Notes on Path Integral Methods

DYSON-SCHWINGER EQUATIONS

GENERATING FUNCTIONAL $Z[J]=\int \mathcal{D}\phi \exp\left\{i\left(S[\phi]+\int d^4x\, J(x)\phi(x)\right)\right\}$

GENERATES ALL POSSIBLE DIAGRAMS INCLUDING VACUUM BUBBLES

$\hat{Z}[J]=\dfrac{Z[J]}{Z[0]}$ GENERATES ONLY DIAGRAMS WITHOUT VACUUM BUBBLES $\longleftrightarrow G^{(m)}(x_1,\dots,x_m)$

$$\int \mathcal{D}\phi \exp\left\{i\left(S[\phi]+\int d^4x\, J(x)\phi(x)\right)\right\}=\left[\begin{array}{c}\text{SUBSTITUTION IN PATH INTEGRAL}\\ \underbrace{\phi(x)}_{\int\text{OVER}} \longrightarrow \phi(x)+\underbrace{f(x)}_{\text{FIX}}\\ \text{JACOBIAN}=1\end{array}\right]=$$

$$=\int \mathcal{D}\phi \exp\left\{i\left(S[\phi+f]+\int d^4x\, J(x)\left(\phi(x)+f(x)\right)\right)\right\}\equiv Z_f[J]$$

LIKE WITH $\mathcal{Z}=\int_{-\infty}^{\infty}F(x)\,dx=\int_{\forall x}\underbrace{d(x+c)}_{dx}F(x+c)$

NOW WE HAVE $\int_{\forall\phi}\mathcal{D}\phi \exp\left\{i\left(S[\phi+f]+\dots\right)\right\}=\int_{\forall\phi}\mathcal{D}\phi \exp\left\{i\left(S[\phi]+\dots\right)\right\}$

$$\Rightarrow\quad Z_f[J]=Z[J]\longrightarrow 0=Z_f[J]-Z[J]=\left[\text{FOR SMALL } f\right]=$$

$$=\int d^4y\,\frac{\delta Z[J]}{\delta\phi(y)}\underbrace{\delta\phi(y)}_{f(y)}+\frac{1}{2!}\int d^4y\int d^4z\,\frac{\delta^2 Z[J]}{\delta\phi(y)\delta\phi(z)}\underbrace{\delta\phi(y)\delta\phi(z)+\dots}_{\mathcal{O}(f^2)}=$$

$$=\int\mathcal{D}\phi\int d^4y\left[\frac{\delta}{\delta\phi(y)}\exp\left\{i\left(S[\phi]+\int d^4x\, J(x)\phi(x)\right)\right\}\right]f(y)+\mathcal{O}(f^2)$$

$$\int d^4y\,\Omega(y)f(y)\approx 0\quad\text{FOR }\forall f\Rightarrow\Omega(y)=0\longrightarrow$$

$$0=\Omega(y)=\int\mathcal{D}\phi\,\frac{\delta}{\delta\phi(y)}\exp\left\{i\left(S[\phi]+\int d^4x\, J(x)\phi(x)\right)\right\}=$$

$$=\int\mathcal{D}\phi\; i\left(\frac{\delta S[\phi]}{\delta\phi(y)}+J(y)\right)\exp\left\{i\left(S[\phi]+\int d^4x\, J(x)\phi(x)\right)\right\}=$$

$$=i\left(\frac{\delta S[\phi]}{\delta\phi(y)}+J(y)\right)\int\mathcal{D}\phi\exp\left\{i\left(S[\phi]+\int d^4x\, J(x)\phi(x)\right)\right\}=$$

$$=i\left(\left.\frac{\delta S[\phi]}{\delta\phi(y)}\right|_{\phi(y)\to\frac{1}{i}\frac{\delta}{\delta J(x)}}+J(y)\right)Z[J]$$

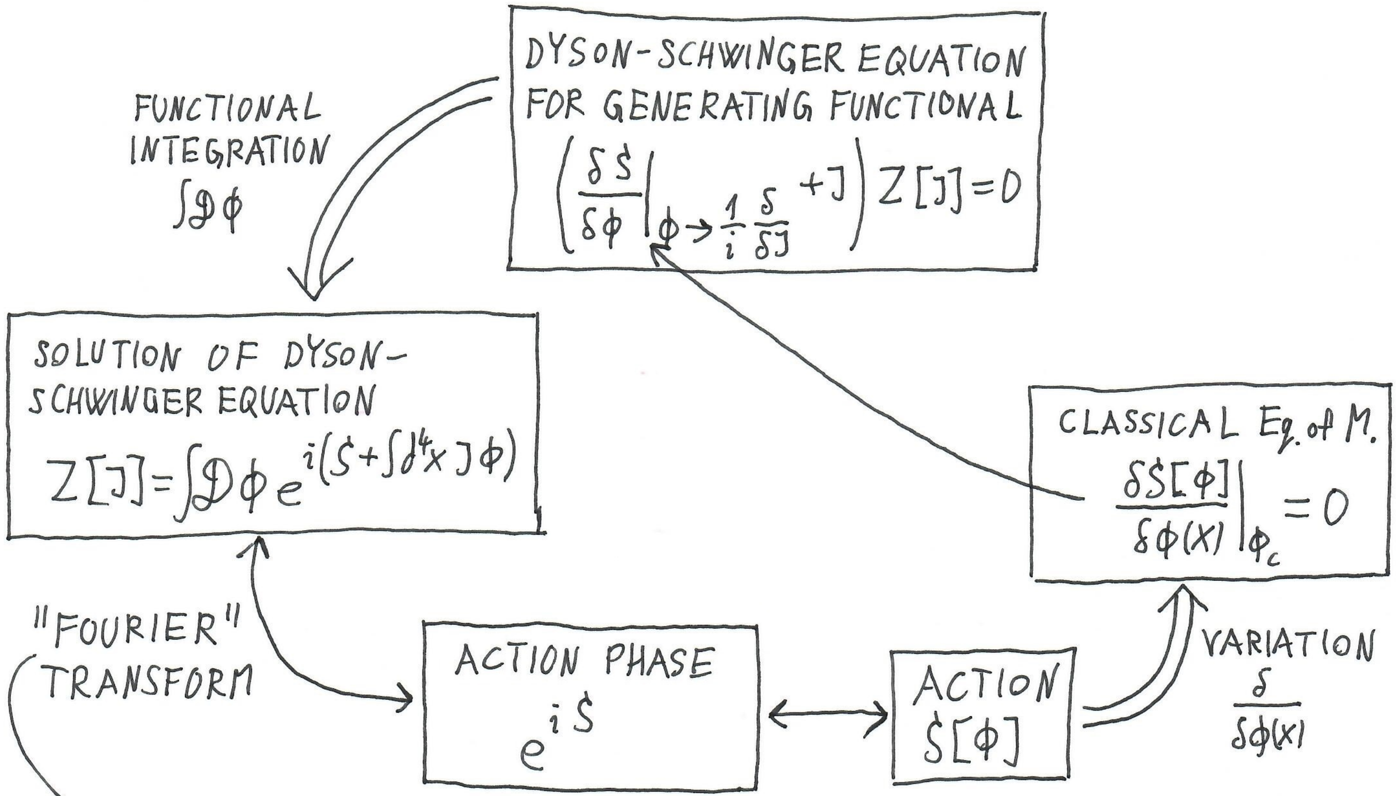
DYSON-SCHWINGER EQUATION
$$\left(\frac{\delta S[\phi]}{\delta \phi(x)}\bigg|_{\phi(x)\to \frac{1}{i}\frac{\delta}{\delta J(x)}} + J(x)\right) Z[J] = 0$$

LOOKS LIKE A "VARIATIONAL" EQUATION FOR GENERATING FUNCTIONAL

$$\text{OPERATOR}\left[\frac{\delta}{\delta J(x)}\right] Z[J] \overset{\forall J(x)}{=} 0$$

$$\left(\text{OPERATOR}\left[\frac{\partial}{\partial x^{\mu}}\right] F[x] \overset{\forall x}{=} 0 \quad \text{LIKE DIFFERENTIAL EQUATION}\right)$$

THIS OPERATOR IS GIVEN BY $\frac{\delta S}{\delta \phi} + J$ WHICH IS A CLASSICAL Eq. of M. WITH SOURCE J FOR FIELD ϕ



$$e^{iS[\phi]} = \int \mathcal{D}J\; Z[J]\, e^{-i\int d^4x\, J\phi}$$

$$Z[J] = \int \mathcal{D}\phi \int \mathcal{D}J'\, Z[J']\, e^{-i\int d^4x\, J'\phi}\, e^{i\int d^4x\, J\phi} =$$

$$= \int \mathcal{D}J'\, Z[J'] \underbrace{\int \mathcal{D}\phi\, e^{-i\int d^4x\, \phi\,(J'-J)}}_{\text{SOMETHING LIKE } \delta(J'-J)}$$

$$S_{FREE}[\phi]=\int d^4x\left(\frac{1}{2}\eta^{\mu\nu}\phi_{|\mu}\phi_{|\nu}-\frac{1}{2}m^2\phi^2\right)$$

$$\frac{\delta}{\delta\phi(y)}\frac{\partial\phi(x)}{\partial x}=-\delta(x-y)\frac{\partial}{\partial x}$$

$$\int dx\,\frac{\delta}{\delta\phi(y)}\frac{\partial\phi(x)}{\partial x}f(x)=\frac{\delta}{\delta\phi(y)}\int dx\,\frac{\partial\phi(x)}{\partial x}f(x)\overset{P.P.}{=}$$

$$\overset{P.P.}{=}-\frac{\delta}{\delta\phi(y)}\int dx\,\phi(x)\frac{\partial f(x)}{\partial x}=-\int dx\,\delta(x-y)\frac{\partial}{\partial x}f(x)$$

$$\frac{\delta S_{FREE}[\phi]}{\delta\phi(x)}=\int d^4y\Big(\frac{1}{2}\eta^{\mu\nu}\left[-\delta(x-y)\partial_\mu\right]\phi(y)_{|\nu}+$$

$$+\frac{1}{2}\eta^{\mu\nu}\left[-\delta(x-y)\partial_\nu\right]\phi(y)_{|\mu}-m^2\delta(x-y)\phi(y)\Big)=$$

$$=-\eta^{\mu\nu}\phi(x)_{|\mu\nu}-m^2\phi(x)=-(\Box+m^2)\phi(x)\equiv-\int d^4z\,K_{KG}(x,z)\,\phi(z)$$

KLEIN-GORDON KERNEL

FOR $S[\phi]=\int d^4x\left(\frac{1}{2}\eta^{\mu\nu}\phi_{|\mu}\phi_{|\nu}-\frac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4\right)$ THE D.S. EQUATION IS

$$\left[-\int d^4y\,K_{KG}(x,y)\frac{1}{i}\frac{\delta}{\delta J(y)}-\frac{1}{3!}\lambda\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3+J(x)\right]Z[J]=0$$

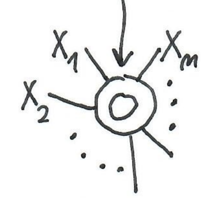
FOR $S[\phi]=\int d^4x\left(\frac{1}{2}\eta^{\mu\nu}\phi_{|\mu}\phi_{|\nu}-\frac{1}{2}m^2\phi^2-\frac{1}{3!}g\phi^3\right)$ THE D.S. EQUATION IS

$$\left[-\int d^4y\,K_{KG}(x,y)\frac{1}{i}\frac{\delta}{\delta J(y)}-\frac{1}{2!}g\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^2+J(x)\right]Z[J]=0$$

WE WILL WRITE THESE EQUATIONS INTERMS OF FEYNMAN DIAGRAMS
GENERATING FUNCTIONAL CONTAINS INFORMATION ABOUT ALL DIAGRAMS

GREEN FUNCTIONS WITH VACUUM BUBLES

Σ OF $\forall$ DIAGRAMS

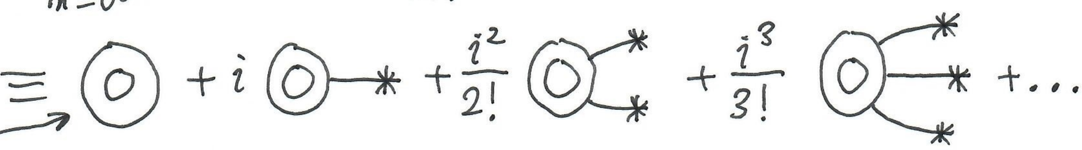
$$\tilde{G}^{(m)}(x_1,\ldots,x_m)=\left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_m)}Z[J]\right]_{J=0}\equiv\Bigg($$  $$\Bigg)$$

EXPANSION OF GENERATING FUNCTIONAL

$$Z[J]=\sum_{m=0}^{\infty}\int d^4x_1\dots\int d^4x_m\, Z^{(m)}(x_1,\dots,x_m)\,J(x_1)\dots J(x_m)$$

$$\left\{\tilde{G}^{(m)}(x_1,\dots,x_m)=\left(\frac{1}{i}\right)^m\left[\frac{\delta^m Z[J]}{\delta J(x_1)\dots\delta J(x_m)}\right]_{J=0}=\left(\frac{1}{i}\right)^m m!\,Z^{(m)}(x_1,\dots,x_m)\right\}$$

$$Z[J]=\sum_{m=0}^{\infty}\int d^4x_1\dots\int d^4x_m\,\frac{i^m}{m!}\tilde{G}^{(m)}(x_1,\dots,x_m)\,J(x_1)\dots J(x_m)\equiv$$

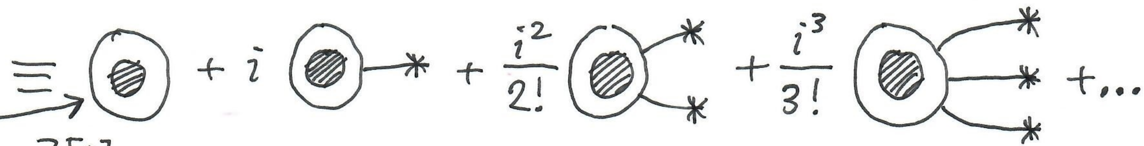


VACUUM BUBLES — 0-POINT FUNCTION — $Z[0]$

M-POINT GREEN FUNCTIONS (WITHOUT VACUUM BUBBLES)

$$G^{(m)}(x_1,\dots,x_m)=\left(\frac{1}{i}\right)^m\left[\frac{\delta^m\hat{Z}[J]}{\delta J(x_1)\dots\delta J(x_m)}\right]_{J=0},\quad \hat{Z}[J]=\frac{Z[J]}{Z[0]}$$

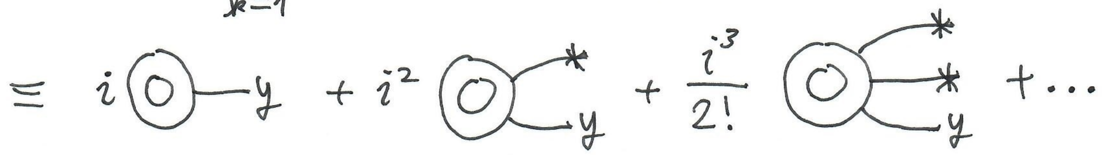
$$\hat{Z}[J]=\sum_{m=0}^{\infty}\int d^4x_1\dots\int d^4x_m\,\frac{i^m}{m!}G^{(m)}(x_1,\dots,x_m)\,J(x_1)\dots J(x_m)\equiv$$



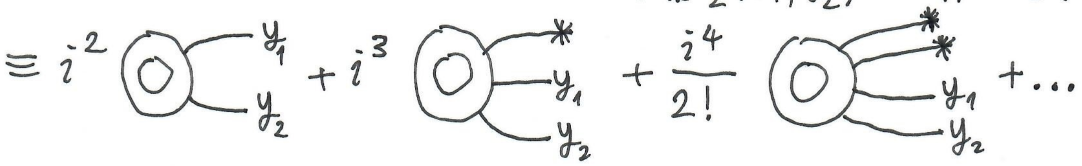
$\hat{Z}[0]=\frac{Z[0]}{Z[0]}=1$

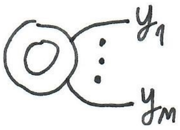
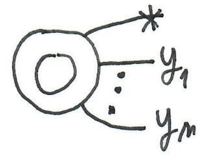
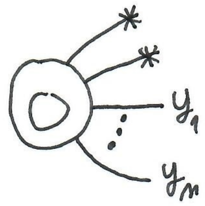
$$\frac{\delta}{\delta J(y)}Z[J]=\frac{\delta}{\delta J(y)}\sum_{k=0}^{\infty}\int d^4x_1\dots\int d^4x_k\,\frac{i^k}{k!}\tilde{G}^{(k)}(x_1,\dots,x_k)\,J(x_1)\dots J(x_k)=$$

$$=\sum_{k=1}^{\infty}\int d^4x_1\dots\int d^4x_{k-1}\,\frac{i^k}{k!}\,k\,\tilde{G}^{(k)}(x_1,\dots,x_{k-1},y)\,J(x_1)\dots J(x_{k-1})\equiv$$



$$\frac{\delta^2}{\delta J(y_1)\delta J(y_2)}Z[J]=\sum_{k=2}^{\infty}\int d^4x_1\dots\int d^4x_{k-2}\,\frac{i^k}{k!}\,k(k-1)\,\tilde{G}^{(k)}(x_1,\dots,x_{k-2},y_1,y_2)\,J(x_1)\dots J(x_{k-2})\equiv$$



$$\frac{\delta^m Z[J]}{\delta J(y_1)\dots\delta J(y_m)} = i^m$$  $$+\, i^{m+1}$$  $$+\, \frac{i^{m+2}}{2!}$$  $$+\dots$$

D.S. Eq.

$$-\int d^4y\, K_{KG}(x,y)\frac{1}{i}\frac{\delta Z[J]}{\delta J(y)} - \frac{\lambda}{3!}\left(\frac{1}{i}\right)^3\frac{\delta^3 Z[J]}{\delta J(x)^3} + J(x)Z[J] = 0$$

$$\Big/ \int d^4x \int d^4z\, J(z)\,\Delta_F(z-x)\dots(x)$$

$$-\underbrace{\int d^4x \int d^4y \int d^4z\, J(z)\,\Delta_F(z-x)\,K_{KG}(x,y)}\,\frac{1}{i}\frac{\delta Z[J]}{\delta J(y)} -$$

$$\int d^4x\, \Delta_F(z-x)\,K_{KG}(x,y) = (\Box_y + m^2)\underbrace{\Delta_F(z-y)}_{iB^{-1}(z,y)} = i\underbrace{(\Box_y + m^2)B^{-1}(z,y)}_{-\delta(y-z)} = \frac{1}{i}\,\delta(y-z)$$

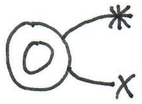
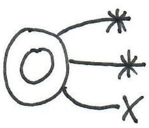
$$-\frac{\lambda}{3!}\int d^4x \int d^4z\, J(z)\,\Delta_F(z-x)\left(\frac{1}{i}\right)^3\frac{\delta^3 Z[J]}{\delta J(x)^3} + Z[J]\int d^4x \int d^4z\, J(z)\,\Delta_F(z-x)\,J(x) =$$

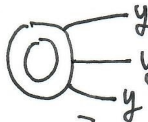
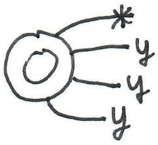
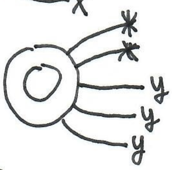
$$= -\int d^4x\, J(x)\frac{1}{i^2}\frac{\delta Z[J]}{\delta J(x)} - \frac{\lambda}{3!}\int d^4x \int d^4y\, J(x)\,\Delta_F(x-y)\left(\frac{1}{i}\right)^3\frac{\delta^3 Z[J]}{\delta J(y)^3} +$$

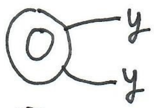
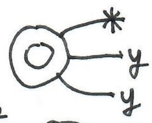
$\{$ FOR ϕ^3 INTERACTION $\frac{g}{2!}$ — // — — // — $\left(\frac{1}{i}\right)^2\frac{\delta^2 Z[J]}{\delta J(y)^2}$ $\}$

$$+Z[J]\int d^4x \int d^4y\, J(x)\,\Delta_F(x-y)\,J(y) = 0$$

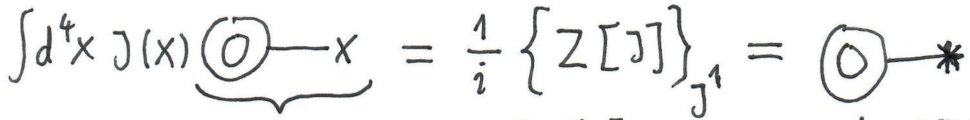
$$+\frac{-i\lambda}{3!}\frac{1}{i} \quad \Big\{ \text{OR } \frac{-ig}{2!}\frac{1}{i} \Big\}$$

$$-\int d^4x\, J(x)\frac{1}{i}\Big[$$  $+\, i$  $+\, \frac{i^2}{2!}$  $$+\dots\Big] - \frac{\lambda}{3!}\int d^4x \int d^4y\, J(x)\,\cdot$$

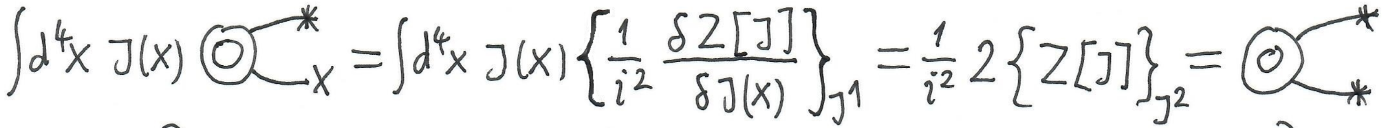
$$\cdot\,\Delta_F(x-y)\Big[$$  $+\, i$  $+\, \frac{i^2}{2!}$  $$+\dots\Big] +$$

$\{$ OR  $+\, i$  $+\dots$ FOR ϕ^3 INTERACTION $\}$

$$+(*\!\!-\!\!-\!\!*)\Big[$$  $+\, i$  $+\, \frac{i^2}{2!}$  $$+\dots\Big] = 0$$

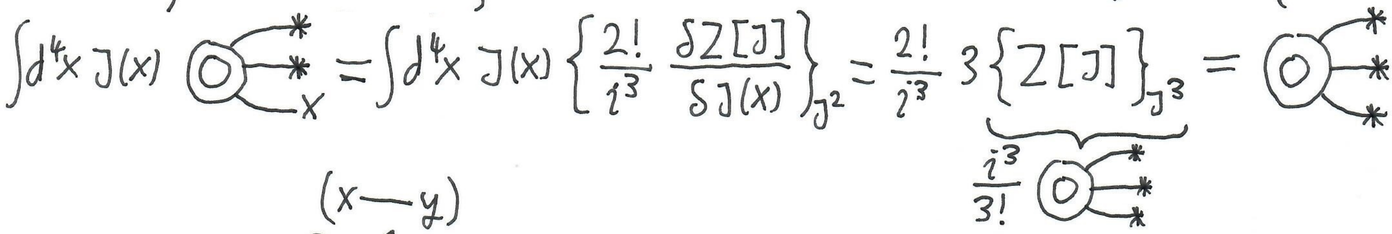


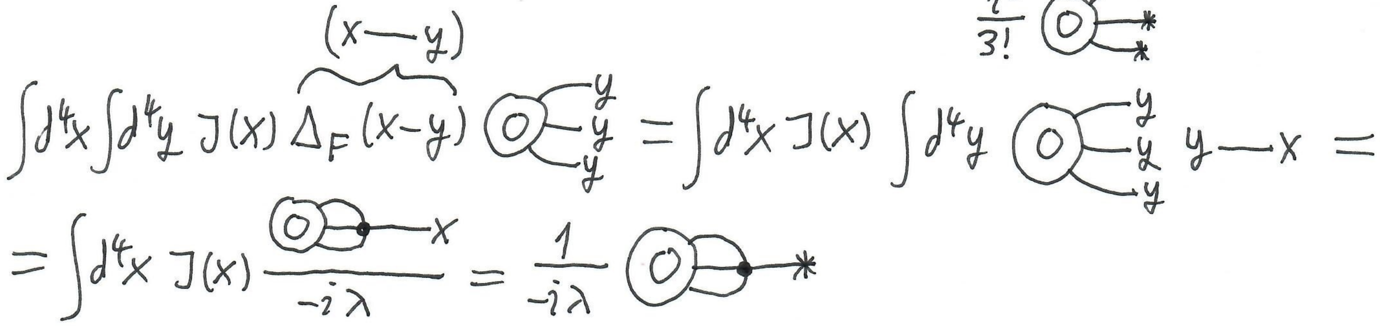
$$\tilde{G}^{(1)}(x) = \frac{1}{i}\,\frac{\delta Z[J]}{\delta J(x)}\bigg|_{J=0} \equiv \left\{\frac{1}{i}\,\frac{\delta Z[J]}{\delta J(x)}\right\}_0$$



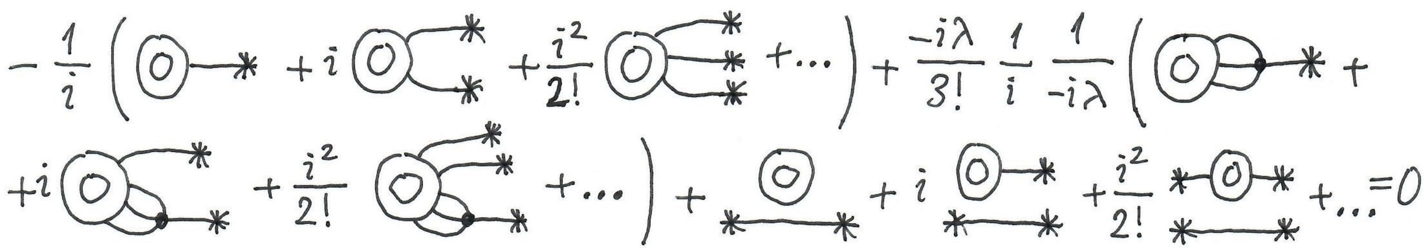
$$f(\xi)=\sum_k f_k \xi^k \to \frac{df}{d\xi}=\sum_k f_k k\,\xi^{k-1}=\sum_k f_{k+1}(k+1)\,\xi^k$$

$$\xi\left\{\frac{df}{d\xi}\right\}_{\xi^m} = \xi\, f_{m+1}(m+1)\,\xi^m = (m+1)\,\{f\}_{\xi^{m+1}}$$

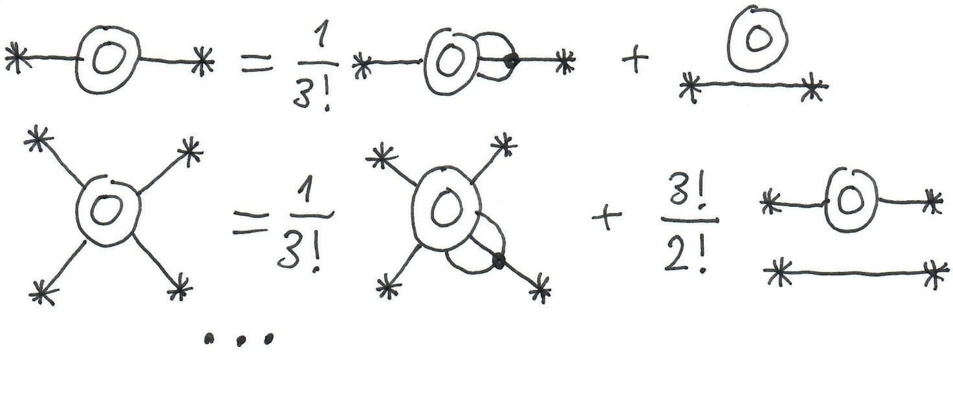




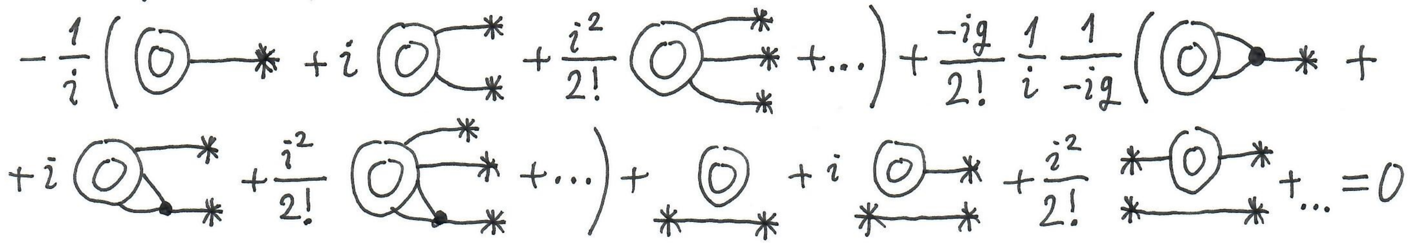
D.S. Eq. FOR ϕ^4 INTERACTION



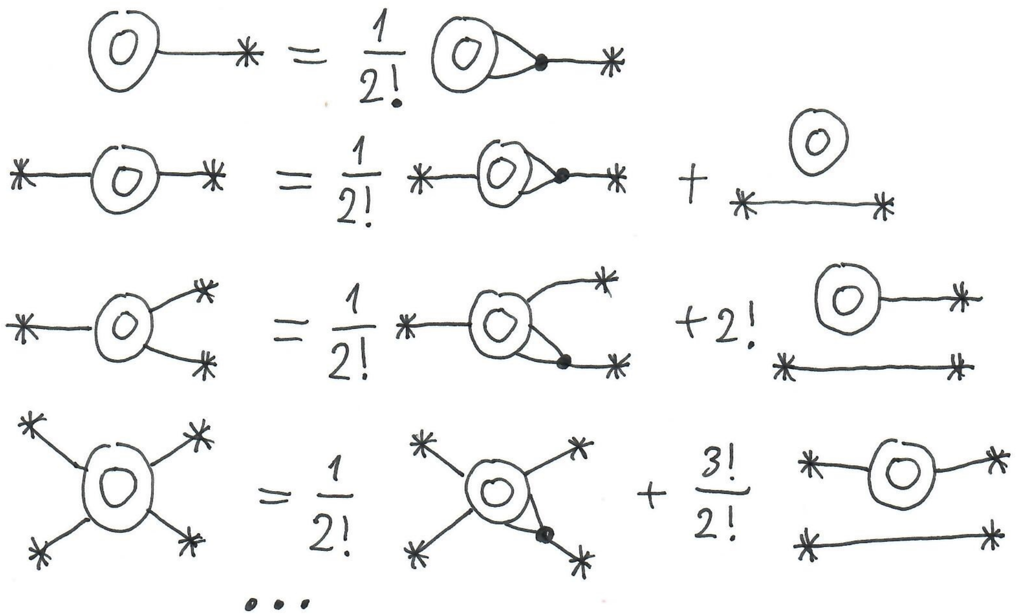
BY COMPARING PARTS WITH THE SAME NUMBER OF LEGS —✱ (ORDER OF $J(x)$) AND USING THE FACT THAT 1, 3, 5, ... - POINT FUNCTIONS ARE ZERO FOR ϕ^4 INTERACTION WE OBTAIN



D.S. Eq. FOR ϕ^3 INTERACTION

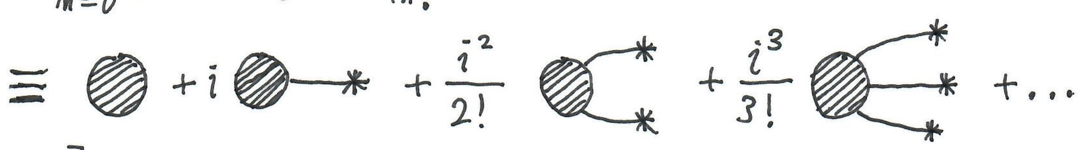


NOW WE HAVE TO TAKE INTO ACCOUNT ALSO 1,3,5,... – POINT FUNCTIONS



SUBSTITUTION TO NEW GENERATING FUNCTIONAL $Z[J] = e^{W[J]}$

$$W[J] = \sum_{m=0}^{\infty} \int d^4x_1 \ldots \int d^4x_m \frac{i^m}{m!} W^{(m)}(x_1, \ldots, x_m) J(x_1) \ldots J(x_m) \equiv$$



$$\left[\frac{\delta}{\delta J(x)} \equiv \delta_x\right] \qquad \delta_x Z = (\delta_x W) e^W = (\delta_x W) Z$$

$$\delta_{x_1}\delta_{x_2} Z = \left[(\delta_{x_1}\delta_{x_2} W) + (\delta_{x_1} W)(\delta_{x_2} W)\right] Z$$

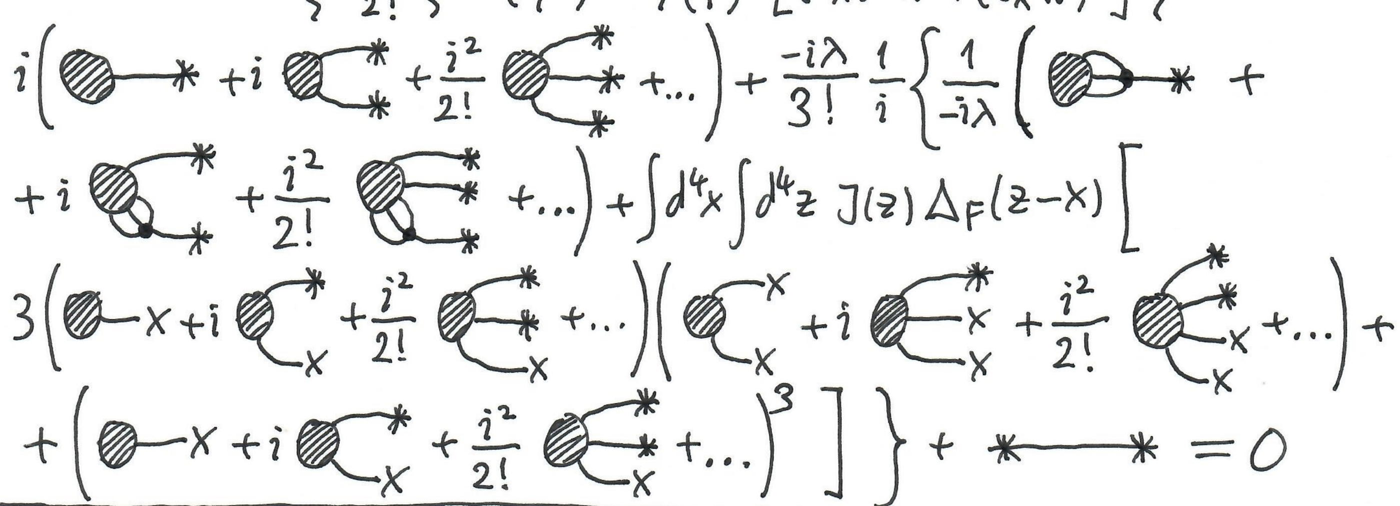
$$\delta_{x_1}\delta_{x_2}\delta_{x_3} Z = \Big[(\delta_{x_1}\delta_{x_2}\delta_{x_3} W) + (\delta_{x_1}\delta_{x_2} W)(\delta_{x_3} W) + (\delta_{x_1}\delta_{x_3} W)(\delta_{x_2} W) +$$
$$+ (\delta_{x_2}\delta_{x_3} W)(\delta_{x_1} W) + (\delta_{x_1} W)(\delta_{x_2} W)(\delta_{x_3} W)\Big] Z$$

D. S. Eq. $\left\{-\frac{g}{2!}\right\} \longleftarrow$ FOR ϕ^3 INT. $\longrightarrow \left\{\left(\frac{1}{i}\delta_y\right)^2\right\}$

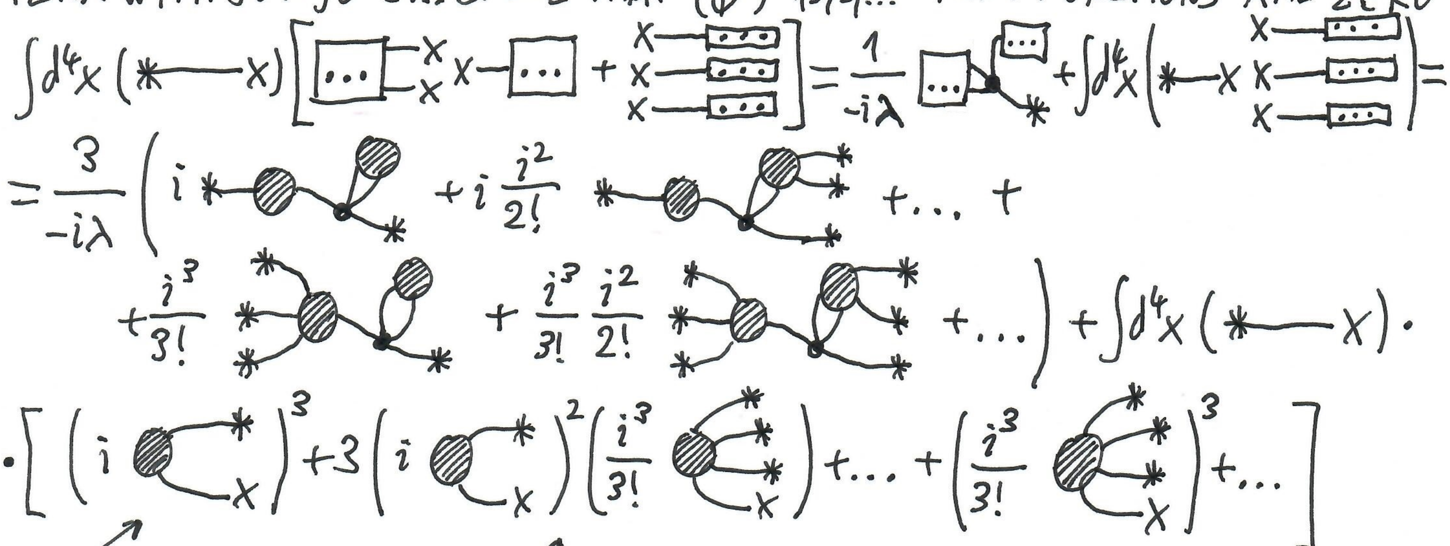
$$i\int d^4x\, J(x)\frac{1}{i}\delta_x Z - \frac{\lambda}{3!}\int d^4x\int d^4y\, J(x)\Delta_F(x-y)\left(\frac{1}{i}\delta_y\right)^3 Z + (*\!\!-\!\!-\!\!*)Z = 0$$

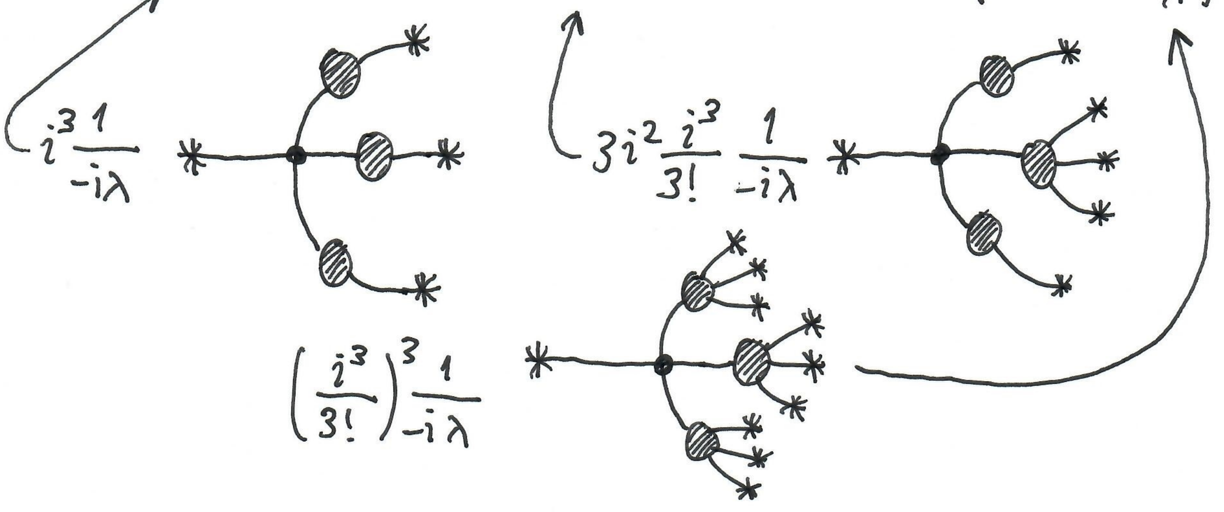
$$i\int d^4x\, J(x)\frac{1}{i}\delta_x W - \frac{\lambda}{3!}\int d^4x\int d^4z\, J(z)\Delta_F(z-x)\left(\frac{1}{i}\right)^3\Big[(\delta_x)^3 W + 3\left((\delta_x)^2 W\right)(\delta_x W) + (\delta_x W)^3\Big] + *\!\!-\!\!-\!\!* = 0$$

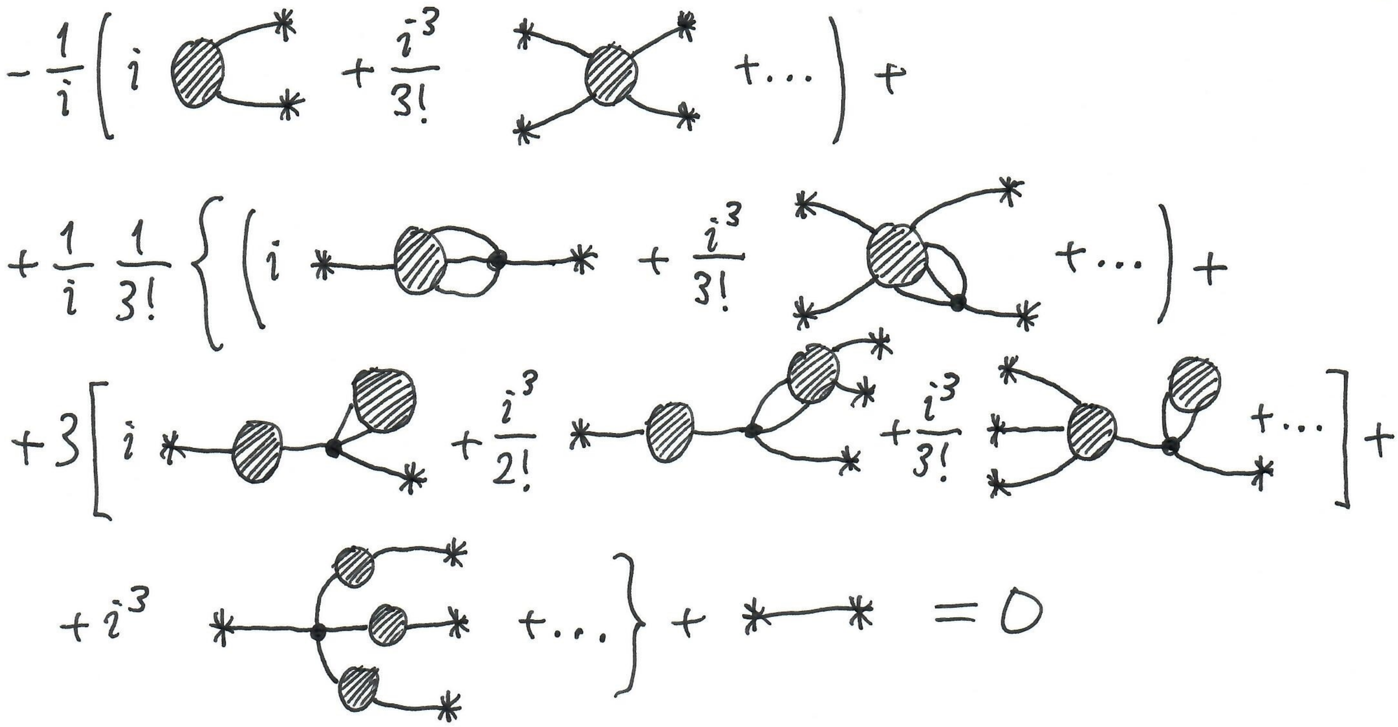
$$\left\{-\frac{g}{2!}\right\} \leftarrow (\phi^3) \rightarrow \left\{\left(\frac{1}{i}\right)^2\left[(\delta_x)^2 W + (\delta_x W)^2\right]\right\}$$

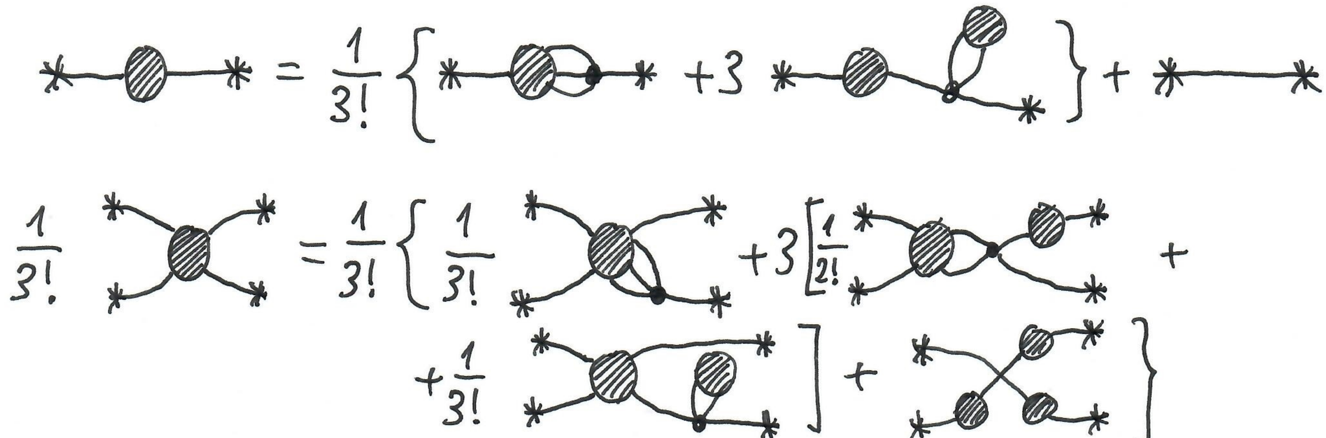


TERM WITH $\int d^4x\int d^4z\ldots$ & USE THAT (ϕ^4) 1,3,5,...-POINT FUNCTIONS ARE ZERO

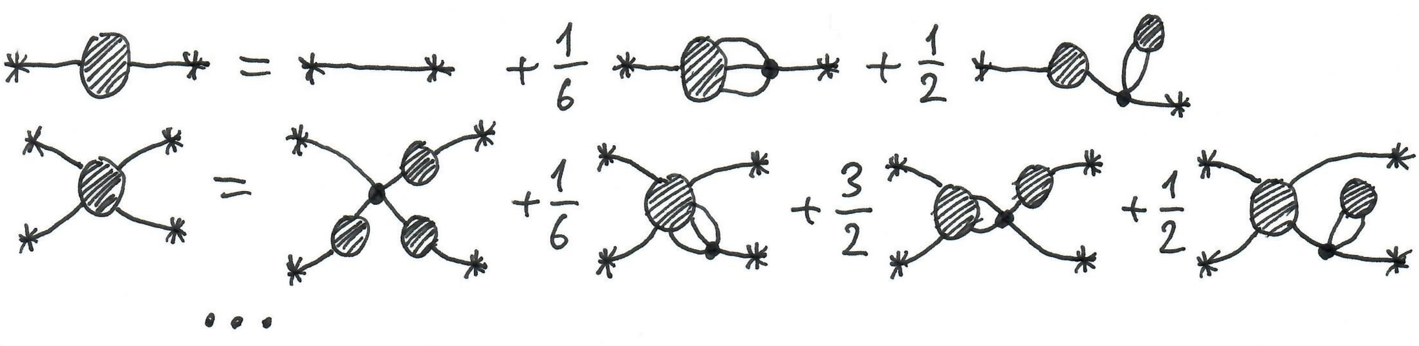








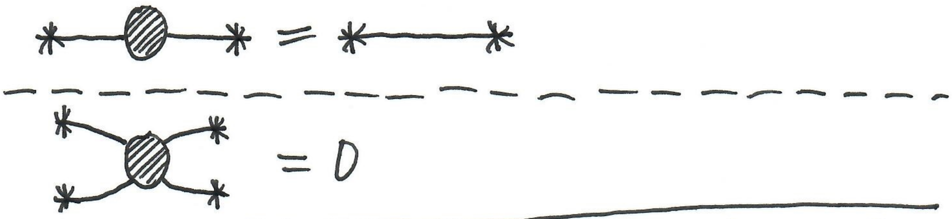
THESE ARE RECCURENCE RELATIONS FOR FINDING $W^{(n)}$ IN PERTURBATIVE WAY $\{W^{(n)}\}_{(-i\lambda)^k} = \sum_{n_1,n_2,\dots} a_{n_1 n_2 \dots}\{W^{(n_1)}W^{(n_2)}\dots\}_{(-i\lambda)^{k-1}}$



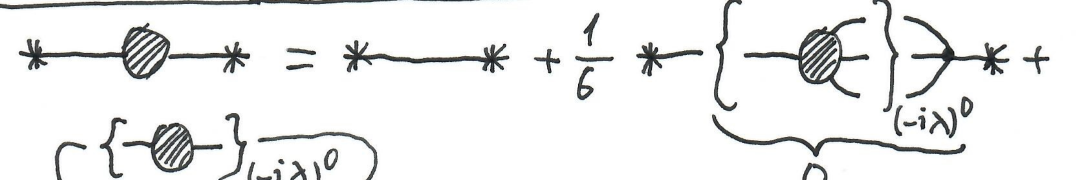
SOLVING THESE RELATIONS PERTUBATIVELY WE CAN FIND n-POINT FUNCTIONS $W^{(n)}(x_1,\dots,x_n)$ AS SUMS OF ALL FEYNMAN DIAGRAMS – THIS PRODUCES ONLY CONNECTED DIAGRAMS!

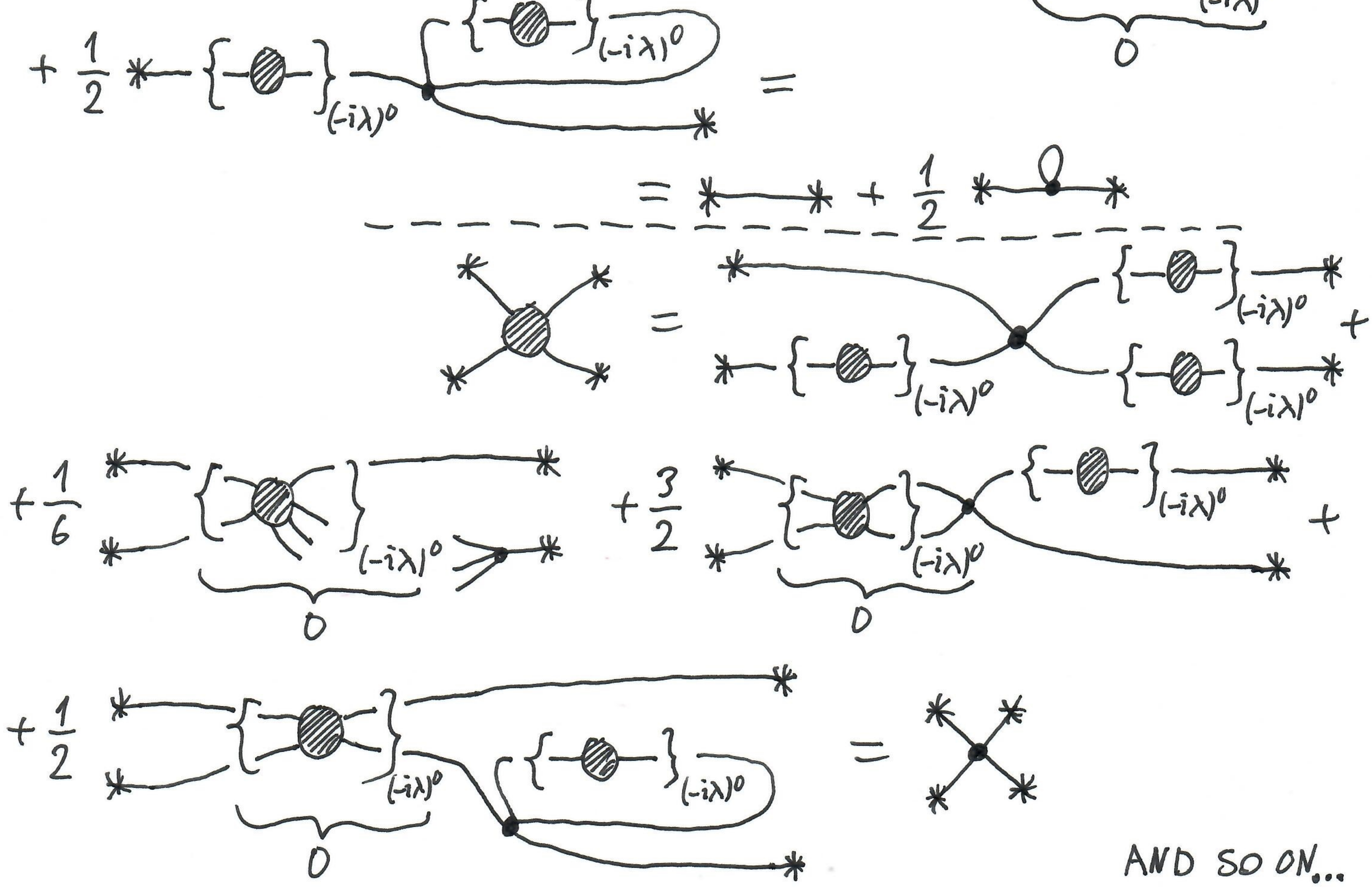
PERTURBATIVE SOLUTION:

ORDER $(-i\lambda)^0$:



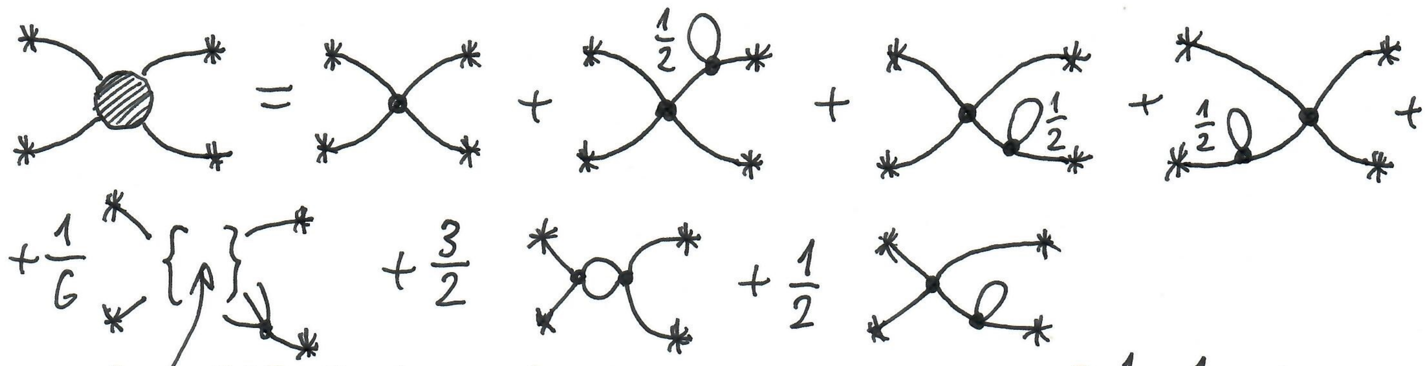
ORDER $(-i\lambda)^1$:





AND SO ON...

ORDER $(-i\lambda)^2$: ONLY FOR 4-POINT FUNCTION



$= 0$ / MORE THAN ONE VERTEX

$3 \cdot \frac{1}{2} + \frac{1}{2} = 2$