

Notes on Path Integral Methods

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DYSON-SCHWINGER EQUATIONS

GENERATING FUNCTIONAL $Z[J] = \int \mathcal{D}\phi \exp \left\{ i \left(S[\phi] + \int d^4x J(x) \phi(x) \right) \right\}$

GENERATES ALL POSSIBLE DIAGRAMS INCLUDING VACUUM BUBBLES

$$\hat{Z}[J] = \frac{Z[J]}{Z[0]} \quad \begin{array}{l} \text{GENERATES ONLY} \\ \text{DIAGRAMS WITHOUT} \\ \text{VACUUM BUBBLES} \end{array} \longleftrightarrow G^{(m)}(x_1, \dots, x_m)$$

$$\begin{aligned} \int \mathcal{D}\phi \exp \left\{ i \left(S[\phi] + \int d^4x J(x) \phi(x) \right) \right\} &= \left[\begin{array}{l} \text{SUBSTITUTION IN PATH INTEGRAL} \\ \phi(x) \rightarrow \phi(x) + f(x) \\ \text{JACOBIAN} = 1 \quad \text{OVER } \phi \quad \text{FIX} \end{array} \right] = \\ &= \int \mathcal{D}\phi \exp \left\{ i \left(S[\phi + f] + \int d^4x J(x) (\phi(x) + f(x)) \right) \right\} \equiv Z_f[J] \end{aligned}$$

LIKE WITH $\int_{-\infty}^{\infty} F(x) dx = \int_{\forall x} \underbrace{d(x+c)}_{dx} F(x+c)$

NOW WE HAVE $\int_{\forall \phi} \mathcal{D}\phi \exp \left\{ i \left(S[\phi + f] + \dots \right) \right\} = \int_{\forall \phi} \mathcal{D}\phi \exp \left\{ i \left(S[\phi] + \dots \right) \right\}$

$$\begin{aligned} \Rightarrow Z_f[J] &= Z[J] \rightarrow 0 = Z_f[J] - Z[J] = [\text{FOR SMALL } f] = \\ &= \int d^4y \frac{\delta Z[J]}{\delta \phi(y)} \underbrace{\phi(y)}_{f(y)} + \frac{1}{2!} \int d^4y \int d^4z \frac{\delta^2 Z[J]}{\delta \phi(y) \delta \phi(z)} \underbrace{\phi(y) \phi(z) + \dots}_{\mathcal{O}(f^2)} = \end{aligned}$$

$$= \int \mathcal{D}\phi \int d^4y \left[\frac{\delta}{\delta \phi(y)} \exp \left\{ i \left(S[\phi] + \int d^4x J(x) \phi(x) \right) \right\} \right] f(y) + \mathcal{O}(f^2)$$

$$\int d^4y \Omega(y) f(y) \approx 0 \quad \text{FOR } \forall f \Rightarrow \Omega(y) = 0 \rightarrow$$

$$\begin{aligned} 0 = \Omega(y) &= \int \mathcal{D}\phi \frac{\delta}{\delta \phi(y)} \exp \left\{ i \left(S[\phi] + \int d^4x J(x) \phi(x) \right) \right\} = \\ &= \int \mathcal{D}\phi i \left(\frac{\delta S[\phi]}{\delta \phi(y)} + J(y) \right) \exp \left\{ i \left(S[\phi] + \int d^4x J(x) \phi(x) \right) \right\} = \\ &= i \left(\frac{\delta S[\phi]}{\delta \phi(y)} + J(y) \right) \int \mathcal{D}\phi \exp \left\{ i \left(S[\phi] + \int d^4x J(x) \phi(x) \right) \right\} = \\ &= i \left(\frac{\delta S[\phi]}{\delta \phi(y)} \right) \Big|_{\phi(y) \rightarrow \frac{1}{i} \frac{\delta}{\delta J(y)}} + J(y) \Big| Z[J] \end{aligned}$$

DYSON-SCHWINGER EQUATION $\left(\frac{\delta S[\phi]}{\delta \phi(x)} \Big|_{\phi(x) \rightarrow \frac{1}{i} \frac{\delta}{\delta J(x)} + J(x)} \right) Z[J] = 0$

LOOKS LIKE A "VARIATIONAL" EQUATION FOR GENERATING FUNCTIONAL

OPERATOR $\left[\frac{\delta}{\delta J(x)} \right] Z[J] \stackrel{\forall J(x)}{=} 0$

$\left(\text{OPERATOR} \left[\frac{\partial}{\partial x^\mu} \right] F[x] \stackrel{\forall x}{=} 0 \right)$ LIKE DIFFERENTIAL EQUATION

THIS OPERATOR IS GIVEN BY $\frac{\delta S}{\delta \phi} + J$ WHICH IS A CLASSICAL Eq. of M. WITH SOURCE J FOR FIELD ϕ

FUNCTIONAL
INTEGRATION

$\int \mathcal{D}\phi$

DYSON-SCHWINGER EQUATION
FOR GENERATING FUNCTIONAL

$\left(\frac{\delta S}{\delta \phi} \Big|_{\phi \rightarrow \frac{1}{i} \frac{\delta}{\delta J} + J} \right) Z[J] = 0$

SOLUTION OF DYSON-SCHWINGER EQUATION

$Z[J] = \int \mathcal{D}\phi e^{i(S + \int d^4x J\phi)}$

CLASSICAL Eq. of M.

$\frac{\delta S[\phi]}{\delta \phi(x)} \Big|_{\phi_c} = 0$

"FOURIER"
TRANSFORM

ACTION PHASE

e^{iS}

ACTION
 $S[\phi]$

VARIATION
 $\frac{\delta}{\delta \phi(x)}$

$e^{iS[\phi]} = \int \mathcal{D}J Z[J] e^{-i \int d^4x J\phi}$

$Z[J] = \int \mathcal{D}\phi \int \mathcal{D}J' Z[J'] e^{-i \int d^4x J'\phi} e^{i \int d^4x J\phi} =$

$= \int \mathcal{D}J' Z[J'] \underbrace{\int \mathcal{D}\phi e^{-i \int d^4x \phi (J' - J)}}_{\text{SOMETHING LIKE } \delta(J' - J)}$

SOMETHING LIKE $\delta(J' - J)$

$$S_{\text{FREE}}[\phi] = \int d^4x \left(\frac{1}{2} \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^2 \right)$$

$$\left\{ \begin{aligned} \frac{\delta}{\delta \phi(y)} \frac{\partial \phi(x)}{\partial x} &= -\delta(x-y) \frac{\partial}{\partial x} \leftarrow \\ \int d^4x \frac{\delta}{\delta \phi(y)} \frac{\partial \phi(x)}{\partial x} f(x) &= \frac{\delta}{\delta \phi(y)} \int d^4x \frac{\partial \phi(x)}{\partial x} f(x) \stackrel{\text{P.P.}}{=} \\ &\stackrel{\text{P.P.}}{=} -\frac{\delta}{\delta \phi(y)} \int d^4x \phi(x) \frac{\partial f(x)}{\partial x} = -\int d^4x \delta(x-y) \frac{\partial}{\partial x} f(x) \end{aligned} \right\}$$

$$\begin{aligned} \frac{\delta S_{\text{FREE}}[\phi]}{\delta \phi(x)} &= \int d^4y \left(\frac{1}{2} \eta^{\mu\nu} [-\delta(x-y) \partial_\mu] \phi(y)_{,\nu} + \right. \\ &\quad \left. + \frac{1}{2} \eta^{\mu\nu} [-\delta(x-y) \partial_\nu] \phi(y)_{,\mu} - m^2 \delta(x-y) \phi(y) \right) = \\ &= -\eta^{\mu\nu} \phi(x)_{,\mu\nu} - m^2 \phi(x) = -(\square + m^2) \phi(x) \equiv -\int d^4z K_{\text{KG}}(x|z) \phi(z) \end{aligned}$$

KLEIN-GORDON KERNEL $\nearrow$

FOR $S[\phi] = \int d^4x \left(\frac{1}{2} \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \right)$ THE D.S. EQUATION IS

$$\left[-\int d^4y K_{\text{KG}}(x|y) \frac{1}{i} \frac{\delta}{\delta J(y)} - \frac{1}{3!} \lambda \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 + J(x) \right] Z[J] = 0$$

FOR $S[\phi] = \int d^4x \left(\frac{1}{2} \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^2 - \frac{1}{3!} g \phi^3 \right)$ THE D.S. EQUATION IS

$$\left[-\int d^4y K_{\text{KG}}(x|y) \frac{1}{i} \frac{\delta}{\delta J(y)} - \frac{1}{2!} g \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^2 + J(x) \right] Z[J] = 0$$

WE WILL WRITE THESE EQUATIONS INTERMS OF FEYNMAN DIAGRAMS
GENERATING FUNCTIONAL CONTAINS INFORMATION ABOUT ALL DIAGRAMS

Σ OF $\forall$ DIAGRAMS

GREEN FUNCTIONS WITH VACUUM BUBLES

$$\tilde{G}^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i} \frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i} \frac{\delta}{\delta J(x_m)} Z[J] \right]_{J=0} \equiv \left(\begin{array}{c} x_1 \quad x_m \\ \downarrow \quad \uparrow \\ \text{O} \\ \uparrow \quad \downarrow \\ x_2 \quad \vdots \end{array} \right)$$

EXPANSION OF GENERATING FUNCTIONAL

$$Z[J] = \sum_{m=0}^{\infty} \int d^4x_1 \dots \int d^4x_m Z^{(m)}(x_1, \dots, x_m) J(x_1) \dots J(x_m)$$

$$\left\{ \tilde{G}^{(m)}(x_1, \dots, x_m) = \left(\frac{1}{i} \right)^m \left[\frac{\delta^m Z[J]}{\delta J(x_1) \dots \delta J(x_m)} \right]_{J=0} = \left(\frac{1}{i} \right)^m m! Z^{(m)}(x_1, \dots, x_m) \right\}$$

$$Z[J] = \sum_{m=0}^{\infty} \int d^4x_1 \dots \int d^4x_m \frac{i^m}{m!} \tilde{G}^{(m)}(x_1, \dots, x_m) J(x_1) \dots J(x_m) \equiv$$

$$\equiv \bigcirc + i \bigcirc - * + \frac{i^2}{2!} \bigcirc \begin{matrix} * \\ * \end{matrix} + \frac{i^3}{3!} \bigcirc \begin{matrix} * \\ * \\ * \end{matrix} + \dots$$

VACUUM BUBBLES — 0-POINT FUNCTION — $Z[0]$

M-POINT GREEN FUNCTIONS (WITHOUT VACUUM BUBBLES)

$$G^{(m)}(x_1, \dots, x_m) = \left(\frac{1}{i} \right)^m \left[\frac{\delta^m \hat{Z}[J]}{\delta J(x_1) \dots \delta J(x_m)} \right]_{J=0}, \quad \hat{Z}[J] = \frac{Z[J]}{Z[0]}$$

$$\hat{Z}[J] = \sum_{m=0}^{\infty} \int d^4x_1 \dots \int d^4x_m \frac{i^m}{m!} G^{(m)}(x_1, \dots, x_m) J(x_1) \dots J(x_m) \equiv$$

$$\equiv \bigcirc + i \bigcirc - * + \frac{i^2}{2!} \bigcirc \begin{matrix} * \\ * \end{matrix} + \frac{i^3}{3!} \bigcirc \begin{matrix} * \\ * \\ * \end{matrix} + \dots$$

$$\hat{Z}[0] = \frac{Z[0]}{Z[0]} = 1$$

$$\frac{\delta}{\delta J(y)} Z[J] = \frac{\delta}{\delta J(y)} \sum_{k=0}^{\infty} \int d^4x_1 \dots \int d^4x_k \frac{i^k}{k!} \tilde{G}^{(k)}(x_1, \dots, x_k) J(x_1) \dots J(x_k) =$$

$$= \sum_{k=1}^{\infty} \int d^4x_1 \dots \int d^4x_{k-1} \frac{i^k}{k!} k \tilde{G}^{(k)}(x_1, \dots, x_{k-1}, y) J(x_1) \dots J(x_{k-1}) \equiv$$

$$\equiv i \bigcirc - y + i^2 \bigcirc \begin{matrix} * \\ y \end{matrix} + \frac{i^3}{2!} \bigcirc \begin{matrix} * \\ * \\ y \end{matrix} + \dots$$

$$\frac{\delta^2}{\delta J(y_1) \delta J(y_2)} Z[J] = \sum_{k=2}^{\infty} \int d^4x_1 \dots \int d^4x_{k-2} \frac{i^k}{k!} k(k-1) \tilde{G}^{(k)}(x_1, \dots, x_{k-2}, y_1, y_2) J(x_1) \dots J(x_{k-2}) \equiv$$

$$\equiv i^2 \bigcirc \begin{matrix} y_1 \\ y_2 \end{matrix} + i^3 \bigcirc \begin{matrix} * \\ y_1 \\ y_2 \end{matrix} + \frac{i^4}{2!} \bigcirc \begin{matrix} * \\ * \\ y_1 \\ y_2 \end{matrix} + \dots$$

$$\frac{\delta^m Z[J]}{\delta J(y_1) \dots \delta J(y_m)} = i^m \text{ (diagram with } m \text{ external lines)} + i^{m+1} \text{ (diagram with } m+1 \text{ external lines)} + \frac{i^{m+2}}{2!} \text{ (diagram with } m+2 \text{ external lines)} + \dots$$

D.S. Eq.

$$-\int d^4 y K_{KG}(x, y) \frac{1}{i} \frac{\delta Z[J]}{\delta J(y)} - \frac{\lambda}{3!} \left(\frac{1}{i} \right)^3 \frac{\delta^3 Z[J]}{\delta J(x)^3} + J(x) Z[J] = 0$$

$$\downarrow \quad \int d^4 x \int d^4 z J(z) \Delta_F(z-x) \dots (x)$$

$$-\int d^4 x \int d^4 y \int d^4 z J(z) \Delta_F(z-x) K_{KG}(x, y) \frac{1}{i} \frac{\delta Z[J]}{\delta J(y)} -$$

$$\left\{ \begin{aligned} \int d^4 x \Delta_F(z-x) K_{KG}(x, y) &= \\ &= (\Box_y + m^2) \underbrace{\Delta_F(z-y)}_{iB^{-1}(z, y)} = \\ &= i \underbrace{(\Box_y + m^2) B^{-1}(z, y)}_{-\delta(y-z)} = \frac{1}{i} \delta(y-z) \end{aligned} \right.$$

$$\begin{aligned} & -\frac{\lambda}{3!} \int d^4 x \int d^4 z J(z) \Delta_F(z-x) \left(\frac{1}{i} \right)^3 \frac{\delta^3 Z[J]}{\delta J(x)^3} + Z[J] \int d^4 x \int d^4 z J(z) \Delta_F(z-x) J(x) = \\ & = -\int d^4 x J(x) \frac{1}{i^2} \frac{\delta Z[J]}{\delta J(x)} - \frac{\lambda}{3!} \int d^4 x \int d^4 y J(x) \Delta_F(x-y) \left(\frac{1}{i} \right)^3 \frac{\delta^3 Z[J]}{\delta J(y)^3} + \\ & \quad \left\{ \text{FOR } \phi^3 \text{ INTERACTION } \frac{g}{2!} \text{ --- } \text{---} \text{---} \left(\frac{1}{i} \right)^2 \frac{\delta^2 Z[J]}{\delta J(y)^2} \right\} \end{aligned}$$

$$+ Z[J] \int d^4 x \int d^4 y J(x) \Delta_F(x-y) J(y) = 0 \quad \rightarrow + \frac{-i\lambda}{3!} \frac{1}{i} \left\{ \text{OR } \frac{-ig}{2!} \frac{1}{i} \right\}$$

$$\begin{aligned} & -\int d^4 x J(x) \frac{1}{i} \left[\text{diagram with 1 external line} + i \text{ (diagram with 2 external lines)} + \frac{i^2}{2!} \text{ (diagram with 3 external lines)} + \dots \right] - \frac{\lambda}{3!} \int d^4 x \int d^4 y J(x) \cdot \\ & \cdot \Delta_F(x-y) \left[\text{diagram with 1 external line} + i \text{ (diagram with 2 external lines)} + \frac{i^2}{2!} \text{ (diagram with 3 external lines)} + \dots \right] + \\ & \quad \left\{ \text{OR } \text{diagram with 1 external line} + i \text{ (diagram with 2 external lines)} + \dots \text{ FOR } \phi^3 \text{ INTERACTION} \right\} \\ & + (* \text{---} *) \left[\text{diagram with 1 external line} + i \text{ (diagram with 2 external lines)} + \frac{i^2}{2!} \text{ (diagram with 3 external lines)} + \dots \right] = 0 \end{aligned}$$

$$\int d^4x \mathcal{J}(x) \underbrace{\bigcirc - x}_{\text{1 leg}} = \frac{1}{i} \{Z[\mathcal{J}]\}_{\mathcal{J}^1} = \bigcirc - *$$

$$\tilde{G}^{(1)}(x) = \frac{1}{i} \frac{\delta Z[\mathcal{J}]}{\delta \mathcal{J}(x)} \Big|_{\mathcal{J}=0} \equiv \left\{ \frac{1}{i} \frac{\delta Z[\mathcal{J}]}{\delta \mathcal{J}(x)} \right\}_0$$

$$\int d^4x \mathcal{J}(x) \underbrace{\bigcirc - x}_{\text{2 legs}}^* = \int d^4x \mathcal{J}(x) \left\{ \frac{1}{i^2} \frac{\delta^2 Z[\mathcal{J}]}{\delta \mathcal{J}(x)^2} \right\}_{\mathcal{J}^1} = \frac{1}{i^2} 2 \{Z[\mathcal{J}]\}_{\mathcal{J}^2} = \underbrace{\bigcirc - x}_{\text{2 legs}}^*$$

$$\left\{ f(\xi) = \sum_k f_k \xi^k \rightarrow \frac{df}{d\xi} = \sum_k f_k k \xi^{k-1} = \sum_k f_{k+1} (k+1) \xi^k \right\}$$

$$\xi \left\{ \frac{df}{d\xi} \right\}_{\xi^m} = \xi f_{m+1} (m+1) \xi^m = (m+1) \{f\}_{\xi^{m+1}}$$

$$\int d^4x \mathcal{J}(x) \underbrace{\bigcirc - x}_{\text{3 legs}}^* = \int d^4x \mathcal{J}(x) \left\{ \frac{2!}{i^3} \frac{\delta^3 Z[\mathcal{J}]}{\delta \mathcal{J}(x)^3} \right\}_{\mathcal{J}^2} = \frac{2!}{i^3} 3 \{Z[\mathcal{J}]\}_{\mathcal{J}^3} = \underbrace{\bigcirc - x}_{\text{3 legs}}^*$$

$\frac{i^3}{3!} \underbrace{\bigcirc - x}_{\text{3 legs}}^*$

$$\int d^4x \int d^4y \mathcal{J}(x) \underbrace{\Delta_F(x-y)}_{(x-y)} \underbrace{\bigcirc - y}_{\text{2 legs}}^y = \int d^4x \mathcal{J}(x) \int d^4y \underbrace{\bigcirc - y}_{\text{2 legs}}^y y \rightarrow x =$$

$$= \int d^4x \mathcal{J}(x) \frac{\underbrace{\bigcirc - x}_{\text{2 legs}}}{-i\lambda} = \frac{1}{-i\lambda} \underbrace{\bigcirc - x}_{\text{2 legs}}$$

D.S. Eq. FOR ϕ^4 INTERACTION

$$-\frac{1}{i} \left(\underbrace{\bigcirc - *}_{\text{1 leg}} + i \underbrace{\bigcirc - *}_{\text{2 legs}}^* + \frac{i^2}{2!} \underbrace{\bigcirc - *}_{\text{3 legs}}^* + \dots \right) + \frac{-i\lambda}{3!} \frac{1}{i} \frac{1}{-i\lambda} \left(\underbrace{\bigcirc - *}_{\text{2 legs}} + \right.$$

$$\left. + i \underbrace{\bigcirc - *}_{\text{3 legs}}^* + \frac{i^2}{2!} \underbrace{\bigcirc - *}_{\text{4 legs}}^* + \dots \right) + \underbrace{* - *}_{\text{1 leg}} + i \underbrace{* - *}_{\text{2 legs}}^* + \frac{i^2}{2!} \underbrace{* - *}_{\text{3 legs}}^* + \dots = 0$$

BY COMPARING PARTS WITH THE SAME NUMBER OF LEGS $\rightarrow *$ (ORDER OF $\mathcal{J}(x)$) AND USING THE FACT THAT 1, 3, 5, ... -POINT FUNCTIONS ARE ZERO FOR ϕ^4 INTERACTION WE OBTAIN

$$* - \underbrace{\bigcirc - *}_{\text{1 leg}} = \frac{1}{3!} * - \underbrace{\bigcirc - *}_{\text{2 legs}} + * - \underbrace{\bigcirc - *}_{\text{3 legs}}$$

$$* - \underbrace{\bigcirc - *}_{\text{4 legs}} = \frac{1}{3!} * - \underbrace{\bigcirc - *}_{\text{5 legs}} + \frac{3!}{2!} * - \underbrace{\bigcirc - *}_{\text{6 legs}}$$

...

D.S. Eq. FOR ϕ^3 INTERACTION

$$-\frac{1}{i} \left(\text{circle with one external line} + i \text{circle with two external lines} + \frac{i^2}{2!} \text{circle with three external lines} + \dots \right) + \frac{-ig}{2!} \frac{1}{i} \frac{1}{-ig} \left(\text{circle with one external line and a tadpole} + \dots \right) + \dots = 0$$

NOW WE HAVE TO TAKE INTO ACCOUNT ALSO 1,3,5,...-POINT FUNCTIONS

$$\begin{aligned} \text{circle with one external line} &= \frac{1}{2!} \text{circle with one external line and a tadpole} \\ * - \text{circle with one external line} &= \frac{1}{2!} * - \text{circle with one external line and a tadpole} + * - \text{circle} \\ * - \text{circle with two external lines} &= \frac{1}{2!} * - \text{circle with two external lines and a tadpole} + 2! \text{circle with two external lines} \\ * - \text{circle with three external lines} &= \frac{1}{2!} * - \text{circle with three external lines and a tadpole} + \frac{3!}{2!} * - \text{circle with three external lines} \\ &\dots \end{aligned}$$

SUBSTITUTION TO NEW GENERATING FUNCTIONAL $Z[J] = e^{W[J]}$

$$W[J] = \sum_{n=0}^{\infty} \int d^4x_1 \dots \int d^4x_n \frac{i^n}{n!} W^{(n)}(x_1, \dots, x_n) J(x_1) \dots J(x_n) \equiv$$

$$\equiv \text{shaded circle} + i \text{shaded circle with one external line} + \frac{i^2}{2!} \text{shaded circle with two external lines} + \frac{i^3}{3!} \text{shaded circle with three external lines} + \dots$$

$$\left[\frac{\delta}{\delta J(x)} \equiv \delta_x \right] \quad \delta_x Z = (\delta_x W) e^W = (\delta_x W) Z$$

$$\delta_{x_1} \delta_{x_2} Z = [(\delta_{x_1} \delta_{x_2} W) + (\delta_{x_1} W)(\delta_{x_2} W)] Z$$

$$\begin{aligned} \delta_{x_1} \delta_{x_2} \delta_{x_3} Z = & [(\delta_{x_1} \delta_{x_2} \delta_{x_3} W) + (\delta_{x_1} \delta_{x_2} W)(\delta_{x_3} W) + (\delta_{x_1} \delta_{x_3} W)(\delta_{x_2} W) + \\ & + (\delta_{x_2} \delta_{x_3} W)(\delta_{x_1} W) + (\delta_{x_1} W)(\delta_{x_2} W)(\delta_{x_3} W)] Z \end{aligned}$$

D.S. Eq. $\left\{ -\frac{g}{2!} \right\} \leftarrow \text{FOR } \phi^3 \text{ INT.} \rightarrow \left\{ \left(\frac{1}{i} \delta_y \right)^2 \right\}$

$$i \int d^4x J(x) \frac{1}{i} \delta_x Z - \frac{\lambda}{3!} \int d^4x \int d^4y J(x) \Delta_F(x-y) \left(\frac{1}{i} \delta_y \right)^3 Z + (* \text{---} *) Z = 0$$

$$i \int d^4x J(x) \frac{1}{i} \delta_x W - \frac{\lambda}{3!} \int d^4x \int d^4z J(z) \Delta_F(z-x) \left(\frac{1}{i} \right)^3 \left[(\delta_x)^3 W + 3((\delta_x)^2 W)(\delta_x W) + (\delta_x W)^3 \right] + * \text{---} * = 0$$

$$\left\{ -\frac{g}{2!} \right\} \leftarrow (\phi^3) \rightarrow \left\{ \left(\frac{1}{i} \right)^2 [(\delta_x)^2 W + (\delta_x W)^2] \right\}$$

$$i \left(\text{diagram 1} + i \text{diagram 2} + \frac{i^2}{2!} \text{diagram 3} + \dots \right) + \frac{-i\lambda}{3!} \frac{1}{i} \left\{ \frac{1}{-i\lambda} \left(\text{diagram 4} + \dots \right) + \int d^4x \int d^4z J(z) \Delta_F(z-x) \left[3 \left(\text{diagram 5} + i \text{diagram 6} + \frac{i^2}{2!} \text{diagram 7} + \dots \right) \left(\text{diagram 8} + i \text{diagram 9} + \frac{i^2}{2!} \text{diagram 10} + \dots \right) + \left(\text{diagram 11} + i \text{diagram 12} + \frac{i^2}{2!} \text{diagram 13} + \dots \right)^3 \right] \right\} + * \text{---} * = 0$$

TERM WITH $\int d^4x \int d^4z \dots$ & USE THAT $(\phi^4)_{1,3,5,\dots}$ - POINT FUNCTIONS ARE ZERO

$$\int d^4x (* \text{---} x) \left[\dots \right] = \frac{1}{-i\lambda} \left[\dots \right] + \int d^4x (* \text{---} x) \left[\dots \right] =$$

$$= \frac{3}{-i\lambda} \left(i * \text{diagram 14} + i \frac{i^2}{2!} * \text{diagram 15} + \dots + \frac{i^3}{3!} * \text{diagram 16} + \frac{i^3}{3!} \frac{i^2}{2!} * \text{diagram 17} + \dots \right) + \int d^4x (* \text{---} x).$$

$$\cdot \left[\left(i \text{diagram 18} \right)^3 + 3 \left(i \text{diagram 19} \right)^2 \left(\frac{i^3}{3!} \text{diagram 20} \right) + \dots + \left(\frac{i^3}{3!} \text{diagram 21} \right)^3 + \dots \right]$$

$$\begin{aligned} & i^3 \frac{1}{-i\lambda} * \text{diagram 22} \\ & 3 i^2 \frac{i^3}{3!} \frac{1}{-i\lambda} * \text{diagram 23} \\ & \left(\frac{i^3}{3!} \right)^3 \frac{1}{-i\lambda} * \text{diagram 24} \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{i} \left(i \text{ (diagram: circle with two external lines) } + \frac{i^3}{3!} \text{ (diagram: circle with four external lines) } + \dots \right) + \\
 & + \frac{1}{i} \frac{1}{3!} \left\{ \left(i \text{ (diagram: two circles connected by a line, with four external lines) } + \frac{i^3}{3!} \text{ (diagram: two circles connected by a line, with six external lines) } + \dots \right) + \right. \\
 & + 3 \left[i \text{ (diagram: circle with a loop, with three external lines) } + \frac{i^3}{2!} \text{ (diagram: circle with a loop, with four external lines) } + \frac{i^3}{3!} \text{ (diagram: circle with a loop, with five external lines) } + \dots \right] + \\
 & \left. + i^3 \text{ (diagram: circle with a loop, with three external lines) } + \dots \right\} + * - * = 0
 \end{aligned}$$

$$* - * = \frac{1}{3!} \left\{ * - * + 3 * - * \right\} + * - *$$

$$\frac{1}{3!} \text{ (diagram: circle with four external lines) } = \frac{1}{3!} \left\{ \frac{1}{3!} \text{ (diagram: circle with four external lines) } + 3 \left[\frac{1}{2!} \text{ (diagram: circle with four external lines) } + \frac{1}{3!} \text{ (diagram: circle with four external lines) } \right] + \dots \right\}$$

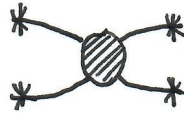
THESE ARE RECURRENCE RELATIONS FOR FINDING $W^{(m)}$ IN PERTURBATIVE WAY $\{W^{(m)}\}_{(-i\lambda)^k} = \sum_{m_1, m_2, \dots} a_{m_1, m_2, \dots} \{W^{(m_1)} W^{(m_2)} \dots\}_{(-i\lambda)^{k-1}}$

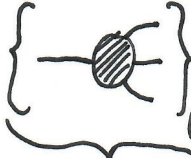
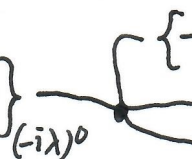
$$\begin{aligned}
 * - * &= * - * + \frac{1}{6} \text{ (diagram: circle with a loop, with two external lines) } + \frac{1}{2} \text{ (diagram: circle with a loop, with three external lines) } \\
 \text{ (diagram: circle with four external lines) } &= \text{ (diagram: two circles connected by a line, with four external lines) } + \frac{1}{6} \text{ (diagram: circle with a loop, with four external lines) } + \frac{3}{2} \text{ (diagram: circle with a loop, with five external lines) } + \frac{1}{2} \text{ (diagram: circle with a loop, with six external lines) } \\
 &\dots
 \end{aligned}$$

SOLVING THESE RELATIONS PERTUBATIVELY WE CAN FIND m -POINT FUNCTIONS $W^{(m)}(x_1, \dots, x_m)$ AS SUMS OF ALL FEYNMAN DIAGRAMS — THIS PRODUCES ONLY CONNECTED DIAGRAMS!

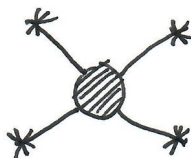
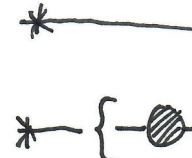
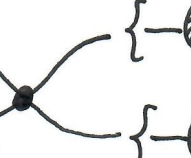
PERTURBATIVE SOLUTION :

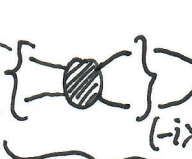
ORDER $(-i\lambda)^0$:  = 

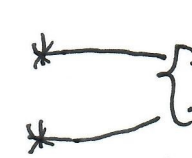
 = 0

ORDER $(-i\lambda)^1$:  =  + $\frac{1}{6}$  $(-i\lambda)^0$ + $\frac{1}{2}$  $(-i\lambda)^0$ =

=  + $\frac{1}{2}$ 

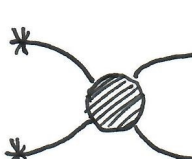
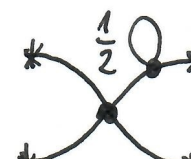
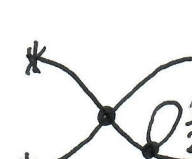
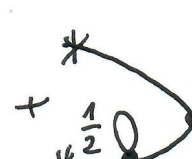
 =  $(-i\lambda)^0$ +  $(-i\lambda)^0$ +

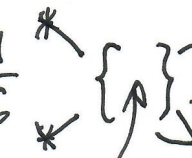
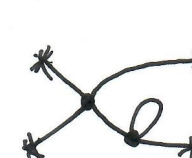
+ $\frac{1}{6}$  $(-i\lambda)^0$ + $\frac{3}{2}$  $(-i\lambda)^0$ +

+ $\frac{1}{2}$  $(-i\lambda)^0$ = 

AND SO ON...

ORDER $(-i\lambda)^2$: ONLY FOR 4-POINT FUNCTION

 =  + $\frac{1}{2}$  +  + $\frac{1}{2}$  +

+ $\frac{1}{6}$  + $\frac{3}{2}$  + $\frac{1}{2}$ 

= 0 MORE THAN ONE VERTEX

$3 \cdot \frac{1}{2} + \frac{1}{2} = 2$