d\varphi\, r^2\sin\vartheta\left(p_A^{(2)}\Big|_{\mathcal{I}_-}\overleftrightarrow{\partial_v} f_{A'}^*\Big|_{\mathcal{I}_-}\right) =$$

$$= -i\int_{-\infty}^{v_0}dv\int_0^{\pi}d\vartheta\int_0^{2\pi}d\varphi\, r^2\sin\vartheta\,\frac{1}{4\pi\sqrt{\omega\omega'}}\frac{1}{r^2}(-1)^l Y_{lm}(\vartheta,\varphi)Y^*_{l'm'}(\vartheta,\varphi)\,T_{\omega l}\cdot$$

$$\cdot\left(\left(\frac{v_0-v}{K}\right)^{i\omega 4M}\overleftrightarrow{\partial_v} e^{+i\omega' v}\right) =$$

$$= \frac{-i(-1)^l T_{\omega l}}{4\pi\sqrt{\omega\omega'}}\delta_{ll'}\delta_{mm'}\int_{-\infty}^{v_0}dv\left[\left(\frac{v_0-v}{K}\right)^{i\omega 4M}\left(\partial_v e^{+i\omega' v}\right) - \left(\partial_v\left(\frac{v_0-v}{K}\right)^{i\omega 4M}\right)e^{+i\omega' v}\right] =$$

PER PARTES

$$= \frac{-i(-1)^l T_{\omega l}}{4\pi\sqrt{\omega\omega'}}\delta_{ll'}\delta_{mm'}\left\{\int_{-\infty}^{v_0}dv\, 2\left(\frac{v_0-v}{K}\right)^{i\omega 4M}\left(\partial_v e^{+i\omega' v}\right) - \underbrace{\left[\left(\frac{v_0-v}{K}\right)^{i\omega 4M}e^{+i\omega' v}\right]_{v=-\infty}^{v=v_0}}_{0-\text{"}\infty^i e^{i\infty}\text{" OSC.}}\right\} =$$

$$= \frac{-i(-1)^l T_{\omega l}}{4\pi\sqrt{\omega\omega'}}\delta_{ll'}\delta_{mm'}\,2i\omega'\int_{-\infty}^{v_0}dv\left(\frac{v_0-v}{K}\right)^{i\omega 4M}e^{+i\omega' v} =$$

SUBSTITÚCIA $\quad x = \dfrac{v_0-v}{K}\quad \displaystyle\int_{-\infty}^{v_0}dv = \int_{\infty}^{0}(-K)\,dx = K\int_0^{\infty}dx$

$v = v_0 - Kx$

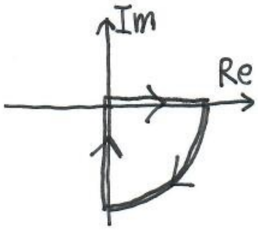
$$= \frac{-i(-1)^l T_{\omega l}}{4\pi\sqrt{\omega\omega'}}\delta_{ll'}\delta_{mm'}\,2i\omega'\,e^{i\omega' v_0}K\int_0^{\infty}dx\; x^{i\omega 4M}e^{-i\omega' Kx} =$$

$y = \omega' K x$

$$= \frac{(-1)^l T_{\omega l}}{2\pi}\sqrt{\frac{\omega'}{\omega}}\,\delta_{ll'}\delta_{mm'}\,e^{i\omega' v_0}K\frac{1}{\omega' K}\left(\frac{1}{\omega' K}\right)^{i\omega 4M}\cdot$$

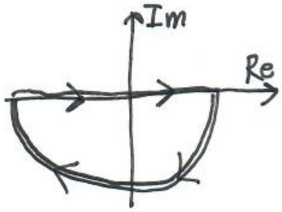
$$\cdot\int_0^{\infty}dy\; y^{i\omega 4M}e^{-iy}$$ POTREBUJEME EULEROVU GAMA FUNKCIU

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x}$$



$$\int_0^\infty dx\, x^{z-1} e^{-x} + \underbrace{\int_{-i\infty}^{0} dx\, x^{z-1} e^{-x}}_{} = 0$$

$$x = iy \qquad \underbrace{i^z \int_{-\infty}^{0} dy\, y^{z-1} e^{-iy}}_{}$$



$$= -\int_0^\infty dy\, y^{z-1} e^{-iy}$$

$$\int_0^\infty dx\, x^{z-1} e^{-ix} = \frac{1}{i^z}\Gamma(z) = e^{-\frac{1}{2} i\pi z}\,\Gamma(z)$$

$$\int_0^\infty dx\, x^{z-1} e^{+ix} = [x=-y] = (-1)^z \underbrace{\int_0^{-\infty} dy\, y^{z-1} e^{-iy}}_{}$$

$$\int_0^\infty dy\, y^{z-1} e^{-iy}$$

$$\int_0^\infty dx\, x^{z-1} e^{+ix} = e^{i\pi z} e^{-\frac{1}{2} i\pi z}\,\Gamma(z) = e^{\frac{1}{2} i\pi z}\,\Gamma(z)$$

$$\alpha_{AA'} = \frac{(-1)^l T_{\omega l}}{2\pi} \sqrt{\frac{\omega'}{\omega}}\, \delta_{ll'} \delta_{mm'}\, e^{i\omega' v_0} K \left(\frac{1}{\omega' K}\right)^{1+i\omega 4M} \cdot$$

$$\cdot e^{-\frac{1}{2} i\pi (1+i\omega 4M)}\, \Gamma(1+i\omega 4M) =$$

$$= e^{\pi\omega 2M} (-i) \frac{(-1)^l T_{\omega l}}{2\pi} \frac{1}{\sqrt{\omega\omega'}} e^{i\omega' v_0} (\omega' K)^{-i\omega 4M}\, i\omega 4M\, \Gamma(i\omega 4M) \cdot$$

$$\cdot \delta_{ll'} \delta_{mm'} =$$

$$= e^{\pi\omega 2M} (-1)^l T_{\omega l} \frac{2M}{\pi} \sqrt{\frac{\omega}{\omega'}}\, e^{i\omega' v_0} e^{-i\omega 4M \ln(\omega' K)}\, \Gamma(i\omega 4M)\, \delta_{ll'} \delta_{mm'}$$

CHÝBA $\delta(\omega' - \omega) \rightarrow p_A^{(2)}$ ZMENILO FREKVENCIU PRI PRECHODE KOLABUJÚCOU HVIEZDOU

$$\beta_{AA'} = -\langle p_A^{(2)}, f_{A'}^{*}\rangle =$$

$$= i \lim_{r\to\infty} \int_{-\infty}^{\infty} dv \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi\, r^2 \sin\vartheta \left(p_A^{(2)}\Big|_{\mathcal{I}_-} \overleftrightarrow{\partial_v} f_{A'}\Big|_{\mathcal{I}_-} \right) =$$

$$= i \int_{-\infty}^{v_0} dv \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sin\vartheta \frac{1}{4\pi\sqrt{\omega\omega'}} Y_{lm}(\vartheta,\varphi) \overbrace{\overline{Y_{l'm'}}(\vartheta,\varphi)}^{(-1)^{m'} Y^{*}_{l',-m'}(\vartheta,\varphi)} (-1)^l T_{\omega l} \cdot$$

$$\cdot \left(\left(\frac{v_0 - v}{K}\right)^{i\omega 4M} \overleftrightarrow{\partial_v} e^{-i\omega' v} \right) =$$

$$= \frac{i(-1)^{l+m'} T_{\omega l}}{4\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{m,-m'} \int_{-\infty}^{v_0} dv \left[\left(\frac{v_0-v}{K}\right)^{i\omega 4M} (\partial_v e^{-i\omega' v}) - \left(\partial_v \left(\frac{v_0-v}{K}\right)^{i\omega 4M}\right) e^{-i\omega' v} \right] =$$

PER PARTES

$$= \frac{i(-1)^{l+m'} T_{\omega l}}{4\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{m,-m'} \left\{ \int_{-\infty}^{v_0} dv\, 2 \left(\frac{v_0-v}{K}\right)^{i\omega 4M} (-i\omega') e^{-i\omega' v} + \text{OKR. ČLEN} \right\} =$$

SUBSTITÚCIA $X = \omega'(v_0 - v)$ $\qquad dv = -\frac{dX}{\omega'}$

$$\int_{-\infty}^{v_0} \longrightarrow \int_{\infty}^{0} = -\int_0^{\infty} \qquad v = v_0 - \frac{X}{\omega'}$$

$$= \frac{(-1)^{l+m'} T_{\omega l}}{2\pi} \sqrt{\frac{\omega'}{\omega}} \delta_{ll'} \delta_{m,-m'} \frac{1}{\omega'} \left(\frac{1}{\omega' K}\right)^{i\omega 4M} e^{-i\omega' v_0} \int_0^{\infty} dX\, X^{i\omega 4M} e^{iX} =$$

$$e^{\frac{1}{2} i\pi(1+i\omega 4M)} \underbrace{\Gamma(1+i\omega 4M)}_{i\omega 4M\, \Gamma(i\omega 4M)}$$

$$= i e^{-\pi\omega 2M} \overbrace{e^{\frac{1}{2} i\pi}}^{i} (-1)^{l+m'} \frac{2M}{\pi} \sqrt{\frac{\omega}{\omega'}} e^{-i\omega' v_0} \left(\frac{1}{\omega' K}\right)^{i\omega 4M} T_{\omega l} \cdot$$

$$\cdot \Gamma(i\omega 4M) \delta_{ll'} \delta_{m,-m'} =$$

$$= -e^{-\pi\omega 2M} (-1)^{l+m'} T_{\omega l} \frac{2M}{\pi} \sqrt{\frac{\omega}{\omega'}} e^{-i\omega' v_0} e^{-i\omega 4M \ln(\omega' K)} \Gamma(i\omega 4M) \delta_{ll'} \delta_{m,-m'}$$

VIDÍME ŽE $|\alpha_{AB}|^2 = (e^{\pi\omega 2M} e^{\pi\omega 2M})^2 |\beta_{AB}|^2$

TO JE JEDINÉ ČO POTREBUJEME NA ODVODENIE SPEKTRA KREOVANÝCH ČASTÍC

$$\Gamma_A \delta_{AB} = \langle p_A^{(2)}, p_B^{(2)} \rangle = \left[p_A^{(2)} = \sum_B \left(\alpha^*_{AB} f_B - \beta_{AB} f_B^* \right) \right]$$

$$= \sum_C \sum_D \langle \alpha^*_{AC} f_C - \beta_{AC} f_C^* , \alpha^*_{BD} f_D - \beta_{BD} f_D^* \rangle =$$

VLASTNOSŤ SKALÁRNEHO SÚČINU $\langle Af, Bg \rangle = AB^* \langle f, g \rangle$

$$= \sum_C \sum_D \left(\alpha^*_{AC} \alpha_{BD} \underbrace{\langle f_C, f_D \rangle}_{\delta_{CD}} + \beta_{AC} \beta^*_{BD} \underbrace{\langle f_C^*, f_D^* \rangle}_{-\delta_{CD}} \right) =$$

$$= \sum_C \left(\alpha^*_{AC} \alpha_{BC} - \beta_{AC} \beta^*_{AC} \right) \qquad \underbrace{|T_{\omega\ell}|^2}_{} \overset{\text{OZN.}}{=} \Gamma_\ell(\omega)$$

$$A=B: \quad \sum_C \left(\underbrace{|\alpha_{AC}|^2}_{e^{\pi\omega 8M}|\beta_{AC}|^2} - |\beta_{AC}|^2 \right) = \cancel{\sum_A} \underbrace{\delta_{AA}}_{\delta_{\ell\ell'}\delta_{mm'}\delta(\omega'-\omega)\big|_{\substack{\ell=\ell' \\ m=m' \\ \omega'=\omega}}} \Gamma_A$$

$$\delta(\omega'-\omega)\big|_{\omega'=\omega} = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-it(\omega'-\omega)}\Big|_{\omega'=\omega} = \frac{\text{CELKOVÝ ČAS}}{2\pi} \equiv \frac{T}{2\pi}$$

$$N_A = \langle 0_{IN} | \hat{a}_A^{OUT+} \hat{a}_A^{OUT} | 0_{IN} \rangle = \sum_B |\beta_{AB}|^2 \equiv N_{\omega\ell m}$$

$$\left(e^{\pi\omega 8M} - 1 \right) N_{\omega\ell m} = \frac{T}{2\pi} \Gamma_\ell(\omega)$$

POČET ČASTÍC S FREKVENCIOU ω VYGENEROVANÝCH ZA JEDNOTKU ČASU

$$\frac{1}{T} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} N_{\omega\ell m} = \frac{1}{2\pi} \frac{\sum_{\ell=0}^{\infty} (2\ell+1) \Gamma_\ell(\omega)}{e^{\pi\omega 8M} - 1}$$

FAKTOR V ČITATELI JE "GREYBODY" FAKTOR ZA "ROZPTYL ŽIARENIA ČIERNEHO TELESA NA GEOMETRII ČIERNEJ DIERY" A MENOVATEĽ ZODPOVEDÁ SPEKTRU ŽIARENIA ČIERNEHO TELESA AK

$$\pi\omega 8M = \frac{\omega}{k_B T_H} \qquad T_H \text{ - HAWKINGOVA TEPLOTA}$$

$$T_H = \frac{1}{8\pi k_B M} = \left[\text{V S.I. JEDNOTKÁCH}\right] = \frac{1}{8\pi M}\frac{\hbar c^3}{k_B G}$$

ČASTO SA TO VYJADRUJE CEZ POVRCHOVÚ GRAVITÁCIU κ DEFINOVANÚ CEZ ROVNICU

$$\xi^\mu \nabla_\mu \xi^\nu = -\kappa \xi^\nu$$

KDE ξ JE ČASOVÝ KILLINGOV 4-VEKTOR

PRE SCHWARZSCHILDOVU METRIKU VYCHÁDZA POVRCHOVÁ GRAVITÁCIA NA HORIZONTE AKO

$$\kappa = \frac{1}{4M} \qquad \text{A POTOM} \qquad \boxed{T_H = \frac{\kappa}{2\pi k_B}}$$

ENERGIA $\leftrightarrow$ HMOTNOSŤ KTORÚ ČIERNA DIERA STRÁCA VYŽAROVANÍM JE DANÁ

$$\frac{dM}{dt} = -\int_0^\infty d\omega\, \omega \left(\frac{1}{T}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} N_{\omega\ell m}\right) = -\int_0^\infty d\omega\, \omega \frac{\frac{1}{2\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\Gamma_\ell(\omega)}{e^{\pi\omega 8M}-1} \sim$$

PRI HRUBOM PRIBLÍŽENÍ

$$\sim \int d\omega \frac{\omega}{e^{\pi\omega 8M}-1} \sim \frac{1}{M^2}\int_0^\infty dx \frac{x}{e^x-1} \sim \frac{1}{M^2} \rightarrow M^2 dM \sim dt$$

$$t \sim M^3 = \left[\text{S.I. JEDNOTKY}\right] = M^3 \frac{G^2}{\hbar c^4} = \left(\frac{M}{M_\odot}\right)^3 \frac{M_\odot^3 G^2}{\hbar c^4} \qquad M_\odot \text{ - HMOTNOSŤ SLNKA}$$

$$\frac{M_\odot^3 G^2}{\hbar c^4} = \frac{(2\cdot 10^{30}\,kg)^3 (6{,}67\cdot 10^{-11}\, m^3 kg^{-1} s^{-2})^2}{1{,}05\cdot 10^{-34}\, kg\, m^2 s^{-1} (3\cdot 10^8\, m\, s^{-1})^4} \sim \frac{10^{30\cdot 3}10^{-22}}{10^{-34}10^{32}}\, s = 10^{66}\, s$$

(JEDNA MILIARDA ROKOV JE $\approx 3\cdot 10^{16}\, s$)

VYPAROVANIE ČIERNEJ DIERY ZNAMENÁ ŽE POLE SPÄTNE VPLÝVA NA ČASOPRIESTOR, ČO JE ĎALŠÍ DÔVOD PREČO JE POTREBNÁ ÚPLNEJŠIA KVANTOVÁ TEÓRIA GRAVITÁCIE

KĽÚČOVOU VLASTNOSŤOU KTORÁ SPÔSOBUJE HAWKINGOV JAV JE EXISTENCIA HORIZONTU. VLNY KTORÉ PRELE-TELI CEZ KOLABUJÚCU HVIEZDU SA NAKOPILI TESNE NAD HORIZONTOM KDE DOŠLO K LIMITNE NEKONEČNE VEĽKÉMU MODRÉMU POSUNU. MODRÝ POSUN SÚVISÍ S DILATÁCIOU ČASU. KVÔLI TOMU Z POHĽADU POZOROVATEĽA NA POVRCHU KOLABUJÚCEJ HVIEZDY TESNE PO SFORMOVANÍ HORIZONTU VZNIKLI ČASTICE ZA LIMITNE NEKONEČNE KRÁTKY OKAMIH A TIE ČASTICE SA Z POHĽADU POZOROVATEĽA NEKONEČNE ĎALEKO OD ČIERNEJ DIERY VYŽARUJÚ POSTUPNE AŽ KÝM SA ČIERNA DIERA NEVYPARÍ. DÁ SA TEDA POVEDAŤ ŽE HAWKING-OVO ŽIARENIE MÁ PÔVOD VEĽMI TESNE NAD HORIZOTOM A BEZ HORIZONTU BY NEBOLO

TECHNICKÝ NÁHĽAD NA TO ŽE BEZ HORIZONTU BY NIČ NEVYŠLO JE NAPRÍKLAD $\int_{-\infty}^{v_0} dv \ldots \rightarrow \int_{-\infty}^{\infty} dv \left(\frac{v_0 - v}{K}\right)^{i\omega 4m} e^{\pm i\omega' v} \xrightarrow{osc.} 0$

ANALÓGIA HAWKINGOVHO JAVU NASTÁVA AJ V INÝCH ČASOPRIESTOROCH AK JE V NICH PRÍTOMNÝ HORIZONT:

- VEČNÁ ČIERNA DIERA (VYŽAROVANIE = POHLCOVANIE)
- ROZPÍNAJÚCI SA VESMÍR (KOZMOLOGICKÝ HORIZONT)
- → POTOM AJ DE SITTEROV ČASOPRIESTOR
- UNRUHOV JAV ZA HORIZONT V RINDLEROVÝCH SÚRADNICIACH

EXISTENCIA HAWKINGOVEJ TEPLOTY MOTIVUJE UVAŽOVAŤ NAD ANLÓGIAMI S TERMODYNAMIKOU

KOMENTÁRE K TERMODYNAMIKE ČIERNYCH DIER

HAWKINGOV JAV SME POČÍTALI PRE BEZHMOTOVÉ SKALÁRNE POLE V SCHWARZSCHILDOVOM ČASOPRIESTORE, DÁ SA VŠAK POČÍTAŤ AJ PRE NABITÉ POLE V SKALÁRNEJ ELEKTRODYNAMIKE V KERROVOM-NEWMANOVOM ČASOPRIESTORE S ROTUJÚCOU A NABITOU ČIERNOU DIEROU: M – HMOTNOSŤ
J – MOMENT HYBNOSTI
Q – NÁBOJ

$$\frac{1}{T} N_{\omega l m q} = \frac{\frac{1}{2\pi}\Gamma_{lmq}(\omega)}{e^{\frac{2\pi}{\kappa}(\omega - m\Omega - q\phi)} - 1}$$

$\kappa = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2}$ POVRCHOVÁ GRAVITÁCIA NA HORIZONTE $(r = r_+)$

$\Omega = \frac{a}{r_+^2 + a^2}$ UHLOVÁ RÝCHLOSŤ HORIZONTU

$\phi = \frac{Q r_+}{r_+^2 + a^2}$ ROZDIEL ELEKTRICKÝCH POTENCIÁLOV MEDZI HORIZONTOM A NEKONEČNOM

$a = \frac{J}{m}$, HORIZONTY $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$

POVRCH HORIZONTU ČIERNEJ DIERY JE $A = 4\pi(r_+^2 + a^2)$

$$\longrightarrow \delta A = \frac{\partial A}{\partial M}\delta M + \frac{\partial A}{\partial J}\delta J + \frac{\partial A}{\partial Q}\delta Q = \dots$$

$$\dots = \frac{16\pi}{r_+ - r_-}\left[(r_+^2 + a^2)\delta M - a\delta J - r_+ Q\delta Q\right]$$

$$\delta M = \underbrace{\frac{1}{16\pi}\frac{r_+ - r_-}{r_+^2 + a^2}}_{\kappa/8\pi}\delta A + \underbrace{\frac{a}{r_+^2 + a^2}}_{\Omega}\delta J + \underbrace{\frac{r_+ Q}{r_+^2 + a^2}}_{\phi}\delta Q$$

TO JE PRVÁ TERMODYNAMICKÁ VETA PRE ČIERNE DIERY

KLASICKÉ TERMODYNAMICKÉ VETY (PRIPOMIENKA):

0. V ROVNOVÁHE $T = \text{KONŠT.}$

1. $dU = \delta Q - \delta W$

2. RAST ENTROPIE $\delta S \geq 0$

$\Big\}\ dU = T dS - p dV$

3. $T \neq 0$

TERMODYNAMICKÉ VETY PRE ČIERNE DIERY:

0. $\kappa = \text{KONŠT.}$ POVRCHOVÁ GRAVITÁCIA JE KONŠTANTNÁ NA CELOM HORIZONTE

1. $$\delta M = \underbrace{\frac{\kappa}{8\pi}\,\delta A}_{T_H\, dS_{BH}} \underbrace{+\,\Omega\,\delta J + \phi\,\delta Q}_{-\delta W}$$

BEKESTEINOVA - HAWKINGOVA ENTROPIA

$$\underbrace{T_H}_{\kappa/2\pi k_B} dS_{BH} = \frac{\kappa}{8\pi}\,\delta A$$

$$\to S_{BH} = \frac{1}{4} k_B A \overset{\text{S.I.}}{=} \frac{1}{4}\frac{k_B c^3}{G\hbar} A$$

$$k_B = 1 : \boxed{S_{BH} = \frac{1}{4} A}$$

2. $\delta A \geq 0$ "HAWKING AREA THEOREM"

NEPLATÍ PRI VYPAROVANÍ ČIERNEJ DIERY PRETOŽE TAM PLATÍ $\delta\left(S_{BH} + S_{OKOLIE}\right) \geq 0$

3. $\kappa \neq 0$ $$\kappa = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2} = \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}$$

PRE $\kappa = 0$ BY SME MALI EXTREMÁLNU ČIERNU DIERU $a^2 + Q^2 = M^2$ S NAHOU SINGULARITOU