Notes on Path Integral Methods

PROPAGATOR FOR HARMONIC OSCILLATOR $L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2$

$$K(x_b,t_b;x_a,t_a)=\int_{x_a,t_a}^{x_b,t_b}\mathcal{D}x(\tau)\,\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}L\,d\tau\right\}=$$

CLASSICAL SOLUTION $\bar{x}(\tau)$: $\ddot{\bar{x}}+\omega^2\bar{x}=0$, $\bar{x}(t_a)=x_a$, $\bar{x}(t_b)=x_b$

SUBSTITUTION $y(\tau)=x(\tau)-\bar{x}(\tau)$: JACOBIAN $=1$, $y(t_a)=y(t_b)=0$

$$\int L\,d\tau=S[x(\tau)]=S[y(\tau)+\bar{x}(\tau)]=S[\bar{x}(\tau)]+$$

$$+\int d\tau\,\frac{\delta S[x]}{\delta x(\tau)}\bigg|_{\bar{x}}y(\tau)+\quad\leftarrow=0$$ STATIONARY PHASE CLASSICAL Eq. of M.

$$+\frac{1}{2}\int d\tau_1\int d\tau_2\,\frac{\delta^2 S[x]}{\delta x(\tau_1)\delta x(\tau_2)}\bigg|_{\bar{x}}y(\tau_1)y(\tau_2)+$$

$$+\frac{1}{6}\int d\tau_1\int d\tau_2\int d\tau_3\,\frac{\delta^3 S[x]}{\delta x(\tau_1)\delta x(\tau_2)\delta x(\tau_3)}\bigg|_{\bar{x}}y(\tau_1)y(\tau_2)y(\tau_3)+\ldots$$

THERE ARE ONLY QUADRATIC TERMS IN THE ACTION $\Rightarrow$ $0=$ ⊛

$\frac{1}{2}m(\dot{\bar{x}}+\dot{y})^2-\frac{1}{2}m\omega^2(\bar{x}+y)^2=$ TERM GIVING $S[\bar{x}(\tau)]+$

$+$ ZERO TERM (BECAUSE OF Eq. of M.) $+$

$+\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2+$ ZERO TERMS (BECAUSE OF QUADRATIC ACTION) ⊛

⊛ $\Rightarrow$ IN THIS PARTICULAR CASE THE FORMULA FOR STATIONARY PHASE METHOD GIVES ACCURATE RESULT, NOT ONLY APPROXIMATION

$$\to\lim_{\varepsilon\to0}\sum_{j=0}^{N}\varepsilon\left(\frac{1}{2}m\,\dot{y}(t_j)^2-\frac{1}{2}m\omega^2y(t_j)^2\right),\quad \varepsilon=\frac{t_b-t_a}{N}\,(=\delta T),\quad t_j=t_a+j\varepsilon$$

$$=\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N+1}{2}}\int dy_1\ldots dy_N\,\exp\left\{\frac{i}{\hbar}S[\bar{x}(\tau)]+\frac{i}{\hbar}\sum_{j=0}^{N}\varepsilon\left[\frac{1}{2}m\left(\frac{y_{j+1}-y_j}{\varepsilon}\right)^2-\frac{1}{2}m\omega^2y_j^2\right]\right\}=$$

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$$\Rightarrow \exp\left\{\frac{i}{\hbar}S[\bar{x}(\tau)]\right\}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N+1}{2}}\int d^N\eta\,\exp\{-\eta^T\sigma\,\eta\}$$

WHERE

$$\eta=\begin{pmatrix}y_1\\ \vdots\\ y_N\end{pmatrix}\qquad \sigma=-\frac{im}{2\hbar\varepsilon}\begin{pmatrix}2&-1&0&\dots&0&0\\-1&2&-1&\dots&0&0\\0&-1&2&\dots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&2&-1\\0&0&0&\dots&-1&2\end{pmatrix}+\frac{i\varepsilon}{2\hbar}m\omega^2\hat{1}$$

$$\int d^N\eta\, e^{-\eta^T\sigma\eta}=\left[\begin{array}{l}\sigma=U^+\sigma_{DIAG}U\\ \det U=1\\ \xi=U\eta,\ J=1\end{array}\right]=\int d^N\xi\, e^{-\xi^T\sigma_{DIAG}\xi}=$$

$$=\left[\sigma_{DIAG}=\mathrm{diag}(\sigma_1,\dots,\sigma_N)\right]=\prod_{\alpha=1}^{N}\sqrt{\frac{\pi}{\sigma_\alpha}}=\frac{\pi^{N/2}}{\sqrt{\det\sigma}}$$

$$\left(-\frac{2\hbar\varepsilon}{im}\right)^N\det\sigma=\det\left\{\begin{pmatrix}2&-1&0&\dots&0&0\\-1&2&-1&\dots&0&0\\0&-1&2&\dots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&2&-1\\0&0&0&\dots&-1&2\end{pmatrix}-\varepsilon^2\omega^2\hat{1}\right\}=$$

$$=\begin{vmatrix}2-\varepsilon^2\omega^2&-1&0&\dots&0&0\\-1&2-\varepsilon^2\omega^2&-1&\dots&0&0\\0&-1&2-\varepsilon^2\omega^2&\dots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&2-\varepsilon^2\omega^2&-1\\0&0&0&\dots&-1&2-\varepsilon^2\omega^2\end{vmatrix}\ \overset{\text{DENOTE}}{\equiv}\varphi_N$$

FIND φ_n THEN TAKE THE LIMIT $\lim\limits_{n\to\infty}\left(\frac{m}{2i\hbar\varepsilon}\right)^n\varphi_n$

DETERMINANT EXPANSION BY MINORS $\Rightarrow$ RECURRENT RELATION FOR φ_n (HERE WITH RESPECT TO THE FIRST ROW)

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$$\Rightarrow \varphi_{j+1}=(2-\varepsilon^2\omega^2)\varphi_j-\varphi_{j-1}$$

RECURRENT RELATION $\xrightarrow{\text{CONTINUOUS LIMIT}}$ DIFFERENTIAL Eq.

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$f(t_j) = \varepsilon \varphi_j$ & TAKE $\lim\limits_{\varepsilon \to 0}$

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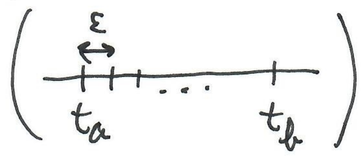
$$\longrightarrow \frac{d^2 f}{dt^2} + \omega^2 f = 0$$

SOLUTION OF DIFF. Eq. & INIT. COND. IS $f(t) = \frac{1}{\omega} \sin(\omega t)$

$$\lim_{n \to \infty} \varphi_n = \frac{1}{\varepsilon} \frac{1}{\omega} \sin[\omega(t_b - t_a)] \Big|_{\varepsilon \to 0}$$



$K(x_b, t_b; x_a, t_a) =$

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$$= \sqrt{\frac{m\omega}{2\pi i \hbar \sin[\omega(t_b - t_a)]}} \exp\left\{\frac{i}{\hbar} S[\bar{x}(\tau)]\right\}$$

WE NEED ACTION EVALUATED FOR CLASSICAL SOLUTION

$\bar{x}(\tau) = A \sin(\omega\tau) + B \cos(\omega\tau) \equiv A s + B c$

BOUNDARY CONDITIONS

$$A s_a + B c_a = x_a$$
$$A s_b + B c_b = x_b$$

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REMAINS ONLY THE BOUNDARY TERM $\int d(x\dot{x})$

CLASSICAL ACTION

$$\text{P.P.} \; -x\ddot{x} \overset{\text{Eq.o.M.}}{=} +x\omega^2 x$$

$$S[\bar{x}(\tau)] = \int_{t_a}^{t_b} d\tau \, \frac{1}{2} m \left[\dot{x}(\tau)^2 - \omega^2 x(\tau)^2\right] = \frac{1}{2} m \left[x\dot{x}\right]_{t_a}^{t_b} =$$

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$$= \frac{m\omega}{2 s_{b-a}} \left[(x_a^2 + x_b^2) c_{b-a} - 2 x_a x_b\right]$$

WE HAVE FOUND QM PROPAGATOR FOR HARMONIC OSCILLATOR

$$K(x_b, t_b; x_a, t_a) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin[\omega(t_b - t_a)]}} \;\cdot$$

$$\cdot \exp\left\{\frac{i m\omega}{2\hbar \sin[\omega(t_b - t_a)]}\left[(x_a^2 + x_b^2)\cos[\omega(t_b - t_a)] - 2 x_a x_b\right]\right\}$$

STATISTICAL MECHANICS

PARTITION FUNCTION $Z = \sum_n e^{-\beta E_n}$, $\beta = \frac{1}{kT}$

TEMPERATURE

BOLTZMANN CONSTANT

$\hat{H}|n\rangle = E_n|n\rangle$

PROBABILITY OF ENERGY LEVEL n : $P_n = \frac{1}{Z} e^{-\beta E_n}$

PROBABILITY DENSITY FOR POSITION OF PARTICLE AT ENERGY LEVEL n

$$P_n(x) = \frac{1}{Z} e^{-\beta E_n} \psi_n(x)\psi_n^*(x)$$

$$Z = \sum_n e^{-\beta E_n} = \sum_n \int dx\, e^{-\beta E_n} \psi_n(x)\psi_n^*(x) =$$

$$\left\{ \begin{array}{l} \text{DENSITY MATRIX } \hat{\rho}_\beta = e^{-\beta\hat{H}} \text{ (THERMAL)} \\ \rho_\beta(x,y) \equiv \langle x|\hat{\rho}_\beta|y\rangle = \langle x|e^{-\beta\hat{H}}\sum_n |n\rangle\langle n|y\rangle = \\ = \sum_n e^{-\beta E_n}\langle x|n\rangle\langle n|y\rangle = \sum_n e^{-\beta E_n}\psi_n(x)\psi_n^*(y) \end{array} \right\}$$

$$= \int dx\, \rho_\beta(x,x)$$

RECALL PROPAGATOR $K(x,t;y,0) = \langle x|e^{-\frac{i}{\hbar}\hat{H}t}|y\rangle$

& SEE SIMILARITY TO DENSITY MATRIX $\rho_\beta(x,y) = \langle x|e^{-\beta\hat{H}}|y\rangle =$

$$= K(x,-i\hbar\beta;y,0) =$$

$$= \int \mathcal{D}q \exp\left\{\frac{i}{\hbar}\int_0^{-i\hbar\beta} dt\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]\right\} =$$

$$\left\{ \begin{array}{ll} \text{WICK ROTATION} \quad \tau = it & \frac{d}{dt} = i\frac{d}{d\tau} \\ \int_0^{-i\hbar\beta} \longrightarrow \int_0^{\hbar\beta} & \frac{1}{2}m\dot{q}^2 = -\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 \end{array} \right\}$$

$$= \int \mathcal{D}q \exp\Big\{-\frac{1}{\hbar}\underbrace{\int_0^{\beta\hbar} d\tau\left[\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q)\right]}_{\text{EUCLIDEAN ACTION} \Leftarrow \text{WICK ROTATION}}\Big\} =$$

$$= \rho_\beta(x,y) \overset{\text{DENOTE}}{\equiv} K_E(x,\beta\hbar;y,0)$$

GROUND STATE ($n=0$) ENERGY $\Leftarrow$ ZERO TEMPERATURE LIMIT

$$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln Z \quad , \quad Z = \int dx\, \rho_\beta(x,x)$$

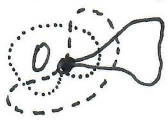
CAN BE CALCULATED IN SIMPLER WAY:

$$\underbrace{\ln \rho_\beta(0,0)}_{\sum_n e^{-\beta E_n} |\psi_n(0)|^2} \xrightarrow{\beta\to\infty} \ln\left[e^{-\beta E_0} |\psi_0(0)|^2\right] \xrightarrow{\beta\to\infty} -\beta E_0 + 2\ln|\psi_0(0)| \longrightarrow$$

$$\xrightarrow{\beta\to\infty} -\beta E$$

$$\Rightarrow E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln \rho_\beta(0,0) \quad , \quad \rho_\beta(0,0) = \int_0 \mathcal{D}q\, e^{-\frac{1}{\hbar} S_E[q]}$$

$\int_0 \mathcal{D}q \equiv \sum$ OVER $\forall$ PATHS STARTING & ENDING AT $x=y=0$



ALL LOOPS CROSSING THE ORIGIN

EUCLIDEAN ACTION $\quad S_E[q] = \int_0^{\beta\hbar} d\tau \left[\frac{1}{2} m \left(\frac{dq}{d\tau}\right)^2 + V(q)\right]$

$\rho_\beta(x,y) = K(x,-i\hbar\beta; y,0) \Rightarrow$ IF WE KNOW PROPAGATOR WE HAVE DENSITY MATRIX FOR FREE

FREE PARTICLE $\quad K(x,T;y,0) = \sqrt{\frac{m}{2\pi i\hbar T}}\, e^{\frac{im}{2\hbar T}(x-y)^2}$

$$\rho_\beta(x,y) = \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\frac{m}{2\hbar^2\beta}(x-y)^2}$$

HARMONIC OSCILLATOR

$$K(x,T;y,0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \exp\left\{\frac{im\omega}{2\hbar\sin(\omega T)}\left[(x^2+y^2)\cos(\omega T) - 2xy\right]\right\}$$

$$\left\{ \begin{aligned} &\sin[\omega(-i\hbar\beta)] = -i\,\mathrm{sh}(\hbar\omega\beta) \ , \ \cos[\omega(-i\hbar\beta)] = \mathrm{ch}(\hbar\omega\beta), \\ &\mathrm{th}\frac{a}{2} = \frac{\mathrm{sh}\,a}{1+\mathrm{ch}\,a} = \frac{\mathrm{ch}\,a - 1}{\mathrm{sh}\,a} \ , \ (x+y)^2\,\mathrm{th}\frac{a}{2} + (x-y)^2 \frac{1}{\mathrm{th}\frac{a}{2}} = (x+y)^2\left(\mathrm{coth}\,a - \frac{1}{\mathrm{sh}\,a}\right) + \\ &+ (x-y)^2\left(\mathrm{coth}\,a + \frac{1}{\mathrm{sh}\,a}\right) = (2x^2+2y^2)\,\mathrm{coth}\,a - 4xy\frac{1}{\mathrm{sh}\,a} \end{aligned} \right\}$$

$$\rho_\beta(x,y) = \sqrt{\frac{m\omega}{2\pi\hbar\,\mathrm{sh}(\hbar\omega\beta)}} \exp\left\{-\frac{m\omega}{4\hbar}\left[(x+y)^2\,\mathrm{th}\frac{\hbar\omega\beta}{2} + (x-y)^2\,\mathrm{coth}\frac{\hbar\omega\beta}{2}\right]\right\}$$

GROUND STATE ENERGY OF ANHARMONIC OSCILLATOR
PERTURBATIVE CALCULATION FOR SMALL λ

HERE SET $\hbar = 1$ & DENOTE $(\ )^{\bullet} = \frac{d}{d\tau}$ AFTER WICK ROTATION

$$K_E(0,\beta;0,0) = \int \mathcal{D}q \, \exp\left\{-\int_0^\beta d\tau \left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2 + \frac{1}{4!}\lambda q^4\right)\right\}$$

HARMONIC OSCILLATOR PROPAGATOR WITH SOURCE $J(\tau)$

$$K_E^0[J] \equiv \int \mathcal{D}q \, \exp\left\{-\int_0^\beta d\tau \left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2 - Jq\right)\right\} \equiv \int \mathcal{D}q \, e^{-S_E^0[J]}$$

FROM IT WE CAN BUILD $K_E(0,\beta;0,0)$ WITH INTERACTION TERM $\frac{\lambda}{4!}q^4$

$$\frac{\delta}{\delta J(\tau_1)} K_E^0[J] = \int \mathcal{D}q \, q(\tau_1) \, e^{-S_E^0[J]}$$

$$F\left[\frac{\delta}{\delta J}\right] K_E^0[J] = \int \mathcal{D}q \, F[q] \, e^{-S_E^0[J]}$$

$$\exp\left\{-\int_0^\beta d\tau \frac{\lambda}{4!}\left(\frac{\delta}{\delta J(\tau)}\right)^4\right\} K_E^0[J] =$$

$$= \int \mathcal{D}q \, \exp\left\{-\int_0^\beta d\tau \frac{\lambda}{4!} q^4\right\} e^{-S_E^0[J]}$$

$$K_E(0,\beta;0,0) = \left[\exp\left\{-\int_0^\beta d\tau \frac{\lambda}{4!}\left(\frac{\delta}{\delta J(\tau)}\right)^4\right\} K_E^0[J]\right]_{J=0}$$

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SUBSTITUTION $\quad q = q_c + y \quad$ CLASSICAL SOLUTION

SIGN CHANGED BECAUSE $\tau = it \longrightarrow +m\ddot{q}_c = m\omega^2 q_c - J$

$$q_c(0) = q_c(\beta) = 0$$

$$K_E^0[J] = \int \mathcal{D}y \, \exp\left\{-\int_0^\beta d\tau \left(\frac{1}{2}m\dot{q}_c^2 + \frac{1}{2}m\omega^2 q_c^2 - Jq_c\right) + \right.$$

$+$TERM LINEAR IN y $= 0 \Leftarrow$ Eq. of M. $-$

$$\left. - \int_0^\beta d\tau \left(\frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\omega^2 y^2\right)\right\} \equiv C \, \exp\left\{-S_{E,c}^0[J]\right\}$$

WHERE →

WHERE →

$$C = \int \mathcal{D}y \, \exp\left\{ -\int_0^\beta d\tau \left(\frac{1}{2} m \dot{y}^2 + \frac{1}{2} m\omega^2 y^2 \right) \right\}$$ DOES NOT DEPEND ON J

$$S^0_{E,c}[J] = \int_0^\beta d\tau \left(\frac{1}{2} m \dot{q}_c^2 + \frac{1}{2} m\omega^2 q_c^2 - J q_c \right) = -\frac{1}{2} \int_0^\beta d\tau \, J q_c$$

$$d(q_c \dot{q}_c) - q_c \ddot{q}_c$$

Eq. of M. → $-\frac{1}{2} m q_c \left(\omega^2 q_c - \frac{J}{m} \right) = -\frac{1}{2} m\omega^2 q_c^2 + \frac{1}{2} J q_c$

BOUNDARY TERM → $[q_c \dot{q}_c]_{\tau=0}^{\tau=\beta} = 0$ ⇐ BOUNDARY CONDITIONS

CLASSICAL SOLUTION THROUGH GREEN FUNCTION $q_c(\tau) = \int_0^\beta d\tau' G(\tau,\tau') J(\tau')$

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$$G(\tau,\tau') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} G_k e^{ik(\tau-\tau')}$$ (homework

spoilers) $$G_k = -\frac{1}{m} \frac{1}{k^2+\omega^2}$$

$$S^0_{E,c}[J] = -\frac{1}{2} \int_0^\beta d\tau \, J q_c = -\frac{1}{2} \int_0^\beta d\tau \int_0^\beta d\tau' J(\tau) G(\tau,\tau') J(\tau')$$

$$K^0_E[J] = C \exp\left\{ -S^0_{E,c}[J] \right\} = C - \frac{1}{2} C \int_0^\beta d\tau \int_0^\beta d\tau' J(\tau) G(\tau,\tau') J(\tau') +$$

$$+ \frac{1}{8} C \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_4 \, J(\tau_1) G(\tau_1,\tau_2) J(\tau_2) J(\tau_3) G(\tau_3,\tau_4) J(\tau_4) + \ldots$$

$$K_E(0,\beta;0,0) = \left[\exp\left\{ -\int_0^\beta d\tau \frac{\lambda}{4!} \left(\frac{\delta}{\delta J(\tau)} \right)^4 \right\} K^0_E[J] \right]_{J=0}$$

IN CALCULATION UP TO ONLY THE FIRST ORDER IN λ (NEGLECT $\mathcal{O}(\lambda^2)$) THE EXPONENTIAL REDUCES TO $1 - \int_0^\beta d\tau \left(\lambda/4!\right) \left(\delta/\delta J(\tau)\right)^4$ AND SINCE IN THE END WE TAKE $J=0$ THE ONLY INTERESTING PART COMES FROM THE TERM IN $K^0_E[J]$ CONTAINING FOUR J-S $J(\tau_1) \ldots J(\tau_4)$

$$K_E(0,\beta;0,0) = C - \frac{\lambda}{4!}\frac{C}{8}\Big(\;$$  $$\Big) + \mathcal{O}(\lambda^2)$$

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USE

$$\left\{ \int d\tau \frac{\delta}{\delta J(\tau)} \int d\xi\, f(\xi) J(\xi) = \int d\tau \int d\xi\, f(\xi) \frac{\delta J(\xi)}{\delta J(\tau)} = \int d\tau \int d\xi\, f(\xi)\,\delta(\xi - \tau) \right\}$$

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$$= 4! \int_0^\beta d\tau\; G(\tau,\tau)^2$$

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$$K_E(0,\beta;0,0) = C - \frac{\lambda}{4!}\frac{C}{8}\; 4!\left(-\frac{1}{2m\omega}\right)^2 \beta + \mathcal{O}(\lambda^2)$$

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GROUND STATE ENERGY

$$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln K_E(0,\beta;0,0) =$$ CENSORED

(homework spoilers) $$= \underbrace{E_0^{H.O.}}_{\frac{\omega}{2}} + \frac{\lambda}{32 m^2 \omega^2}$$

THE RESULT DEPENDS ON (MAKES SENSE WITH) ORDER OF LIMITS TAKEN

1) $\lim_{\lambda\to 0}$: $\ln\left(1 - \frac{\lambda\beta}{32 m^2\omega^2} + \ldots\right) \to -\frac{\lambda\beta}{32 m^2 \omega^2} + \mathcal{O}(\lambda^2)$

2) $\lim_{\beta\to\infty}$ (EASY)

REMINDER: SCHRÖDINGER, HEISENBERG & INTERACTION PICTURE

SCHRÖDINGER PICTURE

$$\frac{\partial}{\partial t}\hat{A}_{(S)} = 0 \quad , \quad i\hbar\frac{\partial}{\partial t}|j\rangle_{(S)} = \hat{H}\,|j\rangle_{(S)}$$

HEISENBERG PICTURE

$$\left[\begin{array}{l} \hat{A}_{(H)} = e^{\frac{i}{\hbar}\hat{H}t}\hat{A}_{(S)}e^{-\frac{i}{\hbar}\hat{H}t} \\ |j\rangle_{(H)} = e^{\frac{i}{\hbar}\hat{H}t}|j\rangle_{(S)} \end{array}\right.$$

$$\frac{\partial}{\partial t}\hat{A}_{(H)} = \frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}\hat{H}t}\hat{A}_{(S)}e^{-\frac{i}{\hbar}\hat{H}t} + e^{\frac{i}{\hbar}\hat{H}t}\hat{A}_{(S)}\left(-\frac{i}{\hbar}\hat{H}\right)\overset{[\hat{H},\hat{H}^n]=0}{e^{-\frac{i}{\hbar}\hat{H}t}} =$$

$$= \frac{i}{\hbar}\left[\hat{H},\hat{A}_{(H)}\right]$$

$$\frac{\partial}{\partial t}|j\rangle_{(H)} = \frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}\hat{H}t}|j\rangle_{(S)} + e^{\frac{i}{\hbar}\hat{H}t}\frac{1}{i\hbar}\hat{H}|j\rangle_{(S)} = 0$$

INTERACTION PICTURE $\quad \hat{H} = \hat{H}_0 + \hat{H}_I$

$$\left[\begin{array}{l} \hat{A}_{(I)} = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{A}_{(S)}e^{-\frac{i}{\hbar}\hat{H}_0 t} = e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}t}\hat{A}_{(H)}e^{\frac{i}{\hbar}\hat{H}t}e^{-\frac{i}{\hbar}\hat{H}_0 t} \\ |j\rangle_{(I)} = e^{\frac{i}{\hbar}\hat{H}_0 t}|j\rangle_{(S)} = e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}t}|j\rangle_{(H)} \end{array}\right.$$

$$\frac{\partial}{\partial t}\hat{A}_{(I)} = \frac{i}{\hbar}\left[\hat{H}_0,\hat{A}_{(I)}\right]$$

$$\frac{\partial}{\partial t}|j\rangle_{(I)} = \frac{i}{\hbar}\hat{H}_0|j\rangle_{(I)} + e^{\frac{i}{\hbar}\hat{H}_0 t}\frac{1}{i\hbar}\hat{H}\overbrace{|j\rangle_{(S)}}^{e^{-\frac{i}{\hbar}\hat{H}_0 t}|j\rangle_{(I)}} = \frac{i}{\hbar}\Big(\hat{H}_0 - \overset{\text{HAM. IN INT. PICTURE}}{\hat{H}_{(I)}}\Big)|j\rangle_{(I)} =$$

$$= \frac{1}{i\hbar}\hat{H}_I|j\rangle_{(I)}$$

INT. HAM. IN INT. PICTURE