

Notes on Path Integral Methods

55	$W[J] \longleftrightarrow$ connected diagrams
57	Legendre transform $W[J] \longrightarrow \Gamma[\Psi]$
58	1PI (one particle irreducible) diagrams
60	summary of generating functionals

$$\begin{array}{c} * \\ * \end{array} \begin{array}{c} * \\ * \end{array} = \begin{array}{c} * \\ * \end{array} \begin{array}{c} * \\ * \end{array} + 2 \begin{array}{c} * \\ * \end{array} \begin{array}{c} * \\ * \end{array} + \frac{3}{2} \begin{array}{c} * \\ * \end{array} \begin{array}{c} * \\ * \end{array} + \mathcal{O}((-i\lambda)^3)$$

$$W^{(4)}(X_1, \dots, X_4) = \left[\frac{1}{i} \frac{\delta}{\delta J(X_1)} \dots \frac{1}{i} \frac{\delta}{\delta J(X_4)} W[J] \right]_{J=0}$$

$$\begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} = \left(\frac{1}{i} \right)^4 \left[\delta_{X_1} \dots \delta_{X_4} \left(\begin{array}{c} * \\ * \end{array} + \frac{i^2}{2!} \begin{array}{c} * \\ * \end{array} + \frac{i^4}{4!} \begin{array}{c} * \\ * \end{array} + \dots \right) \right]_{J=0} = \frac{1}{4!} \delta_{X_1} \dots \delta_{X_4} \left(\begin{array}{c} * \\ * \end{array} \right)$$

$$\delta_{X_1} \dots \delta_{X_4} \left(\begin{array}{c} * \\ * \end{array} \right) = 3! \left(\begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} \right)$$

$$2 \cdot 3! / 4! = 1/2$$

$$\delta_{X_1} \dots \delta_{X_4} \left(\begin{array}{c} * \\ * \end{array} \right) = 2^2 \cdot 2 \left(\begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} \right)$$

$$\frac{3}{2} \cdot 8 / 4! = 1/2$$

$$\begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} = \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \frac{1}{2} \left(\begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} \right) + \frac{1}{2} \left(\begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} + \begin{array}{c} X_1 \\ X_2 \end{array} \begin{array}{c} X_3 \\ X_4 \end{array} \right) + \mathcal{O}((-i\lambda)^3)$$

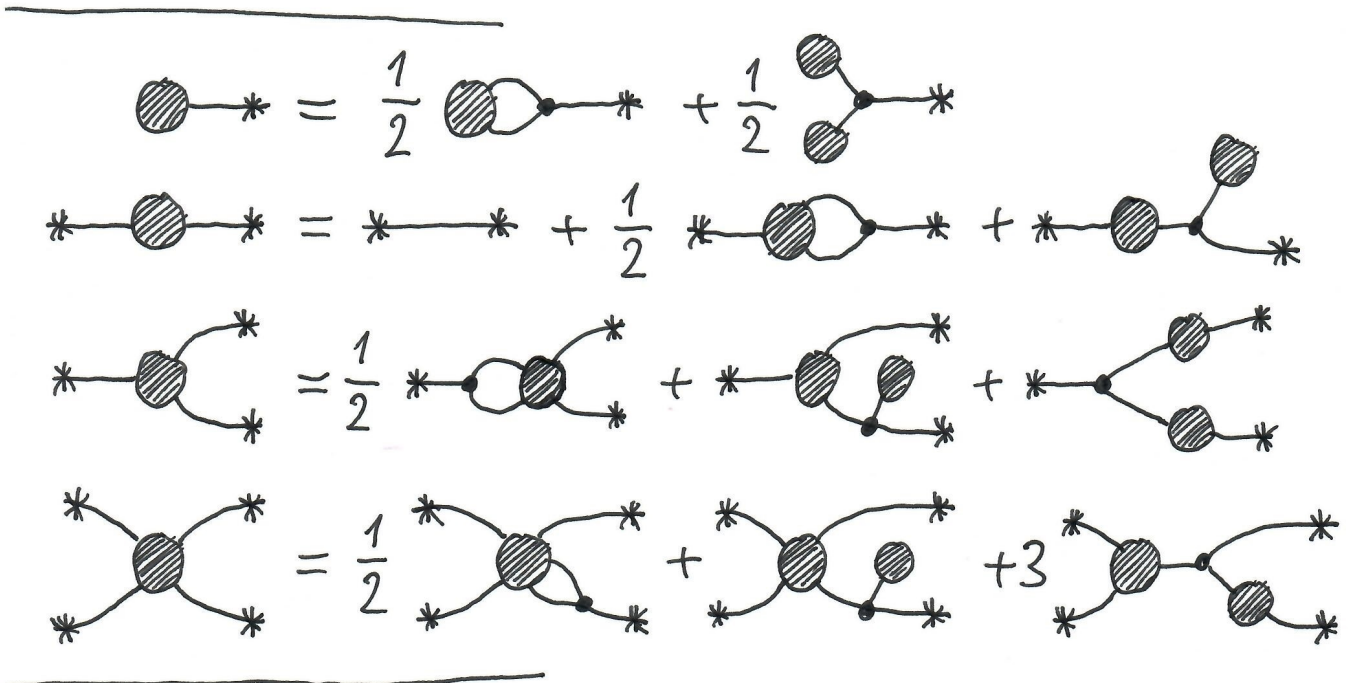
D.S.Eq. FOR ϕ^3 INTERACTION

CENSORED

(homework spoilers)

COMPARE PARTS WITH THE SAME NUMBER OF EXTERNAL
LEGS (ORDERS OF J IN EXPANSION OF $W[J]$)

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PERTURBATIVE SOLUTION

ORDER $(-i\lambda)^0$

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ORDER $(-i\lambda)^1$

CENSORED

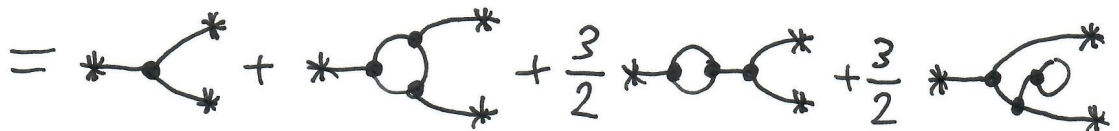
ORDER $(-i\lambda)^2$

(homework
spoilers)

3-POINT FUNCTION UP TO ORDER $(-i\lambda)^3$



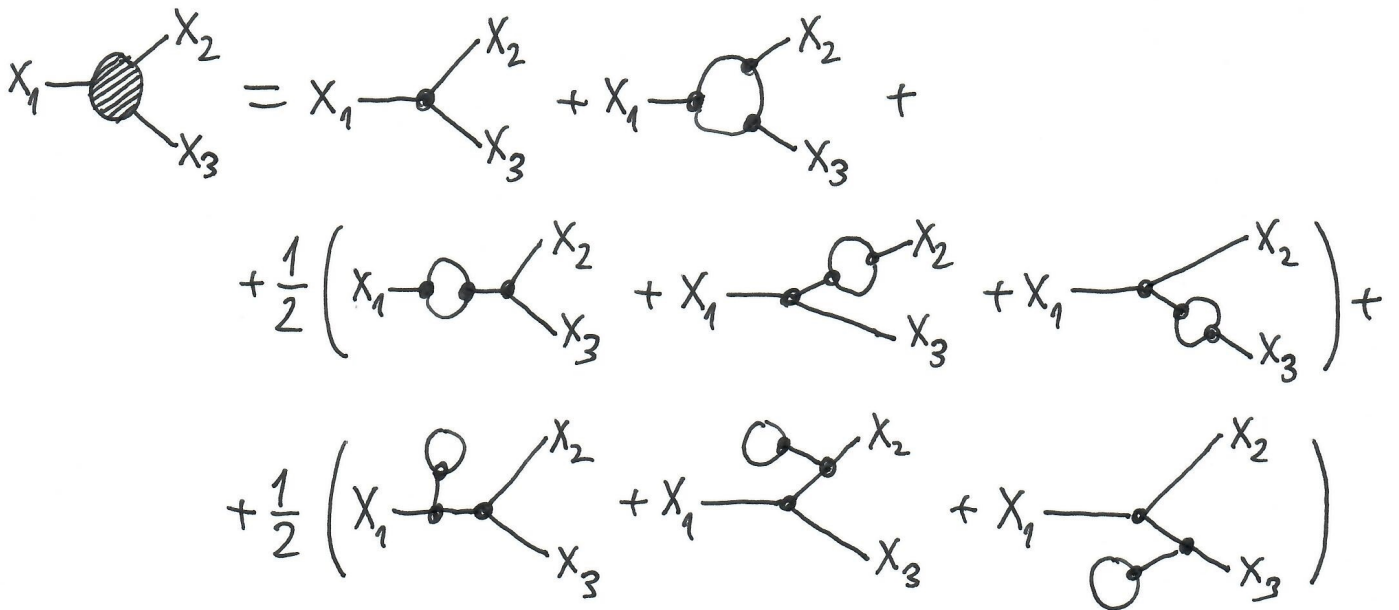
CENSORED (homework spoilers)



$$W^{(3)}(x_1, x_2, x_3) = \frac{1}{3!} \delta_{x_1} \delta_{x_2} \delta_{x_3} \left(\text{diagram of shaded vertex} \right)$$

CENSORED

CENSORED (homework spoilers)



FUNCTIONAL $W[J]$ GENERATES ONLY CONNECTED DIAGRAMS

$$\underbrace{W^{(1)}(x)} = \left(\frac{1}{i} \delta_x W \right)_{J=0} = \frac{1}{i} \left(\delta_x \ln Z \right)_{J=0} = \left(\frac{1}{Z} \frac{1}{i} \delta_x Z \right)_{J=0} = \underbrace{G^{(1)}(x)}$$



$$\boxed{\text{blob with loop} = \text{blob}}$$

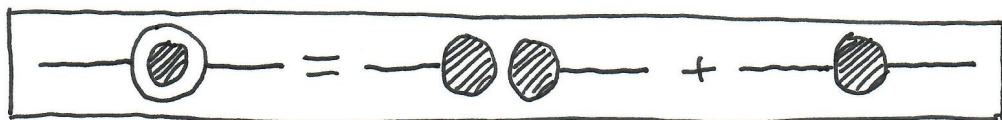


$$\underbrace{W^{(2)}(x, y)} = \left(\frac{1}{i} \delta_x \frac{1}{i} \delta_y W \right)_{J=0} = \left[\frac{1}{i} \delta_x \left(\frac{1}{Z} \frac{1}{i} \delta_y Z \right) \right]_{J=0} =$$

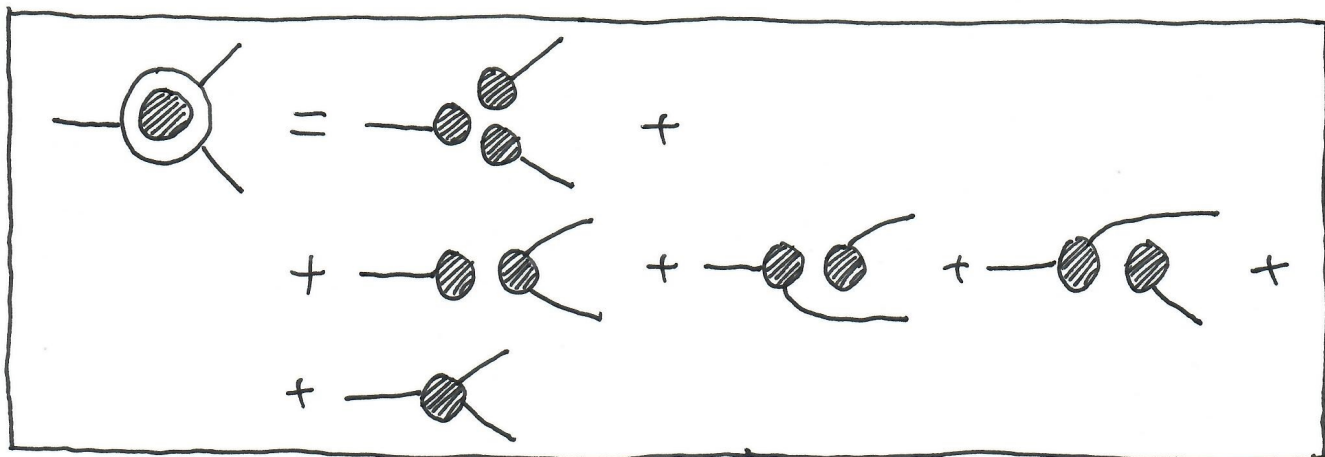
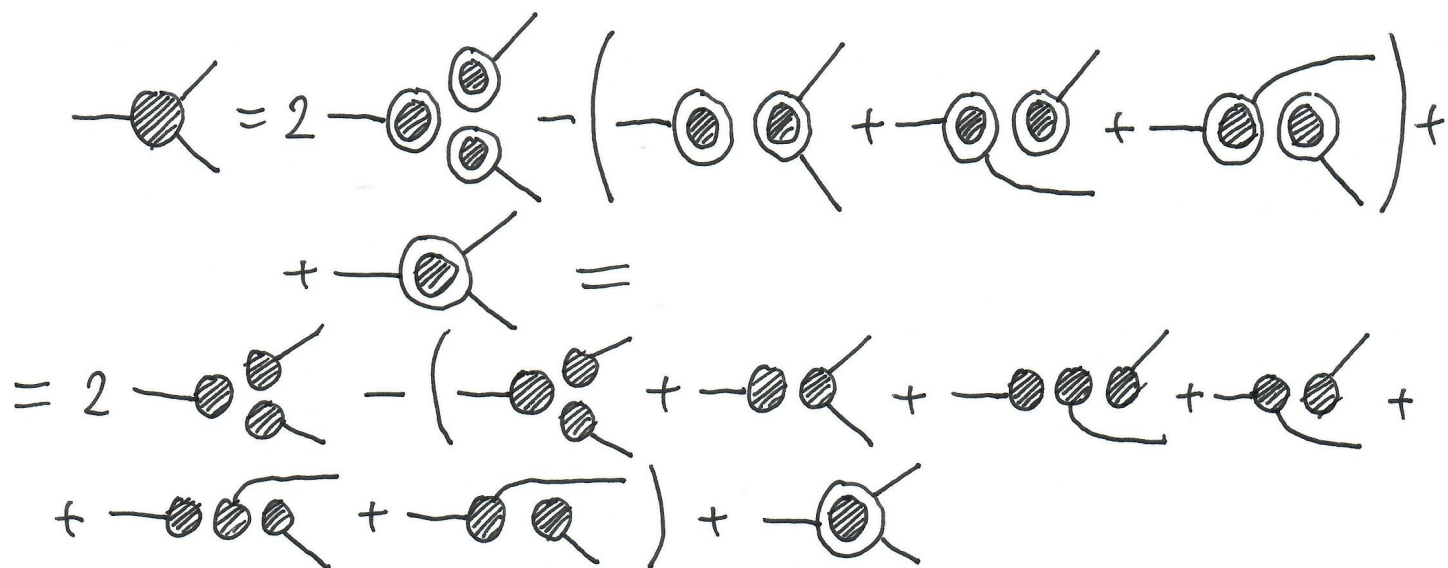
$$= \left[-\frac{1}{Z^2} \left(\frac{1}{i} \delta_x Z \right) \left(\frac{1}{i} \delta_y Z \right) + \frac{1}{Z} \frac{1}{i} \delta_x \frac{1}{i} \delta_y Z \right]_{J=0} =$$

$$= - \underbrace{G^{(1)}(x) G^{(1)}(y)} + \underbrace{G^{(2)}(x, y)}$$





$$\begin{aligned}
 W^{(3)}(x,y,z) &= \left\{ \frac{1}{i} \delta_x \left[-\frac{1}{Z^2} \left(\frac{1}{i} \delta_y Z \right) \left(\frac{1}{i} \delta_z Z \right) + \frac{1}{Z} \frac{1}{i} \delta_y \frac{1}{i} \delta_z Z \right] \right\}_{J=0} = \\
 &= \left\{ \frac{2}{Z^3} \left(\frac{1}{i} \delta_x Z \right) \left(\frac{1}{i} \delta_y Z \right) \left(\frac{1}{i} \delta_z Z \right) - \frac{1}{Z^2} \left(\frac{1}{i} \delta_x \frac{1}{i} \delta_y Z \right) \left(\frac{1}{i} \delta_z Z \right) - \right. \\
 &\quad \left. - \frac{1}{Z^2} \left(\frac{1}{i} \delta_y Z \right) \left(\frac{1}{i} \delta_x \frac{1}{i} \delta_z Z \right) - \frac{1}{Z^2} \left(\frac{1}{i} \delta_x Z \right) \left(\frac{1}{i} \delta_y \frac{1}{i} \delta_z Z \right) + \right. \\
 &\quad \left. + \frac{1}{Z} \frac{1}{i} \delta_x \frac{1}{i} \delta_y \frac{1}{i} \delta_z Z \right\}_{J=0} = 2 G^{(1)}(x) G^{(1)}(y) G^{(1)}(z) + \\
 &+ \left(-G^{(1)}(x) G^{(2)}(y,z) - G^{(1)}(y) G^{(2)}(x,z) - G^{(1)}(z) G^{(1)}(x,y) \right) + \\
 &+ G^{(3)}(x,y,z)
 \end{aligned}$$



LEGENDRE TRANSFORM OF GENERATING FUNCTIONAL

$$W[J] \rightarrow \Gamma[\Psi] = -i \int d^4x J(x)\Psi(x) + W[J]$$

INVERTIBILITY

$$\Psi\{J\} \leftrightarrow J\{\Psi\}$$

$$\Gamma = -i \int J\{\Psi\} \Psi +$$

$$+ W[J\{\Psi\}]$$

$$\Psi(x) = \frac{1}{i} \frac{\delta W[J]}{\delta J(x)} = \langle 0 | \hat{\phi}(x) | 0 \rangle_{J \neq 0}$$

$$= \text{diagram with blob and line } x + i \text{diagram with blob, line } x, \text{ and asterisk} + \frac{i^2}{2!} \text{diagram with blob, line } x, \text{ and two asterisks} + \dots$$

$$\frac{\delta \Gamma[\Psi]}{\delta \Psi(x)} = -i \int d^4y \left(\frac{\delta J(y)}{\delta \Psi(x)} \Psi(y) + J(y) \frac{\delta \Psi(y)}{\delta \Psi(x)} \right) + \frac{\delta W[J]}{\delta \Psi(x)} =$$

$$\frac{1}{i} \frac{\delta W[J]}{\delta J(y)} \delta(y-x) + J(y-x)$$

$$= - \int d^4y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \Psi(x)} - i J(x) + \frac{\delta W[J]}{\delta \Psi(x)} = -i J(x) \rightarrow J(x) = i \frac{\delta \Gamma[\Psi]}{\delta \Psi(x)}$$

NEW FUNCTIONAL GENERATES SOME KIND OF M-POINT FUNCTIONS

$$\text{diagram with box and m lines } x_1, \dots, x_m \equiv \Gamma^{(m)}(x_1, \dots, x_m) = \left[\frac{\delta^m \Gamma[\Psi]}{\delta \Psi(x_1) \dots \delta \Psi(x_m)} \right]_{\Psi=0}$$

$$\Gamma[\Psi] = \sum_{m=0}^{\infty} \int d^4x_1 \dots \int d^4x_m \frac{1}{m!} \Gamma^{(m)}(x_1, \dots, x_m) \Psi(x_1) \dots \Psi(x_m) \equiv$$

$$\equiv \text{diagram with box} + \text{diagram with box and line } 0 + \frac{1}{2!} \text{diagram with box and two lines } 0 + \frac{1}{3!} \text{diagram with box and three lines } 0 + \dots$$

$$\delta(x-y) = \frac{\delta J(x)}{\delta J(y)} = \int d^4z \frac{\delta J(x)}{\delta \Psi(z)} \frac{\delta \Psi(z)}{\delta J(y)} = \int d^4z i \frac{\delta^2 \Gamma[\Psi]}{\delta \Psi(x) \delta \Psi(z)} \frac{1}{i} \frac{\delta^2 W[J]}{\delta J(z) \delta J(y)} =$$

$$= \int d^4z i (\text{SOMETHING LIKE } \Gamma^{(2)}(x, z)) i (\text{SOMETHING LIKE } W^{(2)}(z, y)) \equiv$$

$$\equiv \int d^4z \overset{\circ}{\Gamma}^{(2)}(x, z) \overset{*}{W}^{(2)}(z, y) \quad \text{WHERE:}$$

$$\overset{\circ}{\Gamma}^{(m)}(x_1, \dots, x_m) \equiv \text{diagram with box and m lines } x_1, \dots, x_m + \text{diagram with box, m lines } x_1, \dots, x_m, \text{ and line } 0 + \dots + \frac{1}{k!} \text{diagram with box, m lines } x_1, \dots, x_m, \text{ and k lines } 0 + \dots$$

$$\overset{*}{W}^{(m)}(x_1, \dots, x_m) \equiv \text{diagram with blob and m lines } x_1, \dots, x_m + i \text{diagram with blob, m lines } x_1, \dots, x_m, \text{ and asterisk} + \dots + \frac{1}{m!} \text{diagram with blob, m lines } x_1, \dots, x_m, \text{ and m asterisks} + \dots$$

$$\int d^4 z \Gamma^{(2)}(x, z) \tilde{W}^{*(2)}(z, y) = -\delta(x-y) / \frac{1}{i} \frac{\delta}{\delta J(x)}$$

$$\int d^4 z \left(\int d^4 N \underbrace{\frac{\delta \Gamma^{(2)}(x, z)}{\delta \Psi(N)}}_{\Gamma^{(3)}(x, z, N)} \underbrace{\frac{1}{i} \frac{\delta \Psi(N)}{\delta J(x)}}_{\frac{1}{i^2} \frac{\delta^2 W[J]}{\delta J(x) \delta J(N)}} \tilde{W}^{*(2)}(z, y) + \Gamma^{(2)}(x, z) \underbrace{\frac{1}{i} \frac{\delta \tilde{W}^{*(2)}(z, y)}{\delta J(x)}}_{\tilde{W}^{*(3)}(z, y, x)} \right) = 0$$

$$\int d^4 z \int d^4 N \Gamma^{(3)}(x, z, N) \tilde{W}^{*(2)}(N, x) \tilde{W}^{*(2)}(z, y) + \int d^4 z \Gamma^{(2)}(x, z) \tilde{W}^{*(3)}(z, y, x) = 0$$

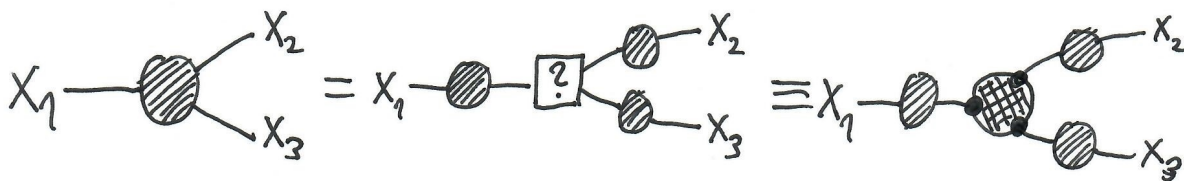
$$\downarrow \quad / \int d^4 x \dots \tilde{W}^{*(2)}(x, N)$$

$$\int d^4 x \int d^4 z \int d^4 N \Gamma^{(3)}(x, z, N) \tilde{W}^{*(2)}(x, N) \tilde{W}^{*(2)}(z, y) \tilde{W}^{*(2)}(N, x) + \int d^4 z \int d^4 x \underbrace{\Gamma^{(2)}(x, z) \tilde{W}^{*(2)}(x, N)}_{-\delta(z-N)} \underbrace{\tilde{W}^{*(3)}(z, y, x)}_{-\tilde{W}^{*(3)}(N, y, x)} = 0$$

$$\tilde{W}^{*(3)}(x_1, x_2, x_3) = \int d^4 y_1 \int d^4 y_2 \int d^4 y_3 \Gamma^{(3)}(y_1, y_2, y_3) \tilde{W}^{*(2)}(y_1, x_1) \tilde{W}^{*(2)}(y_2, x_2) \tilde{W}^{*(2)}(y_3, x_3)$$

$$\downarrow \quad / (\dots) |_{J=0} |_{\Psi=0}$$

$$W^{(3)}(x_1, x_2, x_3) = \int d^4 y_1 \int d^4 y_2 \int d^4 y_3 \Gamma^{(3)}(y_1, y_2, y_3) W^{(2)}(y_1, x_1) W^{(2)}(y_2, x_2) W^{(3)}(y_3, x_3)$$



WE SEE THAT $\Gamma[\Psi]$ GENERATES ONE PARTICLE IRREDUCIBLE DIAGRAMS (1PI) WITHOUT EXTERNAL LEGS

$$\Gamma[\Psi] \equiv \text{diagram 1} + \text{diagram 2} + \frac{1}{2!} \text{diagram 3} + \frac{1}{3!} \text{diagram 4} + \dots$$

$$\Gamma^{(m)}(x_1, \dots, x_m) = \left[\frac{\delta^m \Gamma[\Psi]}{\delta \Psi(x_1) \dots \delta \Psi(x_m)} \right]_{\Psi=0} \equiv \text{diagram with m external legs}$$

$$W^{*(3)}(X_1, X_2, X_3) = \int d^4 y_1 \int d^4 y_2 \int d^4 y_3 \Gamma^{(3)}(y_1, y_2, y_3) W^{*(2)}(y_1, X_1) W^{*(2)}(y_2, X_2) W^{*(2)}(y_3, X_3)$$

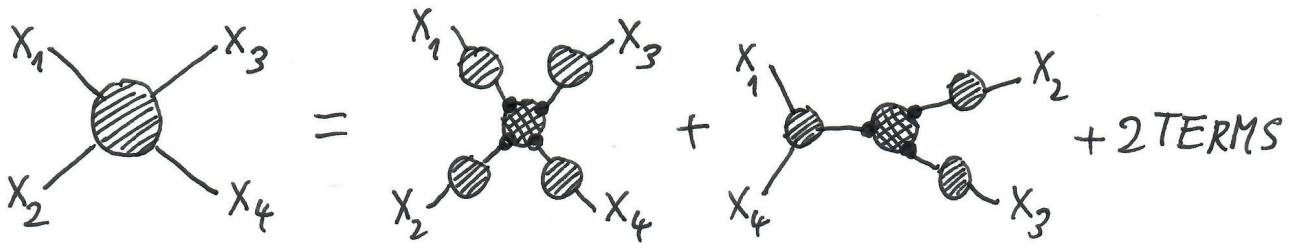
$$/ \frac{1}{i} \frac{\delta}{\delta J(X_4)}$$

$$W^{*(4)}(X_1, \dots, X_4) = \int d^4 y_1 \dots \int d^4 y_4 \underbrace{\frac{\delta \Gamma^{(3)}(y_1, y_2, y_3)}{\delta \Psi(y_4)}}_{\Gamma^{(4)}(y_1, \dots, y_4)} \underbrace{\frac{1}{i} \frac{\delta \Psi(y_4)}{\delta J(X_4)}}_{\frac{1}{i^2} \frac{\delta^2 W[J]}{\delta J(y_4) \delta J(X_4)}} \cdot$$

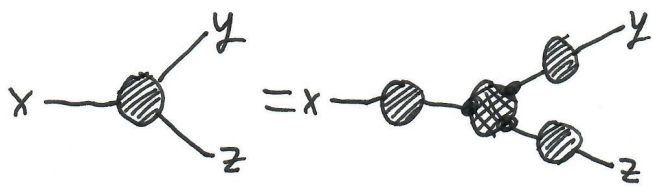
$$\bullet W^{*(2)}(y_1, X_1) W^{*(2)}(y_2, X_2) W^{*(2)}(y_3, X_3) +$$

$$+ \int d^4 y_1 \int d^4 y_2 \int d^4 y_3 \Gamma^{(3)}(y_1, y_2, y_3) [W^{*(3)}(y_1, X_1, X_4) W^{*(2)}(y_2, X_2) W^{*(2)}(y_3, X_3) + 2 \text{TERMS}]$$

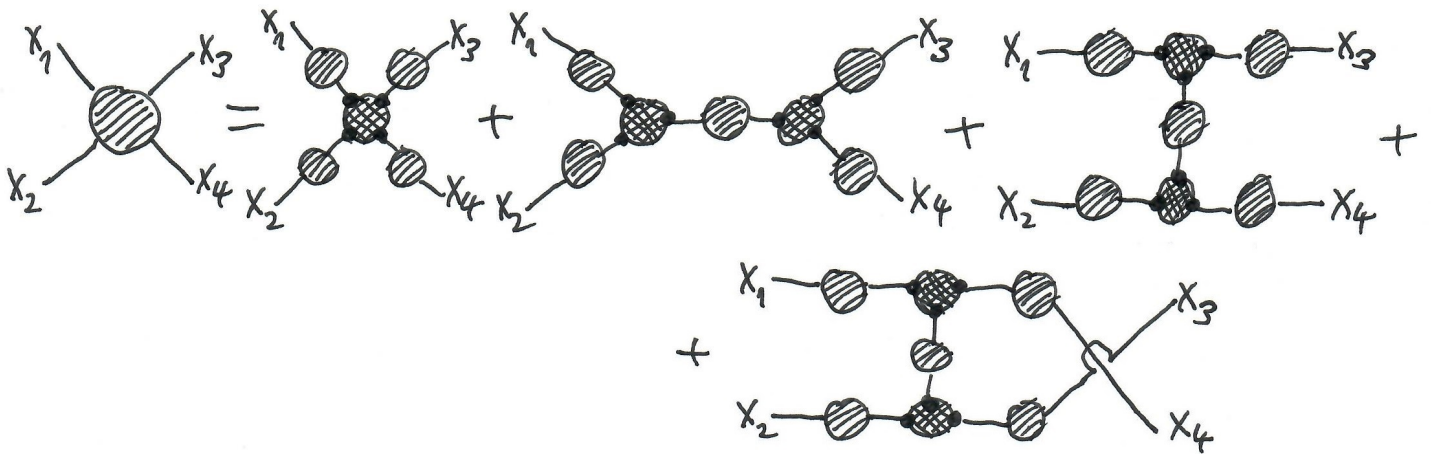
$$\downarrow / (\dots) |_{J=0} |_{\Psi=0}$$



AFTER USING RESULT FROM THE PREVIOUS PAGE



WE FIND



PHASE GIVEN BY ACTION $e^{iS[\phi]}$



FOURIER TRANSFORM $\int \mathcal{D}\phi e^{iS[\phi]} e^{i\int d^4x J\phi}$

GENERATING FUNCTIONAL

$$Z[J] = \text{[circle]} + i \text{[circle with line]} + \frac{i^2}{2!} \text{[circle with two lines]} + \dots$$

GENERATES ALL DIAGRAMS $\tilde{G}^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i} \frac{\delta}{\delta J(x_1)} \dots Z[J] \right]_{J=0}$

$$\frac{Z[J]}{Z[0]} = \text{[shaded circle]} + i \text{[shaded circle with line]} + \frac{i^2}{2!} \text{[shaded circle with two lines]} + \dots$$

GENERATES DIAGRAMS WITHOUT VACUUM BUBBLES - m -POINT GREEN FUNCTIONS

$$\langle 0 | T \{ \hat{\phi}(x_1) \dots \hat{\phi}(x_m) \} | 0 \rangle = G^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i} \frac{\delta}{\delta J(x_1)} \dots \frac{Z[J]}{Z[0]} \right]_{J=0}$$

EXP/LOG SUBSTITUTION $Z = e^W$

$$W[J] = \text{[shaded circle]} + i \text{[shaded circle with line]} + \frac{i^2}{2!} \text{[shaded circle with two lines]} + \dots$$

GENERATES CONNECTED DIAGRAMMS $W^{(m)}(x_1, \dots, x_m) = \left[\frac{1}{i} \frac{\delta}{\delta J(x_1)} \dots W[J] \right]_{J=0}$

LEGENDRE TRANSFORM $\Gamma[\Psi] = -i \int d^4x J\Psi + W[J]$

$$\Psi = \frac{1}{i} \frac{\delta W}{\delta J}$$

$$\Gamma[\Psi] = \text{[shaded circle with cross-hatch]} + \text{[shaded circle with cross-hatch and line]} + \frac{1}{2!} \text{[shaded circle with cross-hatch and two lines]} + \dots$$

GENERATES ONE PARTICLE IRREDUCIBLE DIAGRAMMS WITHOUT EXTERNAL LEGS

$$\Gamma^{(m)}(x_1, \dots, x_m) = \left[\frac{\delta}{\delta \Psi(x_1)} \dots \Gamma[\Psi] \right]_{\Psi=0}$$