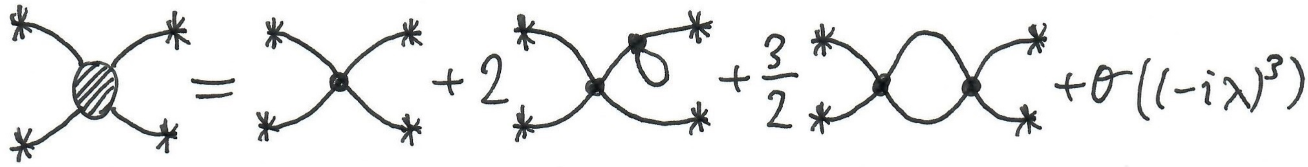
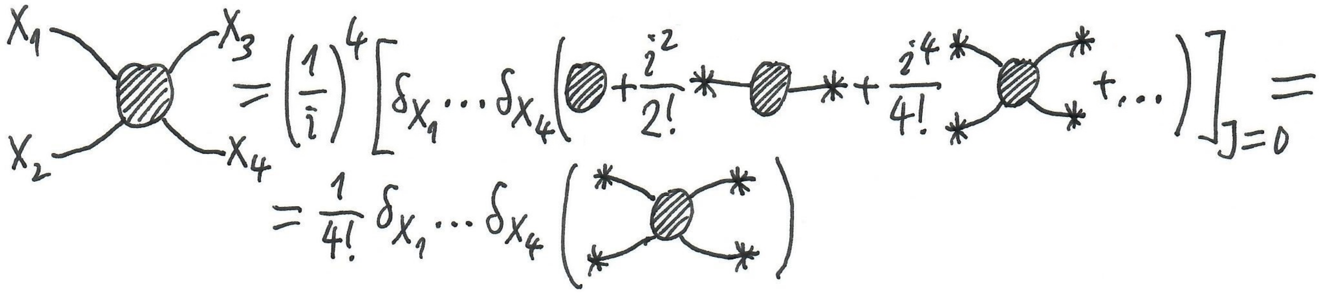
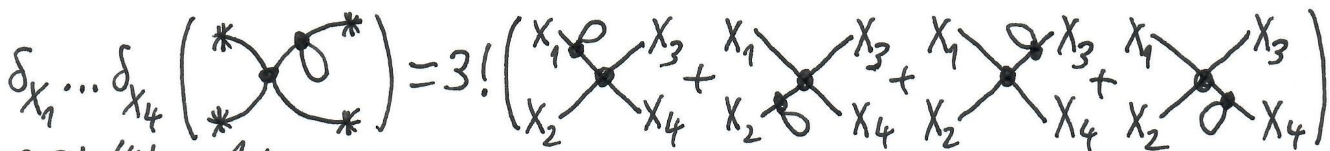
Notes on Path Integral Methods

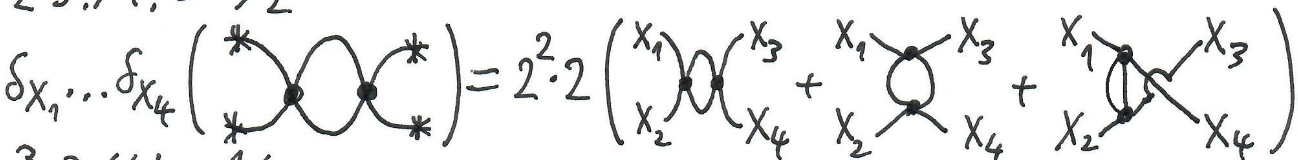


$$W^{(4)}(x_1,\dots,x_4) = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_4)}W[J]\right]_{J=0}$$

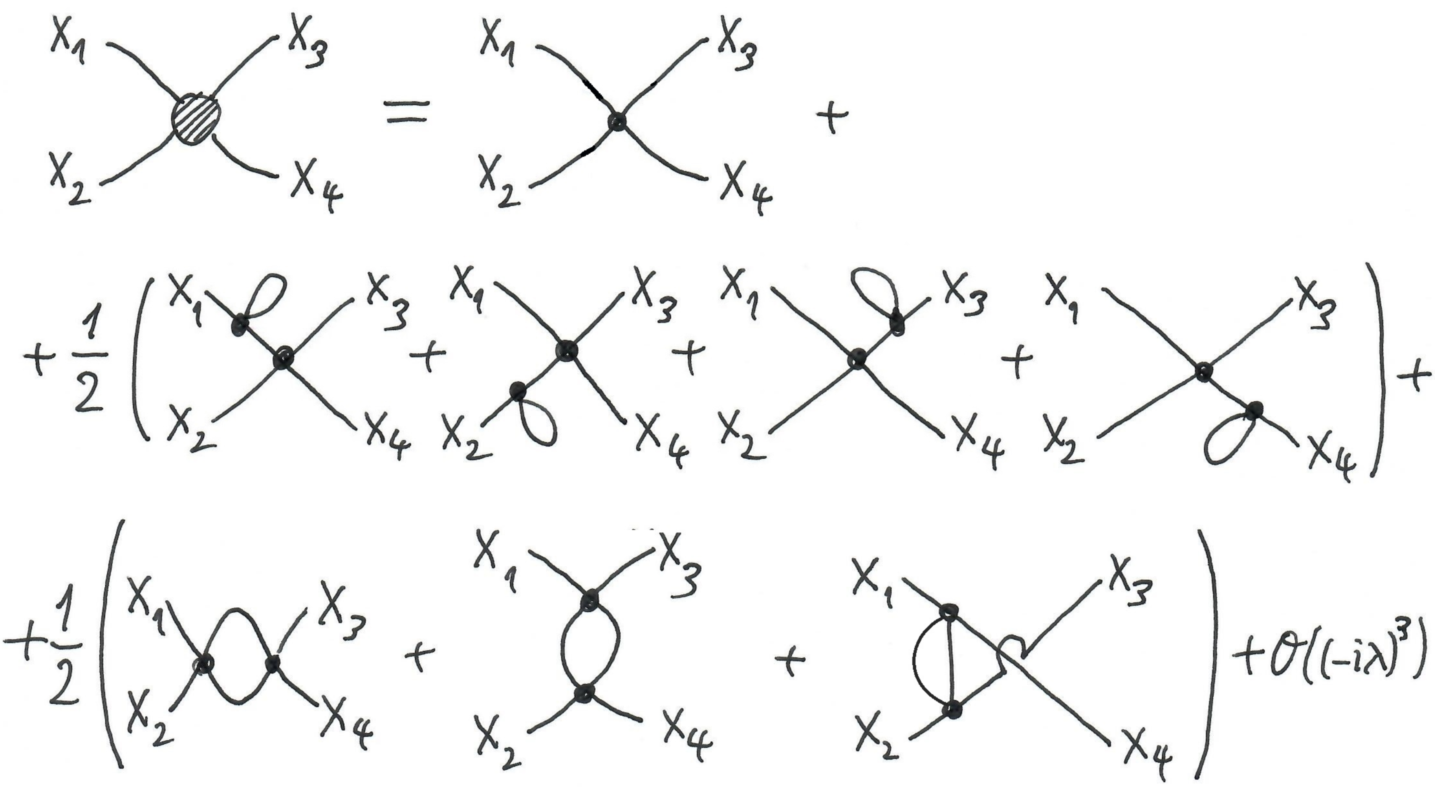




$2\cdot 3!/4! = 1/2$



$\frac{3}{2}\, 8/4! = 1/2$



D.S. Eq. FOR ϕ^3 INTERACTION

CENSORED

(homework spoilers)

COMPARE PARTS WITH THE SAME NUMBER OF EXTERNAL LEGS (ORDERS OF J IN EXPANSION OF W[J])

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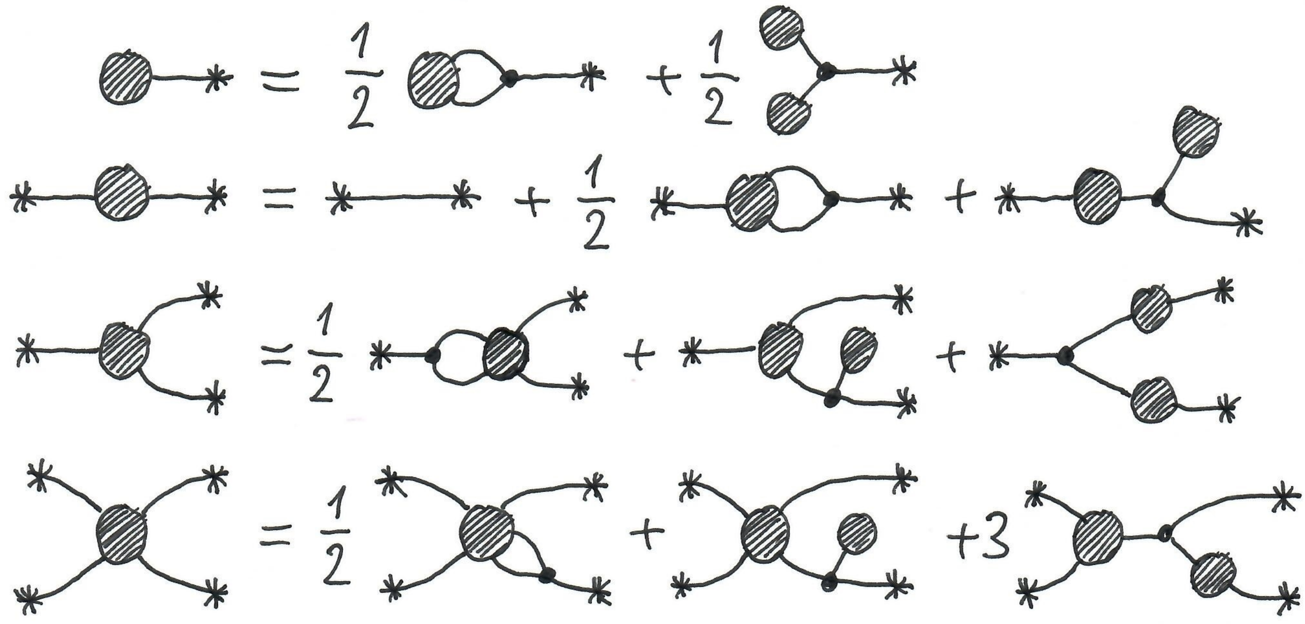


PERTURBATIVE SOLUTION

ORDER $(-i\lambda)^0$

CENSORED

ORDER $(-i\lambda)^1$

CENSORED

(homework

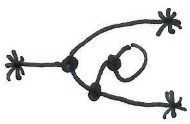
ORDER $(-i\lambda)^2$

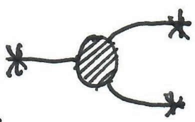
spoilers)

3-POINT FUNCTION UP TO ORDER $(-i\lambda)^3$

 =

CENSORED (homework spoilers)

=  +  $+ \frac{3}{2}$  $+ \frac{3}{2}$ 

$W^{(3)}(x_1, x_2, x_3) = \frac{1}{3!}\, \delta_{x_1} \delta_{x_2} \delta_{x_3}\Big($  $\Big)$

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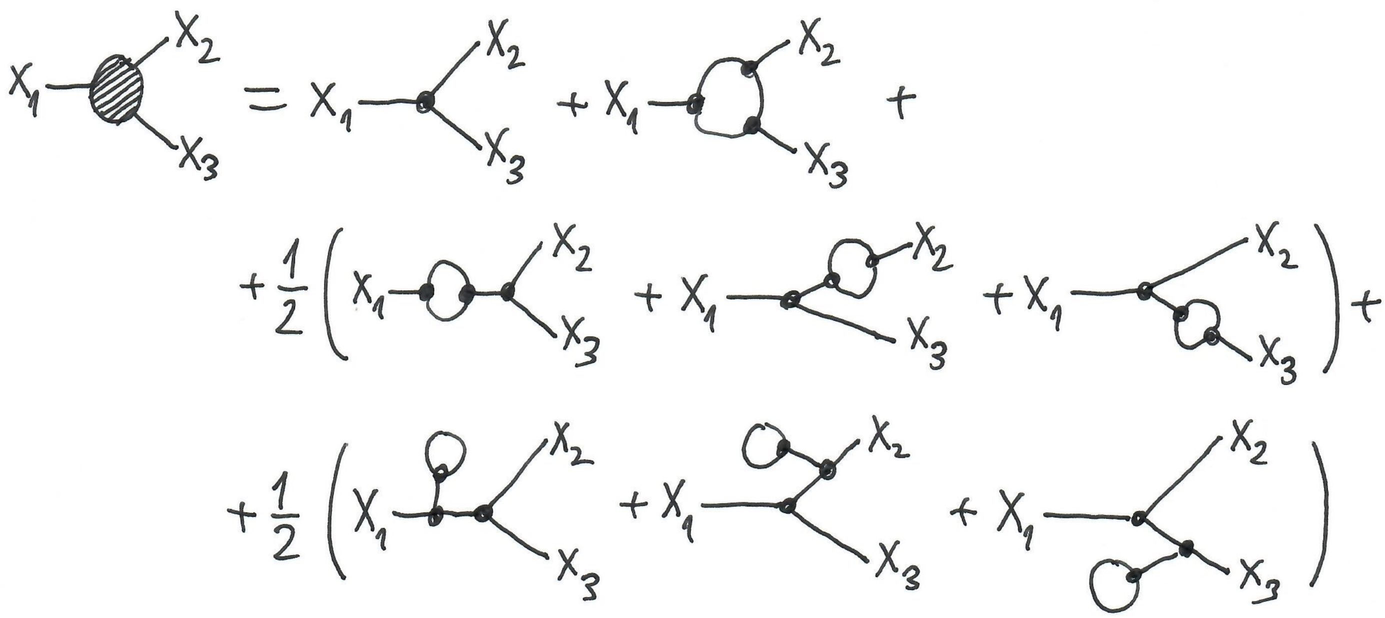
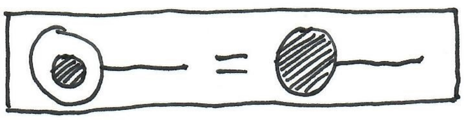


FUNCTIONAL $W[J]$ GENERATES ONLY CONNECTED DIAGRAMS

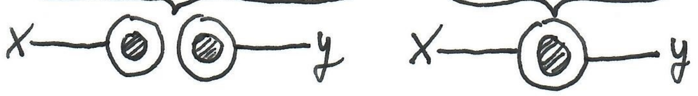
$$\underbrace{W^{(1)}(x)} = \left(\frac{1}{i}\delta_x W\right)_{J=0} = \frac{1}{i}\left(\delta_x \ln Z\right)_{J=0} = \left(\frac{1}{Z}\,\frac{1}{i}\delta_x Z\right)_{J=0} = \underbrace{G^{(1)}(x)}$$

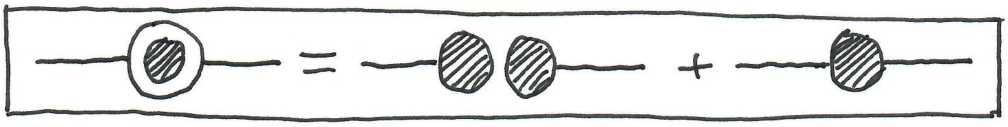


$$\underbrace{W^{(2)}(x,y)} = \left(\frac{1}{i}\delta_x \frac{1}{i}\delta_y W\right)_{J=0} = \left[\frac{1}{i}\delta_x\left(\frac{1}{Z}\,\frac{1}{i}\delta_y Z\right)\right]_{J=0} =$$

$$= \left[-\frac{1}{Z^2}\left(\frac{1}{i}\delta_x Z\right)\left(\frac{1}{i}\delta_y Z\right) + \frac{1}{Z}\,\frac{1}{i}\delta_x\,\frac{1}{i}\delta_y Z\right]_{J=0} =$$

$$= -\underbrace{G^{(1)}(x)\,G^{(1)}(y)} + \underbrace{G^{(2)}(x,y)}$$





$$W^{(3)}(x,y,z) = \left\{\frac{1}{i}\delta_x\left[-\frac{1}{Z^2}\left(\frac{1}{i}\delta_y Z\right)\left(\frac{1}{i}\delta_z Z\right)+\frac{1}{Z}\frac{1}{i}\delta_y\frac{1}{i}\delta_z Z\right]\right\}_{J=0} =$$

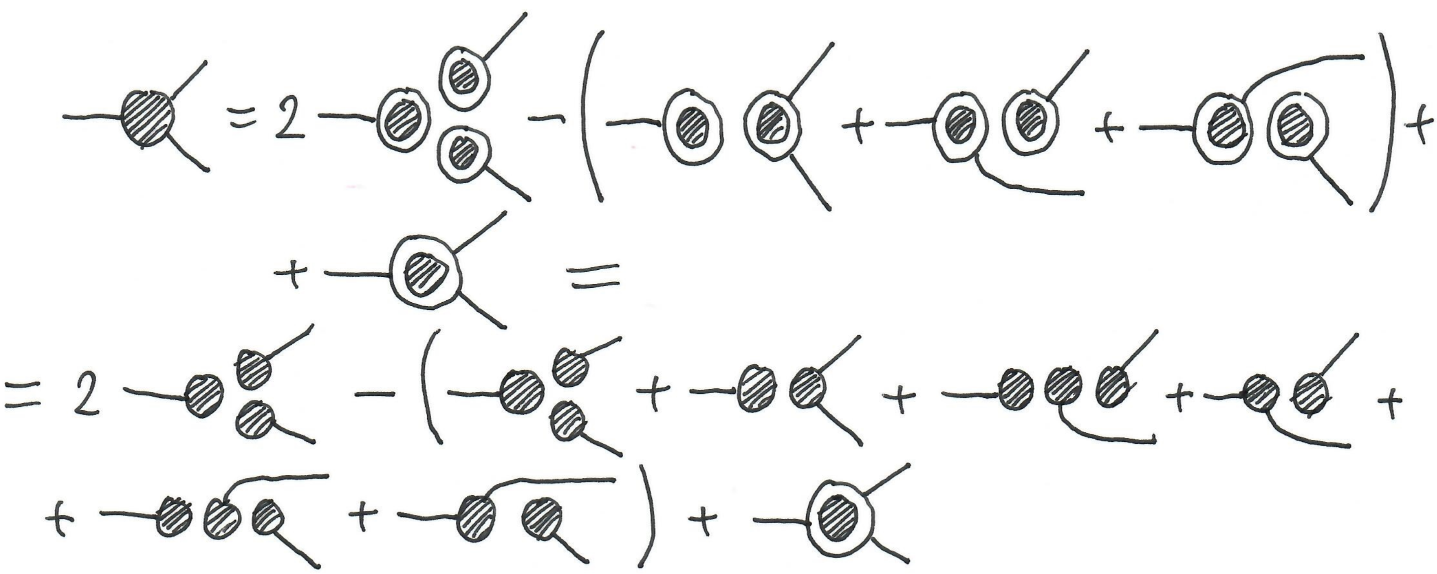
$$= \left\{\frac{2}{Z^3}\left(\frac{1}{i}\delta_x Z\right)\left(\frac{1}{i}\delta_y Z\right)\left(\frac{1}{i}\delta_z Z\right) - \frac{1}{Z^2}\left(\frac{1}{i}\delta_x\frac{1}{i}\delta_y Z\right)\left(\frac{1}{i}\delta_z Z\right) -\right.$$

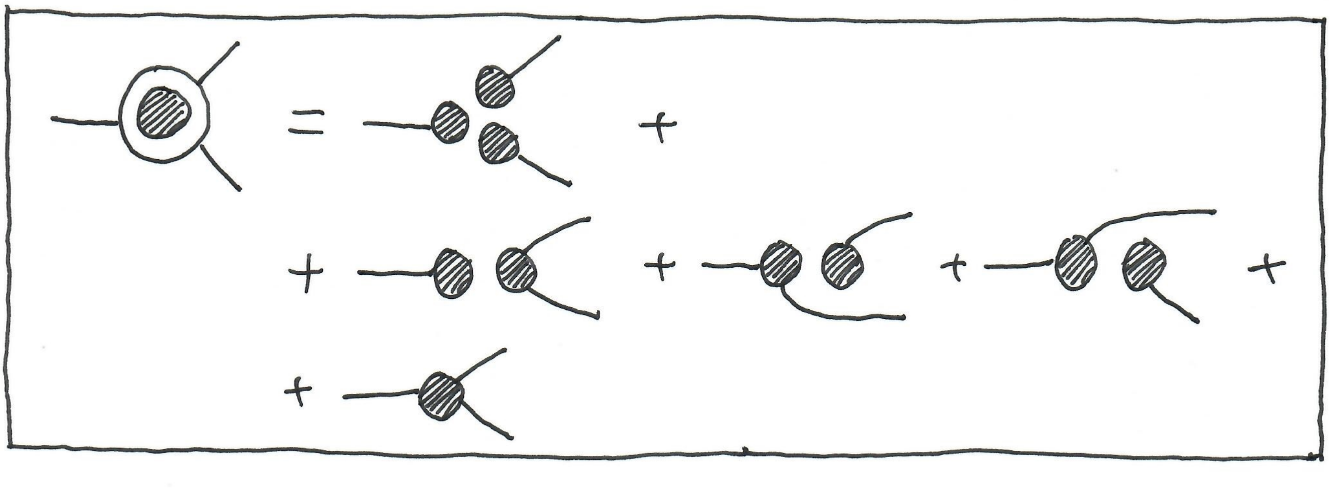
$$- \frac{1}{Z^2}\left(\frac{1}{i}\delta_y Z\right)\left(\frac{1}{i}\delta_x\frac{1}{i}\delta_z Z\right) - \frac{1}{Z^2}\left(\frac{1}{i}\delta_x Z\right)\left(\frac{1}{i}\delta_y\frac{1}{i}\delta_z Z\right) +$$

$$\left. + \frac{1}{Z}\frac{1}{i}\delta_x\frac{1}{i}\delta_y\frac{1}{i}\delta_z Z\right\}_{J=0} = 2G^{(1)}(x)\,G^{(1)}(y)\,G^{(1)}(z) +$$

$$+\left(-G^{(1)}(x)G^{(2)}(y,z) - G^{(1)}(y)G^{(2)}(x,z) - G^{(1)}(z)G^{(1)}(x,y)\right) +$$

$$+ G^{(3)}(x,y,z)$$



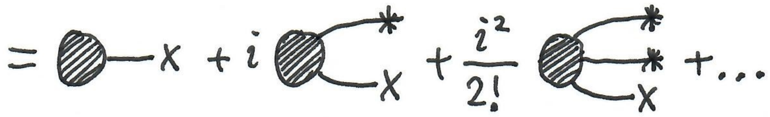


LEGENDRE TRANSFORM OF GENERATING FUNCTIONAL

$$W[J] \longrightarrow \Gamma[\Psi] = -i\int d^4x\, J(x)\Psi(x) + W[J]$$

INVERTIBILITY
$\Psi\{J\} \leftrightarrow J\{\Psi\}$
$\Gamma = -i\int J\{\Psi\}\Psi +$
$+ W[J\{\Psi\}]$

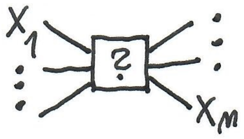
$$\Psi(x) = \frac{1}{i}\frac{\delta W[J]}{\delta J(x)} = \langle 0|\hat{\Phi}(x)|0\rangle_{J\neq 0}$$



$$\frac{\delta\Gamma[\Psi]}{\delta\Psi(x)} = -i\int d^4y \Bigg(\frac{\delta J(y)}{\delta\Psi(x)} \underbrace{\Psi(y)}_{\frac{1}{i}\frac{\delta W[J]}{\delta J(y)}} + J(y)\underbrace{\frac{\delta\Psi(y)}{\delta\Psi(x)}}_{\delta(y-x)} \Bigg) + \frac{\delta W[J]}{\delta\Psi(x)} =$$

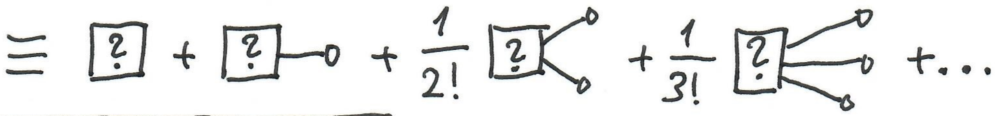
$$= -\int d^4y\, \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\Psi(x)} - iJ(x) + \frac{\delta W[J]}{\delta\Psi(x)} = -iJ(x) \longrightarrow J(x) = i\,\frac{\delta\Gamma[\Psi]}{\delta\Psi(x)}$$

NEW FUNCTIONAL GENERATES SOME KIND OF m-POINT FUNCTIONS



$$\equiv \Gamma^{(m)}(x_1,\dots,x_m) = \left[\frac{\delta^m\Gamma[\Psi]}{\delta\Psi(x_1)\dots\delta\Psi(x_m)}\right]_{\Psi=0}$$

$$\Gamma[\Psi] = \sum_{m=0}^{\infty}\int d^4x_1\dots\int d^4x_m\, \frac{1}{m!}\Gamma^{(m)}(x_1,\dots,x_m)\Psi(x_1)\dots\Psi(x_m) \equiv$$

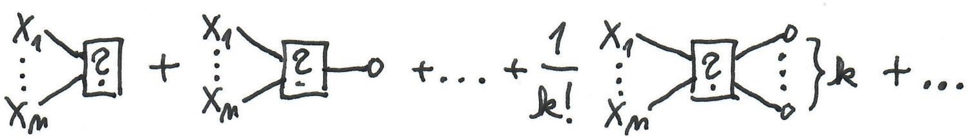


$$\delta(x-y) = \frac{\delta J(x)}{\delta J(y)} = \int d^4z\, \frac{\delta J(x)}{\delta\Psi(z)}\frac{\delta\Psi(z)}{\delta J(y)} = \int d^4z\, i\,\frac{\delta^2\Gamma[\Psi]}{\delta\Psi(x)\delta\Psi(z)}\,\frac{1}{i}\,\frac{\delta^2 W[J]}{\delta J(z)\delta J(y)} =$$

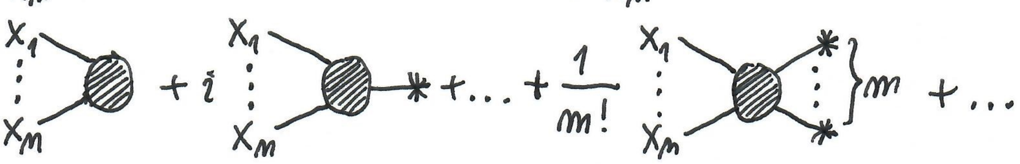
$$= \int d^4z\, i\left(\text{SOMETHING LIKE } \Gamma^{(2)}(x,z)\right) i\left(\text{SOMETHING LIKE } W^{(2)}(z,y)\right) \equiv$$

$$\equiv -\int d^4z\, \overset{\circ}{\Gamma}{}^{(2)}(x,z)\,\overset{*}{W}{}^{(2)}(z,y) \qquad \text{WHERE:}$$

$$\overset{\circ}{\Gamma}{}^{(m)}(x_1,\dots,x_m) \equiv$$



$$\overset{*}{W}{}^{(m)}(x_1,\dots,x_m) \equiv$$



$$\int d^4z\, \overset{\circ}{\Gamma}{}^{(2)}(x,z)\, \overset{*}{W}{}^{(2)}(z,y) = -\delta(x-y) \quad \Big/ \ \frac{1}{i}\frac{\delta}{\delta J(u)}$$

$$\int d^4z \left(\int d^4v\, \underbrace{\frac{\delta \overset{\circ}{\Gamma}{}^{(2)}(x,z)}{\delta\psi(v)}}_{\overset{\circ}{\Gamma}{}^{(3)}(x,z,v)}\, \underbrace{\frac{1}{i}\frac{\delta\psi(v)}{\delta J(u)}}_{\frac{1}{i^2}\frac{\delta^2 W[J]}{\delta J(u)\delta J(v)} = \overset{*}{W}{}^{(2)}(v,u)}\, \overset{*}{W}{}^{(2)}(z,y) + \overset{\circ}{\Gamma}{}^{(2)}(x,z)\, \underbrace{\frac{1}{i}\frac{\delta \overset{*}{W}{}^{(2)}(z,y)}{\delta J(u)}}_{\overset{*}{W}{}^{(3)}(z,y,u)} \right) = 0$$

$$\int d^4z \int d^4v\, \overset{\circ}{\Gamma}{}^{(3)}(x,z,v)\, \overset{*}{W}{}^{(2)}(v,u)\, \overset{*}{W}{}^{(2)}(z,y) + \int d^4z\, \overset{\circ}{\Gamma}{}^{(2)}(x,z)\, \overset{*}{W}{}^{(3)}(z,y,u) = 0$$

$$\downarrow \qquad \Big/ \ \int d^4x \ldots \overset{*}{W}{}^{(2)}(x,w)$$

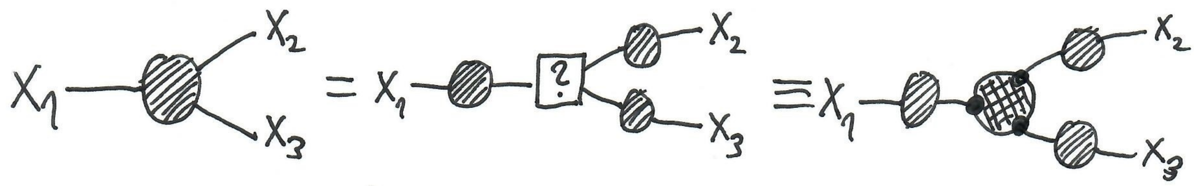
$$\int d^4x \int d^4z \int d^4v\, \overset{\circ}{\Gamma}{}^{(3)}(x,z,v)\, \overset{*}{W}{}^{(2)}(x,w)\, \overset{*}{W}{}^{(2)}(z,y)\, \overset{*}{W}{}^{(2)}(v,u) +$$

$$+ \underbrace{\int d^4z \underbrace{\int d^4x\, \overset{\circ}{\Gamma}{}^{(2)}(x,z)\, \overset{*}{W}{}^{(2)}(x,w)}_{-\delta(z-w)}\, \overset{*}{W}{}^{(3)}(z,y,u)}_{-\overset{*}{W}{}^{(3)}(w,y,u)} = 0$$

$$\overset{*}{W}{}^{(3)}(x_1,x_2,x_3) = \int d^4y_1 \int d^4y_2 \int d^4y_3\, \overset{\circ}{\Gamma}{}^{(3)}(y_1,y_2,y_3)\, \overset{*}{W}{}^{(2)}(y_1,x_1)\, \overset{*}{W}{}^{(2)}(y_2,x_2)\, \overset{*}{W}{}^{(2)}(y_3,x_3)$$

$$\downarrow \qquad \Big/ \ (\ldots)\big|_{J=0}\big|_{\psi=0}$$

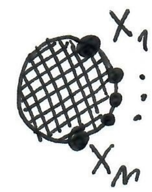
$$W^{(3)}(x_1,x_2,x_3) = \int d^4y_1 \int d^4y_2 \int d^4y_3\, \Gamma^{(3)}(y_1,y_2,y_3)\, W^{(2)}(y_1,x_1)\, W^{(2)}(y_2,x_2)\, W^{(3)}(y_3,x_3)$$



WE SEE THAT $\Gamma[\psi]$ GENERATES ONE PARTICLE IRREDUCIBLE DIAGRAMS (1PI) WITHOUT EXTERNAL LEGS

$\Gamma[\psi] \equiv$

$$\Gamma^{(m)}(x_1,\ldots,x_m) = \left[\frac{\delta^m \Gamma[\psi]}{\delta\psi(x_1)\ldots\delta\psi(x_m)} \right]_{\psi=0} \equiv$$



$$\overset{*}{W}^{(3)}(x_1,x_2,x_3) = \int d^4y_1 \int d^4y_2 \int d^4y_3 \, \overset{\circ}{\Gamma}^{(3)}(y_1,y_2,y_3)\, \overset{*}{W}^{(2)}(y_1,x_1)\, \overset{*}{W}^{(2)}(y_2,x_2)\, \overset{*}{W}^{(2)}(y_3,x_3)$$

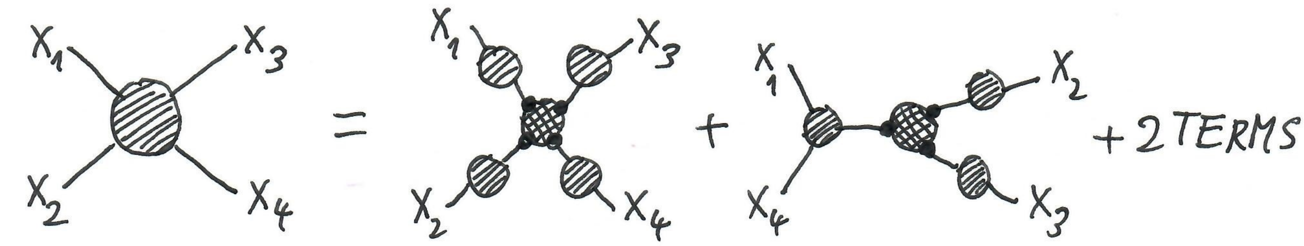
$$\Big/ \ \frac{1}{i}\frac{\delta}{\delta J(x_4)}$$

$$\overset{*}{W}^{(4)}(x_1,\dots,x_4) = \int d^4y_1 \dots \int d^4y_4 \underbrace{\frac{\delta \overset{\circ}{\Gamma}^{(3)}(y_1,y_2,y_3)}{\delta \Psi(y_4)}}_{\overset{\circ}{\Gamma}^{(4)}(y_1,\dots,y_4)} \underbrace{\frac{1}{i}\frac{\delta \Psi(y_4)}{\delta J(x_4)}}_{\frac{1}{i^2}\frac{\delta^2 W[J]}{\delta J(y_4)\delta J(x_4)} = \overset{*}{W}^{(2)}(y_4,x_4)} \bullet$$

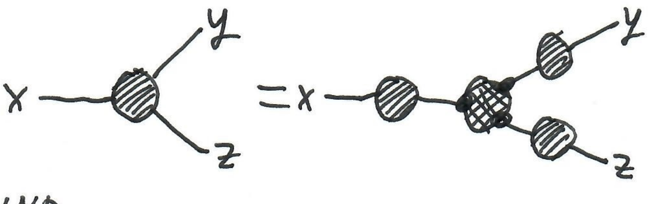
$$\bullet\, \overset{*}{W}^{(2)}(y_1,x_1)\, \overset{*}{W}^{(2)}(y_2,x_2)\, \overset{*}{W}^{(2)}(y_3,x_3) \ +$$

$$+ \int d^4y_1 \int d^4y_2 \int d^4y_3 \, \overset{\circ}{\Gamma}^{(3)}(y_1,y_2,y_3) \left[\overset{*}{W}^{(3)}(y_1,x_1,x_4)\, \overset{*}{W}^{(2)}(y_2,x_2)\, \overset{*}{W}^{(2)}(y_3,x_3) + 2\,\mathrm{TERMS} \right]$$

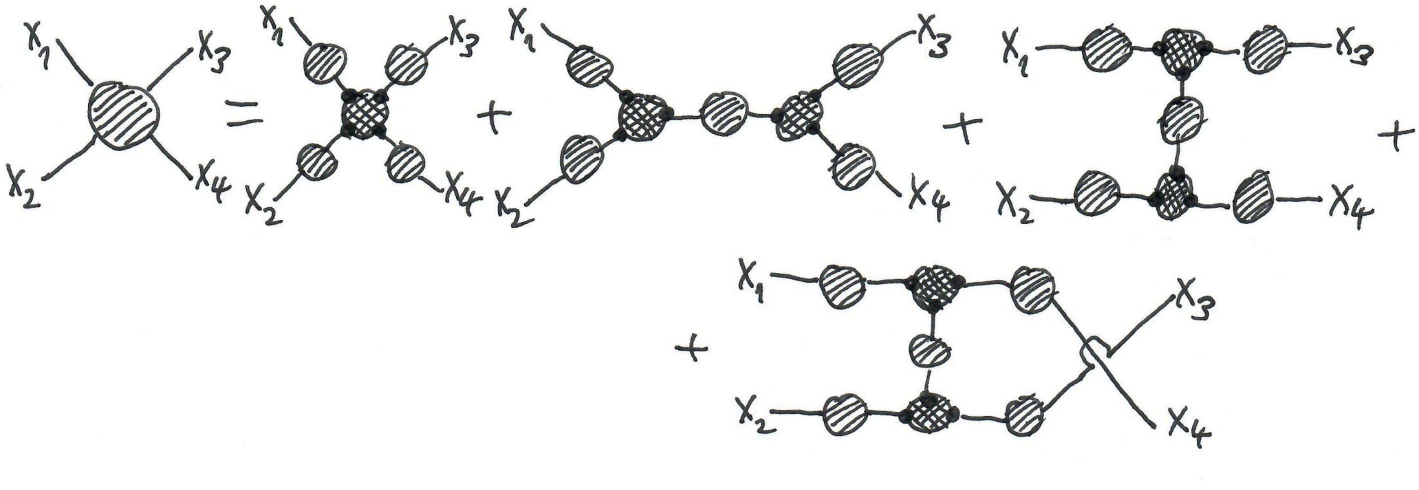
$$\downarrow \qquad \Big/ \ (\dots)\Big|_{J=0}\Big|_{\Psi=0}$$



AFTER USING RESULT FROM THE PREVIOUS PAGE



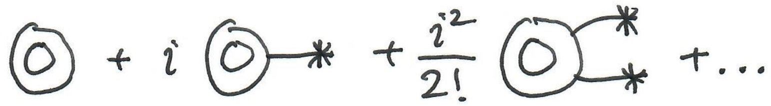
WE FIND



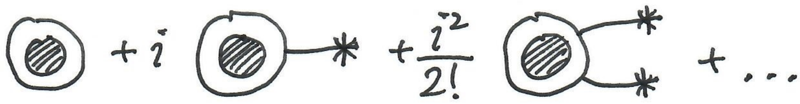
PHASE GIVEN BY ACTION $e^{iS[\phi]}$

FOURIER TRANSFORM $\int \mathcal{D}\phi \, e^{iS[\phi]} e^{i\int d^4x \, J\phi}$

GENERATING FUNCTIONAL

$Z[J] =$ 

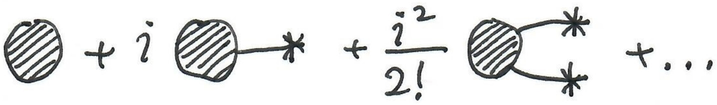
GENERATES ALL DIAGRAMS $\tilde{G}^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)} \dots Z[J]\right]_{J=0}$

$\frac{Z[J]}{Z[0]} =$ 

GENERATES DIAGRAMS WITHOUT VACUUM BUBBLES – m-POINT GREEN FUNCTIONS

$\langle 0 | \mathcal{T}\{\hat{\Phi}(x_1) \dots \hat{\Phi}(x_m)\} | 0 \rangle = G^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)} \dots \frac{Z[J]}{Z[0]}\right]_{J=0}$

EXP/LOG SUBSTITUTION $Z = e^{W}$

$W[J] =$ 

GENERATES CONNECTED DIAGRAMS $W^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)} \dots W[J]\right]_{J=0}$

LEGENDRE TRANSFORM $\Gamma[\Psi] = -i\int d^4x \, J\Psi + W[J]$

$\Psi = \frac{1}{i}\frac{\delta W}{\delta J}$

$\Gamma[\Psi] =$ 

GENERATES ONE PARTICLE IRREDUCIBLE DIAGRAMS WITHOUT EXTERNAL LEGS

$$\Gamma^{(m)}(x_1, \dots, x_m) = \left[\frac{\delta}{\delta \Psi(x_1)} \dots \Gamma[\Psi]\right]_{\Psi=0}$$