Notes on Path Integral Methods

SO FAR WE WORKED WITH PROPAGATORS USING SCHRÖDINGER PICTURE

- POSITION OPERATOR $\hat{q}_{(S)}$ CONST. IN TIME
- POSITION STATE $|q,t\rangle_{(S)} \longleftrightarrow$ AT TIME t PARTICLE WAS AT POSITION q
  EVOLVES IN TIME (Sch. Eq.)

WE WROTE THE PROPAGATOR AS $K(x',T;x,0) = \langle x'|e^{-\frac{i}{\hbar}\hat{H}T}|x\rangle$

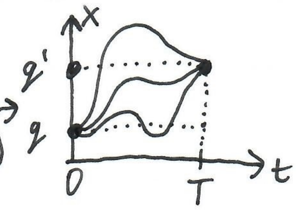
AND WE ACTUALLY MENT $K(q',T;q,0) = {}_{(S)}\langle q',T|e^{-\frac{i}{\hbar}\hat{H}T}|q,0\rangle_{(S)} \;=$

PASSING TO HEISENBERG PICTURE

- POSITION OPERATOR $\hat{q}_{(H)}(t)$ EVOLVES IN TIME $\frac{\partial}{\partial t}\,\cdot = \frac{i}{\hbar}[\hat{H},\cdot]$
- POSITION STATE $|q,t\rangle_{(H)}$ CONST. IN TIME & DESCRIBES STATE SUCH THAT PARTICLE WAS AT TIME t AT POSITION q

$$= {}_{(H)}\langle q',T|q,0\rangle_{(H)} = \int_{q,0}^{q',T} \mathcal{D}q \; e^{\frac{i}{\hbar}S[q]}$$

Σ OVER $\forall$ PATHS



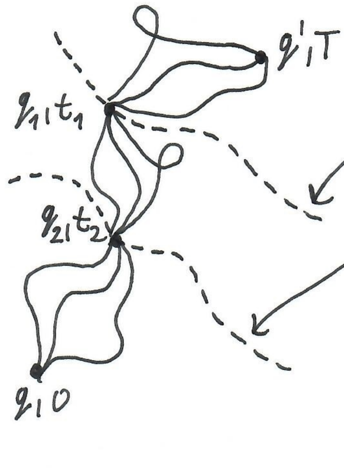
NOW MOVE ON...

$${}_{(H)}\langle q',T|\hat{q}_{(H)}(t_1)\hat{q}_{(H)}(t_2)|q,0\rangle_{(H)} =$$

$$= \int dq_1 \int dq_2 \; {}_{(H)}\langle q',T|q_1,t_1\rangle_{(H)} \underbrace{{}_{(H)}\langle q_1,t_1|\hat{q}_{(H)}(t_1)}_{{}_{(H)}\langle q_1,t_1|\, q_1(t_1)} \underbrace{\hat{q}_{(H)}(t_2)|q_2,t_2\rangle_{(H)}}_{q_2(t_2)\,|q_2,t_2\rangle_{(H)}} {}_{(H)}\langle q_2,t_2|q,0\rangle_{(H)} =$$

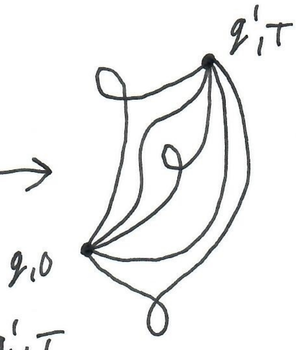
$$= \int dq_1 \int dq_2 \; q_1(t_1)\,q_2(t_2)\; {}_{(H)}\langle q',T|q_1,t_1\rangle_{(H)}\; {}_{(H)}\langle q_1,t_1|q_2,t_2\rangle_{(H)}\; {}_{(H)}\langle q_2,t_2|q,0\rangle_{(H)} =$$

$$= \int dq_1 \int dq_2 \; q_1(t_1)\,q_2(t_2) \int_{q_1,t_1}^{q',T} \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]} \int_{q_2,t_2}^{q_1,t_1} \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]} \int_{q,0}^{q_2,t_2} \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]} \;=$$



EXTRA INTEGRATIONS $\int dq_1$ & $\int dq_2$ GIVES US $\longrightarrow$

PATH INTEGRAL OVER ALL PATHS CONNECTING $q,0$ & q',T



WORKS ONLY WHEN $t_1 > t_2$!!!

$$\Rightarrow \int_{q,0}^{q',T} \mathcal{D}q\, q(t_1)\, q(t_2)\, e^{\frac{i}{\hbar}S[q]} = {}_{(H)}\langle q',T|\, \mathbb{T}\{\hat{q}_{(H)}(t_1)\hat{q}_{(H)}(t_2)\}\, |q,0\rangle_{(H)}$$

ONLY FOR $t_1 > t_2$ — TIME ORDERING

$$\mathbb{T}\{\hat{q}_{(H)}(t_1)\hat{q}_{(H)}(t_2)\} = \begin{cases} \hat{q}_{(H)}(t_1)\hat{q}_{(H)}(t_2) & t_1 > t_2 \\ \hat{q}_{(H)}(t_2)\hat{q}_{(H)}(t_1) & t_1 < t_2 \end{cases}$$

$\Rightarrow$ IF $\hat{Q}$ IS A PRODUCT OF SOME OPERATORS $\hat{A}(t_a)\,\hat{B}(t_b)\ldots$

THEN $\int \mathcal{D}q\, \hat{Q}\, e^{\frac{i}{\hbar}S}$ MUST BE ASSOCIATED WITH $\mathbb{T}\{\hat{Q}\}$

CALCULATE ${}_{(H)}\langle q',T|\hat{Q}_{(H)}|q,-T\rangle_{(H)}$ FOR TIME ORDERED $\hat{Q}_{(H)}$ IN $\lim_{T\to\infty}$

$${}_{(H)}\langle q',T|\hat{Q}_{(H)}|q,-T\rangle_{(H)} = {}_{(S)}\langle q',T|e^{-\frac{i}{\hbar}\hat{H}T}\hat{Q}_{(H)}\, e^{\frac{i}{\hbar}\hat{H}(-T)}|q,-T\rangle_{(S)} =$$

EXPANSION INTO HAMILTONIAN EIGENSTATES $\hat{H}|j\rangle = E_j|j\rangle$

$$|q,t\rangle_{(S)} = \sum_{j=0}^{\infty} {}_{(S)}\langle j,t|q,t\rangle_{(S)}\, |j,t\rangle_{(S)} = \sum_{j=0}^{\infty} \psi_j^*(q,t)\, |j,t\rangle_{(S)}$$

WAVE FUNCTIONS OF EIGENSTATES

$$= \sum_{j=0}^{\infty} \psi_j(q',T)\, \underbrace{{}_{(S)}\langle j,T|e^{-\frac{i}{\hbar}\hat{H}T}}_{e^{-\frac{i}{\hbar}E_jT}\,{}_{(S)}\langle j,T|}\, \hat{Q}_{(H)}\, \underbrace{e^{\frac{i}{\hbar}\hat{H}(-T)} \sum_{k=0}^{\infty}\psi_k^*(q,-T)\, |k,-T\rangle_{(S)}}_{e^{\frac{i}{\hbar}E_k(-T)}\,|k,-T\rangle_{(S)}} =$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \psi_j(q',T)\,\psi_k^*(q,-T)\, e^{-\frac{i}{\hbar}(E_j+E_k)T}\, {}_{(S)}\langle j,T|\hat{Q}_{(H)}|k,-T\rangle_{(S)} \xrightarrow{T\to\infty}$$

$$\xrightarrow{T\to\infty} \psi_0(q',T)\,\psi_0^*(q,-T)\, \underbrace{{}_{(S)}\langle 0,T|e^{-\frac{i}{\hbar}E_0T}}_{\xrightarrow{T\to\infty}\ {}_{(H)}\langle 0,T|}\, \hat{Q}_{(H)}\, \underbrace{e^{-\frac{i}{\hbar}E_0T}\,|0,-T\rangle_{(S)}}_{\xrightarrow{T\to\infty}\ |0,-T\rangle_{(H)}}$$

$$\int_{q,-T}^{q',T} \mathcal{D}q\, Q\, e^{\frac{i}{\hbar}S} = {}_{(H)}\langle q',T|\hat{Q}_{(H)}|q,-T\rangle_{(H)} \xleftrightarrow{T\to\infty} \psi_0(q',T)\,\psi_0^*(q,-T)\, {}_{(H)}\langle 0,T|\hat{Q}_{(H)}|0,-T\rangle_{(H)}$$

BY SETTING $\hat{Q} = \hat{1}$ WE FIND $\psi_0(q',\infty)\,\psi_0^*(q,-\infty) = \int_{q,-\infty}^{q',\infty} \mathcal{D}q\, e^{\frac{i}{\hbar}S}$ (IF ${}_{(H)}\langle 0,\infty|0,-\infty\rangle_{(H)} = 1$)

$$\underbrace{\Psi_0(q',T)\Psi_0^*(q,-T)}_{\int_{q,-T}^{q',T}\mathcal{D}q\, e^{\frac{i}{\hbar}S}} \;{}_{(H)}\langle 0,T|\hat{Q}_{(H)}|0,-T\rangle_{(H)} \overset{T\to\infty}{=} \int_{q,-T}^{q',T}\mathcal{D}q\, Q\, e^{\frac{i}{\hbar}S}$$

IF ${}_{(H)}\langle 0,T|0,-T\rangle_{(H)} \Leftarrow$ VACUUM(T) = VACUUM$(-T)$

$$_{(H)}\langle 0,T|\hat{Q}_{(H)}|0,-T\rangle_{(H)} \overset{T\to\infty}{=} \frac{\int_{q,-T}^{q',T}\mathcal{D}q\, Q\, e^{\frac{i}{\hbar}S}}{\int_{q,-T}^{q',T}\mathcal{D}q\, e^{\frac{i}{\hbar}S}}$$

DEF. n-POINT GREEN FUNCTION

$$G^{(n)}(t_1,\dots,t_n) \equiv \langle 0|\mathcal{T}\{\hat{q}(t_1)\dots\hat{q}(t_n)\}|0\rangle \overset{\Downarrow}{=} \frac{\int\mathcal{D}q\, q(t_1)\dots q(t_n)\, e^{\frac{i}{\hbar}S}}{\int\mathcal{D}q\, e^{\frac{i}{\hbar}S}}$$

NOTATION:

1) SINCE THE FORMULA IS VALID ONLY FOR VACUUM(T)=VACUUM$(-T)$ WE MAY USE SHORTHAND $|0\rangle$ FOR IT
2) BY WRITTING TIME ARGUMENTS IN $\hat{q}(t_1)\dots\hat{q}(t_n)$ WE MAKE IT OBVIOUS THAT WE USE HEISENBERG PICTURE $\hat{q}_{(H)}(t_1)\dots$ AND WE MAY SKIP THE TEDIOUS NOTATION

IMPORTANT NOTE

HERE WE CALCULATE ONLY VACUUM EXPECTATION VALUES $\langle 0|\hat{Q}_{(H)}|0\rangle$ HOWEVER WHEN STUDYING PARTICLE SCATTERINGS WE NEED

$_{OUT}\langle 0|\hat{Q}_{(H)}|0\rangle_{IN}$ WHERE $|0\rangle_{IN} \equiv |0\rangle_{(S)}(t=0)$ SUCH THAT $|0\rangle_{(S)}(t=-T) = |0\rangle$

$|0\rangle_{OUT} \equiv |0\rangle_{(S)}(t=0)$ SUCH THAT $|0\rangle_{(S)}(t=+T) = |0\rangle$

TIME EVOLUTION

$$|0\rangle_{IN} \overset{\downarrow}{=} e^{-\frac{i}{\hbar}\hat{H}(0-(-T))}|0\rangle = e^{-\frac{i}{\hbar}\hat{H}T}|0\rangle$$

$$|0\rangle_{OUT} = e^{-\frac{i}{\hbar}\hat{H}(0-T)}|0\rangle = e^{\frac{i}{\hbar}T}|0\rangle$$

$$_{OUT}\langle 0|\hat{Q}_{(H)}(t=0)|0\rangle_{IN} = \langle 0|e^{-\frac{i}{\hbar}\hat{H}T}\hat{Q}_{(H)}(t=0)\, e^{-\frac{i}{\hbar}\hat{H}T}|0\rangle = \dots$$ PARTICLE PHYSICS

GREEN FUNCTIONS THROUGH GENERATING FUNCTIONAL

$$Z[J] = \int \mathcal{D}q \, \exp\left\{ \frac{i}{\hbar} \left(S[q] + \int dt \, J(t) q(t) \right) \right\} \equiv \int \mathcal{D}q \, e^{\frac{i}{\hbar} S_J[q]}$$

$$\frac{1}{Z[0]} \left(\frac{\hbar}{i} \frac{\delta}{\delta J(t_1)} Z[J] \right)_{J=0} = \frac{\int \mathcal{D}q \, q(t_1) \, e^{\frac{i}{\hbar} S[q]}}{\int \mathcal{D}q \, e^{\frac{i}{\hbar} S[q]}} = \langle 0 | \hat{q}(t_1) | 0 \rangle$$

$$\frac{1}{Z[0]} \left(\frac{\hbar}{i} \frac{\delta}{\delta J(t_1)} \cdots \frac{\hbar}{i} \frac{\delta}{\delta J(t_m)} Z[J] \right)_{J=0} = \langle 0 | \mathcal{T} \{ \hat{q}(t_1) \ldots \hat{q}(t_m) \} | 0 \rangle$$

HARMONIC OSCILLATOR

$$S_J[q] = \int dt \left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 + J q \right) = \left[\begin{array}{l} q = q_c + y \ \text{ BOUNDARY CONDITIONS} \\ \text{ARE SATISFIED BY CLASSICAL} \\ \text{SOLUTION } q_c \Rightarrow y(\pm T) = 0 \end{array} \right] =$$

$$= \int dt \left(\frac{1}{2} m \dot{q}_c^2 - \frac{1}{2} m \omega^2 q_c^2 + J q_c \right) + \int dt \left(\frac{1}{2} 2 m \dot{y} \dot{q}_c - \frac{1}{2} 2 m \omega^2 y q_c + J y \right) +$$

$$+ \int dt \left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} m \omega^2 y^2 \right) =$$

PER PARTES $= 0$

$-m y \ddot{q}_c \overset{\text{Eq. of M.}}{=} y (m \omega^2 q_c - J)$

BOUNDARY TERM $m [y \dot{q}_c]_{-T}^{T} = 0$

$$= S_J[q_c] + S_0[y]$$

$$Z[J] = \underbrace{\int \mathcal{D}y \, e^{\frac{i}{\hbar} S_0[y]}}_{C \,\leftarrow\, \text{DOES NOT DEPEND ON } J} \, e^{\frac{i}{\hbar} S_J[q_c]}$$

$$S_J[q_c] = \int dt \left(\frac{1}{2} m \dot{q}_c^2 - \frac{1}{2} m \omega^2 q_c^2 + J q_c \right) = \frac{1}{2} m [q_c \dot{q}_c]_{-T}^{T} + \int dt \left(-\frac{1}{2} J q_c + J q_c \right)$$

P.P. $\longrightarrow d(q_c \dot{q}_c) - \ddot{q}_c \, dt \qquad \frac{1}{2} m \left(\omega^2 q_c - \frac{J}{m} \right)$ Eq. of M.

THIS BOUNDARY TERM MAY IN PRINCIPLE DEPEND ON SOURCE J BECAUSE THE CLASSICAL SOLUTION q_c DOES THROUGH Eq. of M. BUT WE CHOOSE BOUNDARY CONDITIONS $q_c(\pm T) = 0$ AND THEN $[q_c \dot{q}_c]_{-T}^{T}$ DOES NOT DEPEND ON J

$$Z[J] = \tilde{C} \, e^{\frac{i}{\hbar} \frac{1}{2} \int dt \, J(t) q_c(t)} = Z[0] \, e^{\frac{i}{\hbar} \frac{1}{2} \int dt \, J(t) q_c(t)}$$

CLASSICAL SOLUTION THROUGH GREEN FUNCTION

$$m \ddot{q}_c + m \omega^2 q_c = J \quad , \quad q_c(t) = i \int dt' \, G(t-t') J(t') \, ,$$

$$m \left(\frac{d}{dt^2} + \omega^2 \right) G(t-t') = \frac{1}{i} \delta(t-t') \quad , \quad G(t-t') = \int \frac{dk}{2\pi} G_k e^{ik(t-t')} \, ,$$

$$m \left((ik)^2 + \omega^2 \right) G_k = \frac{1}{i} = -i \quad , \quad G_k = \frac{i}{m} \frac{1}{k^2 - \omega^2}$$

$$G(t-t') = \frac{i}{m}\int \frac{dk}{2\pi}\,\frac{1}{k^2-\omega^2}\,e^{ik(t-t')}$$

$$\frac{Z[J]}{Z[0]} = \exp\left\{\frac{i}{\hbar}\frac{1}{2}\int dt\, J(t)\, q_c(t)\right\} = \exp\left\{-\frac{1}{2\hbar}\int dt\int dt'\, J(t)\, G(t-t')\, J(t')\right\}$$

$$\langle 0|T\{\hat{q}(t_1)\hat{q}(t_2)\}|0\rangle = \frac{1}{Z[0]}(-\hbar^2)\left(\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)} Z[J]\right)_{J=0} =$$

$$= -\hbar^2\left(\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}\left(1-\frac{1}{2\hbar}\int dt\int dt'\, J(t) G(t-t') J(t') + \dots\right)\right)_{J=0} =$$

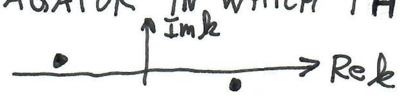
$$= \frac{\hbar}{2}\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}\int dt\int dt'\, J(t)\, G(t-t')\, J(t') =$$

$$= \frac{\hbar}{2}\int dt\int dt'\left(\delta(t-t_1) G(t-t')\delta(t'-t_2) + \delta(t-t_2) G(t-t')\delta(t'-t_1)\right) =$$

$$= \frac{\hbar}{2}\left(G(t_1-t_2) + G(t_2-t_1)\right) = \hbar\, G(t_1-t_2)$$

WE SEE THAT WITH CONVENTION $q_c = \frac{i}{\hbar}\int dt' G(t-t') J(t')$
WE WOULD OBTAIN A NICER LOOKING RESULT
$\langle 0|T\{\hat{q}(t_1)\hat{q}(t_2)\}|0\rangle = G(t_1-t_2)$
$\Rightarrow q_c(t) = \frac{i}{\hbar}\int dt' \langle 0|T\{\hat{q}(t)\hat{q}(t')\}|0\rangle J(t')$

$\langle 0|T\{\hat{q}(t_1)\hat{q}(t_2)\}|0\rangle$ IS FEYNMAN PROPAGATOR IN WHICH THE POLES IN FOURIER SPACE ARE AVOIDED LIKE



AND SINCE FEYNMAN PROPAGATOR $\longleftrightarrow$ GREEN FUNCTION
CONVERGENCE OF GREEN FUNCTION MUST BE IMPROVED IN THE SAME WAY

$$G_k = \dots\frac{1}{k^2-\omega^2}\dots \longrightarrow \dots\frac{1}{k^2-\omega^2+i\varepsilon}\dots$$

AFTER WICK ROTATION ACTION $\longrightarrow$ EUCLIDEAN ACTION
AND $G_k = \dots\frac{1}{k^2-\omega^2}\dots \longrightarrow \dots\frac{1}{k^2+\omega^2}\dots$ LIKE IN CALCULATION
OF GROUND STATE ENERGY OF ANHARMONIC OSCILLATOR
WHERE WE HAD NO ISSUES WITH CONVERGENCE

THERMAL n-POINT GREEN FUNCTIONS

DENSITY MATRIX $\hat{\varrho} = \sum_n P_n |n\rangle\langle n|$, $\mathrm{Tr}\,\varrho = \sum_n \langle n|\hat{\varrho}|n\rangle = \sum_n P_n = 1$

P_n → PROBABILITY OF STATE $|n\rangle$

THERMAL DENSITY MATRIX $\hat{\varrho}_\beta = \frac{1}{Z_\beta} e^{-\beta\hat{H}}$, $Z_\beta = \mathrm{Tr}\, e^{-\beta\hat{H}}$

{ WE HAVE USED (THERMAL) DENSITY MATRIX BEFORE BUT WITH DIFFERENT CONVENTION ($\hat{\varrho}_\beta = e^{-\beta\hat{H}}$), WITH CONVENTION HERE WE HAVE: PARTITION FUNCTION $Z = Z_\beta \int dx\, \varrho_\beta(x,x)$

WHERE $\varrho_\beta(x,y) \equiv \langle x|\hat{\varrho}_\beta|y\rangle$

PROPAGATOR WITH EUCLIDEAN ACTION $K_E(x,\beta\hbar; y,0) = Z_\beta \varrho_\beta(x,y)$

OF COURSE $E_0 = -\lim\limits_{\beta\to\infty} \frac{1}{\beta} \ln Z$ REMAINS THE SAME }

FOR THERMAL DENSITY MATRIX THE PROBABILITY OF STATE $|n\rangle$ IS GIVEN BY THE TEMPERATURE T AND ENERGY SPECTRUM E_n

$$\underline{P_{\beta,n} = \langle n|\hat{\varrho}_\beta|n\rangle = \frac{\langle n|e^{-\beta\hat{H}}|n\rangle}{Z_\beta}} = \frac{\langle n|n\rangle e^{-\beta E_n}}{Z_\beta} = \frac{1}{Z_\beta} e^{-\beta E_n}$$

MEAN VALUE OF QUANTITY Q

$$\langle Q\rangle = \sum_n P_n \langle n|\hat{Q}|n\rangle = \sum_n P_n \sum_k \langle n|k\rangle\langle k|\hat{Q}|n\rangle =$$
$$= \sum_n P_n \sum_k \langle k|\hat{Q}|n\rangle\langle n|k\rangle = \sum_n P_n \mathrm{Tr}\left(\hat{Q}|n\rangle\langle n|\right) =$$
$$= \sum_n P_n \mathrm{Tr}\left(|n\rangle\langle n|\hat{Q}\right) = \mathrm{Tr}\left(\sum_n P_n |n\rangle\langle n|\hat{Q}\right) = \mathrm{Tr}\left(\hat{\varrho}\hat{Q}\right)$$

THE SAME ASSOCIATED WITH TEMPERATURE $\langle Q\rangle_\beta = \mathrm{Tr}\left(\hat{\varrho}_\beta \hat{Q}\right)$

IN ZERO TEMPERATURE LIMIT $\beta \to \infty$

$$P_{\beta,n} \longrightarrow \begin{cases} 1 & n=0 \\ 0 & n\geq 1 \end{cases}, \quad \text{THEN } \langle Q\rangle_\beta = \sum_n P_{\beta,n} \langle n|\hat{Q}|n\rangle \longrightarrow$$
$$\longrightarrow \langle 0|\hat{Q}|0\rangle$$

WITH LOW TEMPERATURE THE MEAN VALUES $\langle Q\rangle$ ARE DOMINATED BY VACUUM MEAN VALUES $\langle 0|\hat{Q}|0\rangle$

M-POINT GREEN FUNCTION $\langle T\{\hat q(t_1)\dots\hat q(t_m)\}\rangle = \langle 0|T\{\hat q(t_1)\dots\hat q(t_m)\}|0\rangle$

THERMAL M-POINT GREEN FUNCTION

$$G_\beta(t_1,\dots,t_m) = \langle T\{\hat q(t_1)\dots\hat q(t_m)\}\rangle_\beta \overset{\lim_{\beta\to\infty}}{=}$$

$$= \mathrm{Tr}\left(\hat\rho_\beta T\{\hat q(t_1)\dots\hat q(t_m)\}\right) \ominus$$

A TRICK FOR CALCULATING ENERGY OF FIRST EXCITED STATE

$$t_1>t_2 : G_\beta(t_1,t_2) = \frac{1}{Z_\beta}\mathrm{Tr}\left(e^{-\beta\hat H}\hat q(t_1)\hat q(t_2)\right) =$$

$$= \frac{1}{Z_\beta}\sum_m \langle m|e^{-\beta\hat H}\hat q(t_1)\hat q(t_2)|m\rangle = \left[\text{HEISENBERG} \longrightarrow \text{SCHRÖDINGER}\right] =$$

$$= \frac{1}{Z_\beta}\sum_m e^{-\beta E_m}\langle m|e^{\frac{i}{\hbar}\hat H t_1}\hat q e^{-\frac{i}{\hbar}\hat H t_1}\underbrace{\ }_{\sum_k |k\rangle\langle k|} e^{\frac{i}{\hbar}\hat H t_2}\hat q e^{-\frac{i}{\hbar}\hat H t_2}|m\rangle =$$

$$= \frac{1}{Z_\beta}\sum_{m,k} e^{-\beta E_m} e^{\frac{i}{\hbar}E_m t_1} e^{-\frac{i}{\hbar}E_k t_1} e^{\frac{i}{\hbar}E_k t_2} e^{-\frac{i}{\hbar}E_m t_2}\langle m|\hat q|k\rangle\langle k|\hat q|m\rangle =$$

$$= \frac{1}{Z_\beta}\sum_{m,k} e^{-\beta E_m} e^{\frac{i}{\hbar}(t_1-t_2)(E_m-E_k)}\langle m|\hat q|k\rangle\langle k|\hat q|m\rangle \xrightarrow{\beta\to\infty}$$

$$\xrightarrow{\beta\to\infty} \underbrace{\left(\lim_{\beta\to\infty}\frac{e^{-\beta E_0}}{Z_\beta}\right)}_{1}\sum_k e^{\frac{i}{\hbar}(t_1-t_2)(E_0-E_k)}|\langle 0|\hat q|k\rangle|^2$$

$$\longrightarrow \lim_{\beta\to\infty} G_\beta(t_1,t_2) = \langle 0|\hat q(t_1)\hat q(t_2)|0\rangle = \qquad \tau = it$$

$$= |\langle 0|\hat q|0\rangle|^2 + e^{\frac{1}{\hbar}(\tau_1-\tau_2)(E_0-E_1)}|\langle 0|\hat q|1\rangle|^2 + \dots$$

$$\lim_{\tau_1-\tau_2\to\infty} e^{-\frac{E_1-E_0}{\hbar}(\tau_1-\tau_2)} = \lim_{t_1-t_2\to\infty}\frac{\langle 0|\hat q(t_1)\hat q(t_2)|0\rangle}{|\langle 0|\hat q|1\rangle|^2}$$

$$\ominus\ \mathrm{Tr}\left(\hat\rho_\beta T\left\{\left[\frac{\hbar}{i}\frac{\delta}{\delta J(t_1)}\dots\frac{\hbar}{i}\frac{\delta}{\delta J(t_m)} e^{\frac{i}{\hbar}\int dt\, J(t)\hat q(t)}\right]_{J=0}\right\}\right) =$$

$$= \left[\frac{\hbar}{i}\frac{\delta}{\delta J(t_1)}\dots\frac{\hbar}{i}\frac{\delta}{\delta J(t_m)}\mathrm{Tr}\left(\hat\rho_\beta T\left\{e^{\frac{i}{\hbar}\int dt\, J(t)\hat q(t)}\right\}\right)\right]_{J=0}$$

$$G_\beta(t_1,\dots,t_n)=\left[\frac{\hbar}{i}\frac{\delta}{\delta J(t_1)}\cdots\frac{\hbar}{i}\frac{\delta}{\delta J(t_n)}Z_\beta[J]\right]_{J=0}$$

THERMAL GENERATING FUNCTIONAL

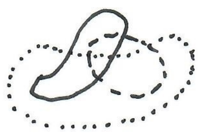
$$Z_\beta[J]=\mathrm{Tr}\left(\hat{\varrho}_\beta\,T\left\{e^{\frac{i}{\hbar}\int dt\,J(t)\hat{q}(t)}\right\}\right)=\int dq\,\langle q|\hat{\varrho}_\beta\,T\left\{e^{\frac{i}{\hbar}\int dt\,J(t)\hat{q}(t)}\right\}|q\rangle=$$

$$=\int dq\,\underbrace{\langle q|\hat{\varrho}_\beta|q\rangle}_{\varrho_\beta(q,q)=\frac{1}{Z_\beta}K_E(q,\beta\hbar;q,0)=}\,e^{\frac{i}{\hbar}\int dt\,J(t)q(t)}$$

WICK ROTATION
$\tau=it$

$$=\frac{1}{Z_\beta}\int_{q,0}^{q,\beta\hbar}\mathcal{D}q\,\exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\left[\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2+V(q)\right]\right\}$$

$$Z_\beta[J]=\frac{1}{Z_\beta}\underbrace{\int dq\int_{q,0}^{q,\beta\hbar}\mathcal{D}q}_{\oint\mathcal{D}q\ \dots\ \sum\ \text{OVER}\ \forall\ \text{LOOPS}}\,e^{+\frac{1}{\hbar}\int d\tau\,J(\tau)q(\tau)}\exp\left\{-\frac{1}{\hbar}\underbrace{\int_0^{\beta\hbar}d\tau\left[\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2+V(q)\right]}_{S_E[q]}\right\}$$



$$Z_\beta[J]=\frac{1}{Z_\beta}\oint\mathcal{D}q\,\exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\left[\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2+V(q)-J(\tau)q(\tau)\right]\right\}$$

IN $\lim_{\beta\to\infty}$ THE SAME AS THE ORDINARY THEORY !!!

IN FORMULA FOR $G_\beta(t_1,\dots,t_n)$ THROUGH VARIATIONS OF $Z_\beta[J]$
WE DO NOT NEED FACTOR $\frac{1}{Z_\beta[0]}$ WHICH WE HAVE
FOR $G(t_1,\dots,t_n)$ THROUGH $Z[J]$ BECAUSE HERE
$Z_\beta[0]=\mathrm{Tr}[\hat{\varrho}_\beta]=1$ THANKS TO GOOD CONVENTIONS

WE MAY ALSO APPRECIATE SIMILARITY BETWEEN
PARTITION FUNCTION AND GENERATING FUNCTIONAL

$$Z_\beta=\sum_n e^{-\beta E_n}\qquad\qquad Z[J]=\int\mathcal{D}q\,e^{\frac{i}{\hbar}S_J[q]}$$

(WE USE THE SAME LETTER Z FOR BOTH)
AND SEE THE ROLE OF WICK ROTATION WHICK TAKES
CARE OF IMAGINARY i ASSOCIATED WITH QM

QUANTUM FIELD THEORY

$q(t) \to \phi(t,\vec{x}) \equiv \phi(x) \qquad (\hbar = c = 1)$

$$K(\phi'(\vec{x}),T;\phi(\vec{x}),0) = \int_{\phi,0}^{\phi',T} \mathcal{D}\phi\, e^{iS[\phi]} \quad , \quad S[\phi] = \int d^4x\, \mathcal{F}(\phi,\partial\phi,\dots)$$

m-POINT GREEN FUNCTIONS THROUGH GENERATING FUNCTIONAL

$$G^{(m)}(x_1,\dots,x_m) = \langle 0|\, T\{\hat{\phi}(x_1)\dots\hat{\phi}(x_m)\}\,|0\rangle =$$

$$= \frac{1}{Z[0]}\left(\frac{1}{i}\right)^m \left(\frac{\delta^m Z[J]}{\delta J(x_1)\dots\delta J(x_m)}\right)_{J=0} \quad , \quad Z[J] = \int \mathcal{D}\phi\, e^{i\left(S[\phi] + \int d^4x\, J(x)\phi(x)\right)}$$

FREE SCALAR FIELD WITH MASS m

$$S[\phi] = \int d^4x \left(\frac{1}{2}\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}m^2\phi^2\right)$$

CLASSICAL SOLUTION $\quad 0 \overset{!}{=} \delta S[\phi] = \int d^4x \left(\eta^{\mu\nu}\phi_{,\mu}\delta\phi_{,\nu} - m^2\phi\,\delta\phi\right) \overset{\text{P.P.}}{=}$

$$\overset{\text{P.P.}}{=} \int d^4x \left(-\eta^{\mu\nu}\phi_{,\mu\nu}\delta\phi - m^2\phi\,\delta\phi\right) \to -\eta^{\mu\nu}\phi_{,\mu\nu} - m^2\phi = 0$$

KLEIN GORDON EQUATION $\quad (\Box + m^2)\phi = 0$

$$Z[J] = \int \mathcal{D}\phi\, \exp\Big\{ i\int d^4x \Big(\frac{1}{2}\underbrace{\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}_{\text{BOUDARY TERM} \longleftarrow \partial_\mu(\eta^{\mu\nu}\phi\,\phi_{,\nu}) - \eta^{\mu\nu}\phi_{,\mu\nu}\phi \longrightarrow -(\Box\phi)\phi} - \frac{1}{2}m^2\phi^2 + J\phi\Big)\Big\} =$$

$$= \int \mathcal{D}\phi\, \exp\Big\{ i\int d^4x \int d^4y\, \phi(x)\left(-\frac{1}{2}\Box_x - \frac{1}{2}m^2\right)\delta(x-y)\,\phi(y) + i\int d^4z\, J(z)\phi(z)\Big\}$$

N-DIMENSIONAL INTEGRAL WITH N×N SYMMETRIC MATRIX B

$$\int d^Nx\, \exp\left\{\frac{i}{2}x^TBx + ix^Tj\right\} = \left[y = x + B^{-1}j\right] =$$

$$= \int d^Ny\, \exp\left\{\frac{i}{2}\left(y^T - j^T(B^{-1})^T\right)B\left(y - B^{-1}j\right) + i\left(y^T - j^T(B^{-1})^T\right)j\right\} =$$

$$= \int d^Ny\, \exp\left\{\frac{i}{2}y^TBy - \frac{i}{2}y^Tj - \frac{i}{2}j^Ty + \frac{i}{2}j^TB^{-1}j + iy^Tj - ij^TB^{-1}j\right\} =$$

$$= \int d^Ny\, \exp\left\{\frac{i}{2}y^TBy - \frac{i}{2}j^TB^{-1}j\right\} = \exp\left\{-\frac{i}{2}j^TB^{-1}j\right\}\int d^Nx\, \exp\left\{\frac{i}{2}x^TBx\right\}$$

$$\underbrace{\int d^N x \exp\left\{\frac{i}{2} x_a B^a_b x^b + i x_c j^c\right\}}_{Z[j]} = \exp\left\{-\frac{i}{2} j_a (B^{-1})^a_b j^b\right\} \underbrace{\int d^N x \exp\left\{\frac{i}{2} x_a B^a_b x^b\right\}}_{Z[0]}$$

$$\Big\downarrow\; j^a \to J(x) \qquad \Big\downarrow\; \text{"}\lim_{N\to\infty}\text{"} \qquad \Big\downarrow\; \sum_a \to \int d^4x \qquad \Big\downarrow$$

$$\underbrace{\int \mathcal{D}\phi \exp\left\{\underbrace{\frac{i}{2}\int d^4x \int d^4y\, \phi(x) B(x,y) \phi(y)}_{iS[\phi]} + i\int d^4z\, \phi(z) J(z)\right\}}_{Z[J]} =$$

$$\left\{ \begin{array}{l} B(x,y) = -(\Box_x + m^2)\,\delta(x-y) \\ \int d^4z\, B^{-1}(x,z) B(z,y) = \delta(x-y) \end{array} \right\}$$

$$= \exp\left\{-\frac{i}{2}\int d^4x \int d^4y\, J(x) B^{-1}(x,y) J(y)\right\} \cdot$$

$$\cdot \underbrace{\int \mathcal{D}\phi \exp\left\{\frac{i}{2}\int d^4x \int d^4y\, \phi(x) B(x,y) \phi(y)\right\}}_{Z[0]}$$

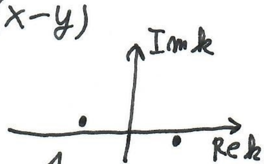
WE AVOID THE PATH INTEGRAL

$$Z[J] = Z[0] \exp\left\{-\frac{i}{2}\int d^4x \int d^4y\, J(x) B^{-1}(x,y) J(y)\right\} =$$

$$\int d^4z\, B^{-1}(x,z) B(z,y) = \delta(x-y) \quad / -(\Box_x + m^2)$$

$$\int d^4z \left[-(\Box_x + m^2) B^{-1}(x,z)\right] B(z,y) = B(x,y)$$

$$\Rightarrow \quad -(\Box_x + m^2) \underbrace{B^{-1}(x,z)}_{\int \frac{d^4k}{(2\pi)^4} B^{-1}_k e^{-ik\cdot(x-y)}} = \delta(x-z)$$



$$-\left((-ik)^2 + m^2\right) B^{-1}_k = 1$$

$$B^{-1}_k = \frac{1}{k^2 - m^2} \longrightarrow \frac{1}{k^2 - m^2 + i\varepsilon}$$

WICK ROTATION $\quad \Box = \partial_0^2 - \Delta \longrightarrow -\partial_0^2 - \Delta$

$$\frac{1}{k^2 - m^2} \longrightarrow \frac{1}{(k_0)^2 + \vec{k}^2 + m^2}$$

$$= Z[0] \exp\left\{\frac{i}{2}\int d^4x \int d^4y\, J(x) \int \frac{d^4k}{(2\pi)^4} \frac{-1}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)} J(y)\right\}$$