Notes on Path Integral Methods

QM: $\hat{H} = \frac{\hat{P}^2}{2m} + V(x)\hat{1}$, $\hat{P} = -i\hbar\frac{\partial}{\partial x}$, $\underbrace{i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle}^{\text{Sch. Eq.}}$

POSITION, MOMENTUM, DIRAC DELTA $\rightarrow$

$\hat{X}|X\rangle = X|X\rangle \qquad \langle x'|x\rangle = \delta(x'-x)$

$\hat{P}|P\rangle = P|P\rangle \qquad \langle P'|P\rangle = \delta(P'-P)$

$$\langle x|P\rangle = \frac{e^{\frac{i}{\hbar}Px}}{\sqrt{2\pi\hbar}}$$

$$\int dx\, |x\rangle\langle x| = \int dP\, |P\rangle\langle P| = \hat{1}$$

$$\delta(x'-x) = \int dP \langle x'|P\rangle\langle P|x\rangle = \int dP\, \frac{e^{\frac{i}{\hbar}P(x'-x)}}{2\pi\hbar} =$$

$$= \int \frac{dy}{2\pi} e^{iy(x'-x)} \qquad \text{O.K.}$$

TIME EVOLUTION $\quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\psi(t_0)\rangle$

PROBABILITY ★ TO FIND PARTICLE AT TIME T AT POSITION x' – PROPAGATOR

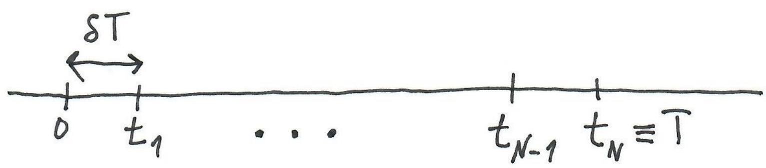
$$K(x',T;x,0) = \langle x'|\psi(T)\rangle = \langle x'|e^{-\frac{i}{\hbar}\hat{H}T}|x\rangle$$

$$\psi(t,x) = \langle x|\psi(t)\rangle = \langle x|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|\psi(t')\rangle =$$

$$= \int dx' \langle x|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|x'\rangle\langle x'|\psi(t')\rangle \equiv$$

$$\equiv \int dx'\, K(x,t;x',t')\,\psi(t',x')$$

PROPAGATOR AS PATH INTEGRAL



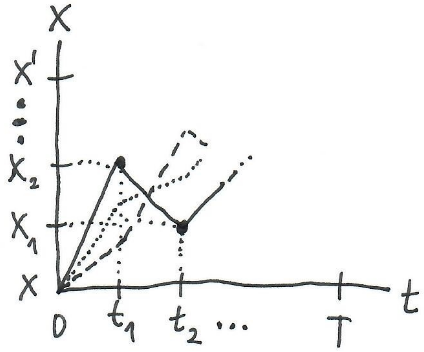
$$K(x',T;x,0) = \langle x'|\underbrace{e^{-\frac{i}{\hbar}\hat{H}\delta T}\cdots e^{-\frac{i}{\hbar}\hat{H}\delta T}}_{N}|x\rangle =$$

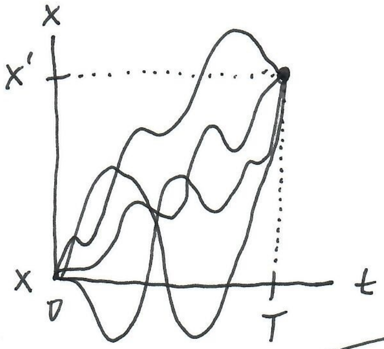
$$= \langle x'|e^{-\frac{i}{\hbar}\hat{H}\delta T}\int dx_{N-1}|x_{N-1}\rangle\langle x_{N-1}|e^{-\frac{i}{\hbar}\hat{H}\delta T}\cdots$$

$$\cdots e^{-\frac{i}{\hbar}\hat{H}\delta T}\int dx_1 |x_1\rangle\langle x_1|e^{-\frac{i}{\hbar}\hat{H}\delta T}|x\rangle$$

★ **probability density**

WE HAVE INTEGRALS $\int dx_1 \dots \int dx_{N-1} \longrightarrow \sum_{\text{PATHS}}$





$$\hat{H}_n \equiv \hat{H}(t=t_n)$$

$$\langle x_{n+1}| e^{-\frac{i}{\hbar}\hat{H}\delta T} |x_n\rangle \approx \langle x_{n+1}| e^{-\frac{i}{\hbar}\hat{H}_n \delta T} |x_n\rangle =$$

$$= \langle x_{n+1}| \left(\hat{1} - \frac{i}{\hbar}\hat{H}_n \delta T + \mathcal{O}(\delta T^2)\right) |x_n\rangle =$$

$$= \langle x_{n+1}|x_n\rangle - \frac{i}{\hbar}\delta T \langle x_{n+1}|\hat{H}_n|x_n\rangle + \mathcal{O}(\delta T^2) =$$

$$= \delta(x_{n+1}-x_n) - \frac{i}{\hbar}\delta T \langle x_{n+1}|\hat{H}_n \int dp_n |p_n\rangle\langle p_n|x_n\rangle + \mathcal{O}(\delta T^2) =$$

$$\left\{ \begin{array}{l} \hat{H}|p_k\rangle = \frac{1}{2m}\hat{p}^2|p_k\rangle + V(x)|p_k\rangle \equiv \underset{\uparrow}{H}|p_k\rangle \\ |p\rangle \text{ IS EIGENSTATE OF } \hat{H} \qquad\qquad \text{EIGENNUMBER} \end{array} \right\}$$

$$= \underbrace{\delta(x_{n+1}-x_n)}_{\int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}p_n(x_{n+1}-x_n)}} - \frac{i}{\hbar}\delta T \int dp_n\, H_n \underbrace{\langle x_{n+1}|p_n\rangle\langle p_n|x_n\rangle}_{\frac{e^{\frac{i}{\hbar}p_n(x_{n+1}-x_n)}}{2\pi\hbar}} + \mathcal{O}(\delta T^2) =$$

$$= \int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}p_n \underbrace{(x_{n+1}-x_n)}_{\dot{x}_n \delta T}} \underbrace{\left(1 - \frac{i}{\hbar}\delta T\, H_n + \mathcal{O}(\delta T^2)\right)}_{\approx e^{-\frac{i}{\hbar}H_n \delta T}} \approx$$

$$\approx \int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T(p_n \dot{x}_n - H_n)}$$

BACK TO PROPAGATOR

$$K(\underset{\substack{\| \\ x_N}}{x'}, T; \underset{\substack{\| \\ x_0}}{x}, 0) = \left[\prod_{n=1}^{N-1} \int dx_n \langle x_{n+1}| e^{-\frac{i}{\hbar}\hat{H}\delta T} |x_n\rangle\right] \langle x_1| e^{-\frac{i}{\hbar}\hat{H}\delta T} |x_0\rangle \Rightarrow$$

$$\Rightarrow \prod_{m=1}^{N-1} \int dx_m \int \frac{dp_m}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T (p_m \dot{x}_m - H_m)} \int \frac{dp_0}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T (p_0 \dot{x}_0 - H_0)} =$$

$$= \int \prod_{i=1}^{N-1} dx_i \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T (p_j \dot{x}_j - H_j)} = \quad \left\{ \prod_{i=1}^{2} f_i e^{g_i} = f_1 e^{g_1} f_2 e^{g_2} = f_1 f_2 e^{g_1+g_2} = \prod_{i=1}^{2} f_i \, e^{\sum_{k=1}^{2} g_k} \right\}$$

$$= \int \prod_{i=1}^{N-1} dx_i \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} (p_k \dot{x}_k - H(p_k, x_k))}$$

NOTATION $\longrightarrow \int \mathcal{D}x(t)\mathcal{D}p(t) \exp\left\{\frac{i}{\hbar}\int_0^T dt\,(p\dot{x} - H(p,x))\right\}$ PATH INTEGRAL

RECALL LEGENDRE TRANSFORM $H = p\dot{x} - L \Rightarrow$ WE SHOULD BE ABLE TO REWRITE THE PATH INTEGRAL AS $\int \ldots \exp\left\{\frac{i}{\hbar}\int dt\, L\right\} \longrightarrow$

$$K(x', T; x, 0) = \int \prod_{i=1}^{N-1} dx_i \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} (p_k \dot{x}_k - H(p_k, x_k))} =$$

$$= \int \prod_{i=1}^{N-1} dx_i \underbrace{\int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar}} e^{-\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} V(x_k)} \underbrace{e^{\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} \left(p_k \dot{x}_k - \frac{p_k^2}{2m}\right)}} \Rightarrow$$

DENOTE

$$\prod_{j=0}^{N-1} \mathcal{I}_j$$

$$\mathcal{I}_j = \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T \left(p\dot{x}_j - \frac{p^2}{2m}\right)}$$

$$\left\{ -\frac{p^2}{2m} + p\dot{x}_j = -\frac{p^2}{2m} + p\dot{x}_j - \left(\frac{\dot{x}_j \sqrt{2m}}{2}\right)^2 + \frac{\dot{x}_j^2 m}{2} = -\left(\frac{p}{\sqrt{2m}} - \frac{\dot{x}_j\sqrt{m}}{\sqrt{2}}\right)^2 + \frac{\dot{x}_j^2 m}{2} = -\frac{1}{2m}(p - \dot{x}_j m)^2 + \frac{\dot{x}_j^2 m}{2} \right\}$$

$$= \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T \frac{1}{2} m \dot{x}_j^2} \underbrace{\int dp\, e^{-\frac{i}{\hbar}\delta T \frac{(p - m\dot{x}_j)^2}{2m}}}_{\sqrt{\frac{\pi}{i}}\sqrt{\frac{\hbar 2m}{\delta T}}}$$

$\leftarrow$ GAUSS/FRESNEL INTEGRAL

RECALL FRESNEL INTEGRALS

$$C(u) - i S(u) = \int_0^u \left[\cos\left(\tfrac{1}{2}\pi x^2\right) - i \sin\left(\tfrac{1}{2}\pi x^2\right)\right] dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-u}^{u} \left[\cos(x^2) - i \sin(x^2)\right] dx \longrightarrow \begin{pmatrix} C(\pm\infty) = \\ = S(\pm\infty) = \pm\frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{u\to\infty} \frac{1}{2}(1-i) = \sqrt{-\frac{i}{2}} = \frac{1}{\sqrt{2\pi}} \int e^{-ix^2} dx$$

$$\Rightarrow K(x', T; x, 0) = \int \prod_{i=1}^{N-1} e^{-\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} V(x_k)} \prod_{j=0}^{N-1} \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}\delta T \frac{1}{2} m \dot{x}_j^2} \sqrt{\frac{\pi}{i}} \sqrt{\frac{\hbar 2m}{\delta T}} =$$

$$= \int \prod_{i=1}^{N-1} dx_i \, e^{-\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} V(x_k)} \prod_{j=0}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \delta T}} \, e^{\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} \frac{1}{2} m \dot{x}_k^2} =$$

$$= C \int \prod_{i=1}^{N-1} dx_i \, e^{\frac{i}{\hbar}\delta T \sum_{k=0}^{N-1} \left(\frac{1}{2} m \dot{x}_k^2 - V(x_k)\right)}$$

$$C \equiv \prod_{j=0}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \delta T}} = \left(\frac{m}{2\pi i \hbar \delta T}\right)^{\frac{N}{2}} \xrightarrow{3D} \left(\frac{m}{2\pi i \hbar \delta T}\right)^{\frac{3N}{2}}$$

$$K(x', T; x, 0) \longrightarrow \int \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar} S[x(t)]\right\}$$

$$S[x(t)] = \int_0^T dt \Big[\underbrace{\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x)}_{L}\Big]$$

FREE PARTICLE (NO PATH INTEGRAL) $\hat{H} = \frac{\hat{P}^2}{2m}$

$$K(y, T; x, 0) = \langle y | e^{-\frac{i}{\hbar}\hat{H} T} | x \rangle =$$

CENSORED (homework spoilers)

CENSORED $= \sqrt{\frac{m}{2\pi i \hbar T}}\, e^{\frac{im}{2\hbar T}(x-y)^2}$

QM $\leftrightarrow$ BROWNIAN MOTION : Sch. Eq. $\leftrightarrow$ DIFFUSION Eq.

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(t,x) + V_Q(x)\psi(t,x)$$

$$\frac{\partial}{\partial t}\rho(t,x) = D\frac{\partial^2}{\partial x^2}\rho(t,x) - V_D(x)\rho(t,x)$$

CENSORED

HEAT KERNEL $W(y,T;x,0) = \frac{1}{\sqrt{4\pi D T}}\, e^{-\frac{(x-y)^2}{4DT}}$

CENSORED (homework spoilers)

$$W(y,T;x,0) = \int_{x,0}^{y,T} \mathcal{D}x \, \exp\Big\{-\underbrace{\int_0^T dt\left[\frac{1}{4D}\left(\frac{dx}{dt}\right)^2 + V_D(x)\right]}_{\text{EUCLIDEAN ACTION !}}\Big\}$$

QM $\xrightarrow{\text{WICK ROTATION}}$ BROWNIAN MOTION

(IMAGINARY TIME)

PROPAGATOR $\longrightarrow$ HEAT KERNEL WITH

WITH ACTION $\longrightarrow$ EUCLIDEAN ACTION

FOKKER-PLANCK Eq. GENERALIZES DIFFUSION Eq.

$$\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\Big[\underbrace{\mu(t,x)}_{\text{DRIFT}}\rho(t,x)\Big] + \frac{\partial^2}{\partial x^2}\Big[\underbrace{D(t,x)}_{\text{DIFFUSION}}\rho(t,x)\Big] =$$

PER PARTES: $\partial_y(\mu\rho)\delta(x-y) \to -\mu\rho\,\partial_y\delta(x-y)$

$$= \int_{-\infty}^{\infty} dy \left\{\left[+\mu(t,y)\frac{\partial}{\partial y} + D(t,y)\frac{\partial^2}{\partial y^2}\right]\delta(x-y)\right\}\rho(t,y)$$

$$\rho(t+\varepsilon,x) = \rho(t,x) + \varepsilon\frac{\partial}{\partial t}\rho(t,x) + \mathcal{O}(\varepsilon^2) =$$

$$= \int_{-\infty}^{\infty} dy \left\{\left[1+\varepsilon\left(\mu(t,y)\frac{\partial}{\partial y} + D(t,y)\frac{\partial^2}{\partial y^2}\right) + \mathcal{O}(\varepsilon^2)\right]\underbrace{\delta(x-y)}_{\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ik(x-y)}}\right\}\rho(t,y) =$$

$$= \int dy \int\frac{dk}{2\pi}\Big\{\underbrace{\Big[1+\varepsilon\big(\mu(t,y)(ik)+D(t,y)(ik)^2\big)+\mathcal{O}(\varepsilon^2)\Big]}_{\approx e^{\varepsilon[\mu(t,y)(ik)+D(t,y)(ik)^2]}}\Big\}e^{-ik(x-y)}\rho(t,y) \approx$$

$$P = ik \qquad \int_{-\infty}^{\infty}\frac{dk}{2\pi} = \int_{-i\infty}^{+i\infty}\frac{dP}{2\pi i}$$

$$\approx \int_{-\infty}^{\infty} dy \int_{-i\infty}^{+i\infty}\frac{dP}{2\pi i}\, e^{\varepsilon\left[\mu(t,y)P + D(t,y)P^2 - P\frac{x-y}{\varepsilon}\right]}\rho(t,y)$$

$T = N\varepsilon$

$$\rho(t+T,x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty}\prod_{i=1}^{N-1} dy_i \int_{-i\infty}^{i\infty}\prod_{j=0}^{N-1}\frac{dP_j}{2\pi i}\exp\Big\{\varepsilon\sum_{k=0}^{N-1}\Big[\mu(t_k,y_k)P_k +$$

$$+ D(t_k,y_k)P_k^2 - P_k\frac{y_{k+1}-y_k}{\varepsilon}\Big]\Big\}\rho(t,y) \equiv$$

$$\equiv \int_{-\infty}^{\infty} dy\, W(x,t+T;y,t)\,\rho(t,y)$$

$$W(y,T;x,0) = \int_{-\infty}^{\infty}\prod_{i=1}^{N-1}dx_i\int_{-i\infty}^{i\infty}\prod_{j=0}^{N-1}\frac{dP_j}{2\pi i}\, e^{\varepsilon\sum_{k=0}^{N-1}\left(\mu_k P_k + D_k P_k^2 - P_k\dot{x}_k\right)}$$

NOTATION $\mu_k \equiv \mu(t_k,x_k)$

$D_k \equiv D(t_k,x_k)$

$$\Rightarrow \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} dx_i \prod_{j=0}^{N-1} \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i}\, e^{\varepsilon(\mu_j p + D_j p^2 - p\dot{x}_j)} =$$

$$p^2 D_j + p(\mu_j - \dot{x}_j) = p^2 D_j + p(\mu_j - \dot{x}_j) + \left(\frac{\mu_j - \dot{x}_j}{2\sqrt{D_j}}\right)^2 - \frac{(\mu_j - \dot{x}_j)^2}{4D_j} =$$

$$= \left(\sqrt{D_j}\, p + \frac{\mu_j - \dot{x}_j}{2\sqrt{D_j}}\right)^2 - \frac{(\mu_j - \dot{x}_j)^2}{4D_j} = D_j\left(p + \frac{\mu_j - \dot{x}_j}{2D_j}\right)^2 - \frac{(\mu_j - \dot{x}_j)^2}{4D_j}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} dx_i \prod_{j=0}^{N-1} \frac{1}{2\pi i}\, e^{-\varepsilon \frac{(\mu_j - \dot{x}_j)^2}{4D_j}} \underbrace{\int_{-i\infty}^{i\infty} dp\, e^{\varepsilon D_j \left(p + \frac{\mu_j - \dot{x}_j}{2D_j}\right)^2}}_{i\frac{\sqrt{\pi}}{\sqrt{\varepsilon D_j}}} =$$

$$i\frac{\sqrt{\pi}}{\sqrt{\varepsilon D_j}} \longleftarrow \qquad p = i\xi, \quad \int_{-i\infty}^{i\infty} dp = i\int_{-\infty}^{\infty} d\xi$$

$$= \left(\frac{1}{4\pi\varepsilon}\right)^{\frac{N}{2}} \left(\prod_{j=0}^{N-1} \frac{1}{\sqrt{D_j}}\right) \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} dx_i\, e^{-\varepsilon \sum_{k=0}^{N-1} \frac{(\mu_k - \dot{x}_k)^2}{4D_k}}$$

$$\longrightarrow \int \mathcal{D}x \, \exp\left\{-\int_0^T dt\, \frac{[\mu(t,x(t)) - \dot{x}(t)]^2}{4D(t,x(t))}\right\}$$

PARTICLE MOVING IN ELECTROMAGNETIC FIELD $\quad L = \frac{1}{2} m \dot{\vec{r}}^{\,2} + q\vec{A}\cdot\dot{\vec{r}} - q\phi$

$$\frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + qA_i\,, \quad \left(\frac{\partial L}{\partial \dot{x}^i}\right)^{\boldsymbol{\cdot}} = m\ddot{x}^i + q\dot{A}_i + qA_{i,j}\dot{x}^j$$

$$\frac{\partial L}{\partial x^i} = q\partial_i(A_j\dot{x}^j - \phi) = q\dot{x}_j A_{j,i} - q\phi_{,i}$$

LORENTZ FORCE

$$\vec{F} = q\left[-\vec{\nabla}\phi - \dot{\vec{A}} + v_i\vec{\nabla}A_i - \vec{A}_{,i}v_i\right] =$$

$$\vec{v}\times(\vec{\nabla}\times\vec{A}) =$$

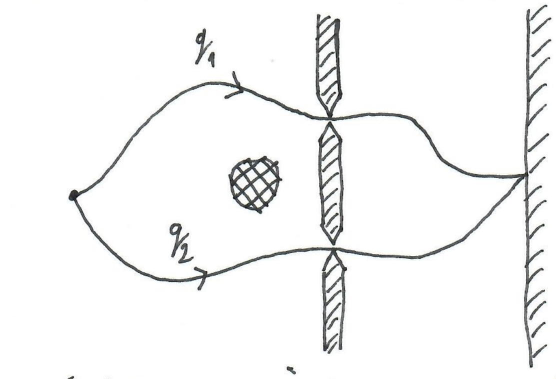
$$\varepsilon_{ijk}v_j\varepsilon_{klm}\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial_l A_m =$$

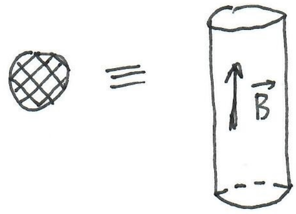
$$= v_j\partial_i A_j - v_j\partial_j A_i$$

$$= v_i\vec{\nabla}A_i - \vec{A}_{,i}v_i$$

$$= q\left[-\vec{\nabla}\phi - \dot{\vec{A}} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right] = q(\vec{E} + \vec{v}\times\vec{B}) \qquad \text{O.K.}$$

AHARONOV–BOHM EFFECT





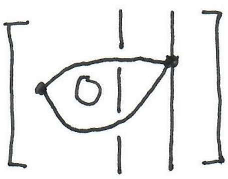
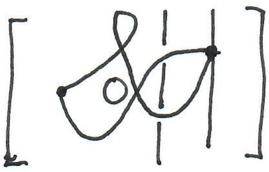
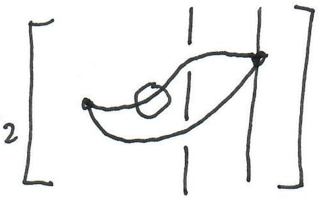
$$S[q] = \int dt \left(L^{(0)} + L^{(\vec{A})}\right) =$$
$$= S^{(0)}[q] + \int dt \; e\vec{A}\cdot\vec{v}$$

$$e^{\frac{i}{\hbar}S[q_1]} + e^{\frac{i}{\hbar}S[q_2]} = e^{\frac{i}{\hbar}S[q_1]}\left(1 + e^{\frac{i}{\hbar}\left(S[q_2]-S[q_1]\right)}\right)$$

$\longrightarrow$ RELATIVE PHASE $\quad \phi_{12} = \frac{1}{\hbar}\left(S[q_2]-S[q_1]\right)$

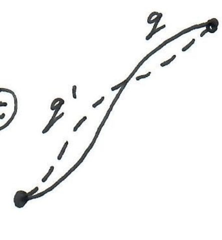
$$\phi_{12} = \phi_{12}^{(0)} + \phi_{12}^{(\vec{A})}$$

$$\phi_{12}^{(\vec{A})} = \frac{e}{\hbar}\left(\int dt\, \vec{A}\cdot\dot{\vec{q}}\,\Big|_{q=q_2} - \int dt\, \vec{A}\cdot\dot{\vec{q}}\,\Big|_{q=q_1}\right) =$$
$$= \frac{e}{\hbar}\left(\int_{q_2(t)} d\vec{q}\cdot\vec{A}(\vec{q}) - \int_{q_1(t)} d\vec{q}\cdot\vec{A}(\vec{q})\right) = \frac{e}{\hbar}\oint d\vec{q}\cdot\vec{A}(\vec{q}) =$$
$$= \frac{e}{\hbar}\iint d\vec{S}\cdot(\vec{\nabla}\times\vec{A}) = \frac{e}{\hbar}\iint d\vec{S}\cdot\vec{B}$$ MAGNETIC FLUX

ϕ_{12}  $=$ ϕ_{12}  $\neq$ ϕ_{12} 

INTERFERENCE OF PATHS & CLASSICAL LIMIT

PROPAGATOR $\quad K = \sum_{q(t)} e^{\frac{i}{\hbar}S[q]} \supset e^{\frac{i}{\hbar}S[q]} + e^{\frac{i}{\hbar}S[q']} \circeq$



TWO CLOSE PATHS $\quad q'(t) = q(t) + \varepsilon(t)$

$$\circeq e^{\frac{i}{\hbar}S[q]} + e^{\frac{i}{\hbar}\left(S[q] + \int dt\, \varepsilon(t)\frac{\delta S[q]}{\delta q(t)}\Big|_q + \mathcal{O}(\varepsilon(t)^2)\right)} =$$
$$= e^{\frac{i}{\hbar}S[q]}\left(1 + e^{\frac{i}{\hbar}\left(\int dt\, \varepsilon(t)\frac{\delta S[q]}{\delta q(t)}\Big|_q + \mathcal{O}(\varepsilon(t)^2)\right)}\right)$$

CONTRIBUTION OF TWO CLOSE PATHS TO PROPAGATOR $\approx$

$$e^{\frac{i}{\hbar}S[q]}\Big(1+\underbrace{e^{\frac{i}{\hbar}\int dt\,\varepsilon(t)\frac{\delta S[q]}{\delta q(t)}\big|_q}}\Big)$$

FOR SMALL $\hbar$ $\exists$ SMALL $\varepsilon(t)$ SUCH THAT $\approx -1 \Rightarrow q'$ ANNIHILATES q

THIS IS DIFFERENT WITH CLASSICAL PATH q_c:

$$\frac{\delta S[q]}{\delta q(t)}\Big|_{q=q_c}=0 \qquad \left(\begin{array}{c}\text{STATIONARY PHASE}\\ \Rightarrow\\ \text{CLASSICAL EQUATION OF MOTION}\end{array}\right)$$

PATH CLOSE TO CLASSICAL PATH $\tilde{q}=q_c+\varepsilon$

$$e^{\frac{i}{\hbar}S[q_c]}+e^{\frac{i}{\hbar}S[\tilde{q}]}=e^{\frac{i}{\hbar}S[q_c]}+e^{\frac{i}{\hbar}\left(S[q_c]+\frac{1}{2}\int dt_1\int dt_2\,\varepsilon(t_1)\varepsilon(t_2)\frac{\delta^2 S[q]}{\delta q(t_1)\delta q(t_2)}\big|_{q_c}+\dots\right)}=$$

$$=e^{\frac{i}{\hbar}S[q_c]}\left(1+1+O(\varepsilon(t)^2)\right)$$

$\Rightarrow$ RELATIVE PHASE OF q_c & $\tilde{q}$ IS SUPPRESSED BY ONE MORE POWER OF ε

$\Rightarrow$ DOMINANT CONTRIBUTION TO THE PATH INTEGRAL COMES FROM PATHS CLOSE TO THE CLASSICAL PATH $\Rightarrow$ CLASSICAL LIMIT FOR $\hbar\to 0$

STATIONARY PHASE METHOD (FOR ORDINARY INTEGRAL)

$$\int_{-\infty}^{\infty}dx\, e^{\frac{i}{\hbar}f(x)} \approx \left[\begin{array}{l}\text{STATIONARY POINT } x_c\\ \frac{df}{dx}\Big|_{x_c}=0 \to f(x)\approx f(x_c)+\frac{1}{2}f''(x_c)(\overbrace{x-x_c}^{y})^2\end{array}\right]\approx$$

$$\approx e^{\frac{i}{\hbar}f(x_c)}\int_{-\infty}^{\infty}dy\, e^{\frac{i}{\hbar}\frac{1}{2}f''(x_c)y^2}=e^{\frac{i}{\hbar}f(x_c)}\frac{\sqrt{\pi}}{\sqrt{-\frac{i}{\hbar}\frac{1}{2}f''(x_c)}}=$$

$$=\sqrt{\frac{2\pi i\hbar}{f''(x_c)}}\,e^{\frac{i}{\hbar}f(x_c)}$$ LEADING ORDER OF WKB (WENTZEL KRAMERS BRILLOUIN) APPROXIMATION

STATIONARY PHASE METHOD FOR PATH INTEGRAL

STATIONARY POINT $\longleftrightarrow$ CLASSICAL SOLUTION $x_c(t)$

$$\frac{\delta S[x]}{\delta x(t)}\Big|_{x=x_c}$$

$\triangleright$

$$\triangleright \quad S[x] = \left[x(t) = x_c(t) + y(t)\right] = S[x_c] + \int dt \underbrace{\left.\frac{\delta S[x]}{\delta x(t)}\right|_{x=x_c}}_{0} y(t) + $$

$$+ \frac{1}{2}\int dt_1 \int dt_2 \left.\frac{\delta^2 S[x]}{\delta x(t_1)\,\delta x(t_2)}\right|_{x=x_c} y(t_1) y(t_2) + \ldots$$

$$\Rightarrow \int \mathcal{D}x\, e^{\frac{i}{\hbar}S[x]} \approx \exp\left\{\frac{i}{\hbar}S[x_c]\right\} \int \mathcal{D}y \exp\left\{\frac{i}{\hbar}\frac{1}{2}\int dt_1 \int dt_2 \left.\frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)}\right|_{x_c} y(t_1) y(t_2)\right\}$$

EXAMPLE: PARTICLE IN POTENTIAL $S[x] = \int dt \left(\frac{1}{2} m \dot{x}^2 - V(x)\right)$

$\left\{\right.$ CLASSICAL Eq. of M. $\quad \delta S = \int dt\, (\underbrace{m \dot{x}\, \delta\dot{x}}_{\text{P.P.} \rightarrow -m\ddot{x}\,\delta x} - V'\delta x) \overset{!}{=} 0 \quad \hookrightarrow m\ddot{x} + V' = 0 \left.\right\}$

$$\left\{ \frac{\delta}{\delta x(t)} \int dt\, \dot{x} f = \frac{\delta}{\delta x(t)} (-1) \int dt\, x \dot{f} = -\dot{f}(t) = -\int dt_1\, \delta(t_1 - t)\, \dot{f}(t_1) \right\}$$

$$\frac{\delta S[x]}{\delta x(t_1)} = -m\ddot{x}(t_1) - V'(x(t_1)) = \int dt\, \delta(t_1 - t)\left[-m\ddot{x}(t) - V'(x(t))\right] =$$

$$= \int dt \left\{ \ddot{\delta}(t_1 - t)\left[-(-1)^2 m x(t)\right] + \delta(t_1 - t)\left[-V'(x(t))\right] \right\}$$

$\uparrow$ DEFINED THROUGH DISTRIBUTIONS LIKE DIRAC δ ITSELF

$$\frac{\delta^2 S[x]}{\delta x(t_1)\,\delta x(t_2)} = -m\, \ddot{\delta}(t_1 - t_2) - \delta(t_1 - t_2) V''(x(t_2))$$

$$\frac{1}{2}\int dt_1 \int dt_2 \left[-m\, \ddot{\delta}(t_1 - t_2) - \delta(t_1 - t_2) V''(x_c(t_2))\right] y(t_1) y(t_2) =$$

$$= \frac{1}{2}\int dt_1 \int dt_2\, \delta(t_1 - t_2)\left[-m\, y(t_1)\ddot{y}(t_2) - V''(x_c(t_2)) y(t_1) y(t_2)\right] =$$

$$= \frac{1}{2}\int dt\, (-m y \ddot{y} - V_c'' y^2) = \frac{1}{2}\int dt\, (m \dot{y}^2 - V''(x_c) y^2)$$

$$S = \int dt \left[\frac{1}{2} m \dot{x}^2 - V(x)\right] \overset{x = x_c + y}{\supset} \int dt \left[\frac{1}{2} m \dot{y}^2 - \frac{1}{2} V''(x_c) y^2\right]$$

$$\int \mathcal{D}x\, e^{\frac{i}{\hbar}S[x]} \approx \exp\left\{\frac{i}{\hbar}S[x_c]\right\} \int \mathcal{D}y \exp\left\{\frac{i}{\hbar}\int dt \left[\frac{1}{2} m \dot{y}^2 - \frac{1}{2} V''(x_c) y^2\right]\right\}$$