Notes on Path Integral Methods

61 .... fermions and Grassmann numbers
63 .... QFT with fermions & Dirac equation
65 .... QED (quantum electrodynamics)
66 .... gauge fixing for electromagnetic field
69 .... Yang–Mills theory

QUANTUM FIELD THEORY WITH FERMIONS

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \int d^4x_1 \dots \int d^4x_n \frac{i^n}{n!} \langle 0| T\{\hat{\Phi}(x_1)\dots\hat{\Phi}(x_n)\}|0\rangle J(x_1)\dots J(x_n)$$

→ n-POINT GREEN FUNCTIONS ARE SYMMETRIC
BUT FOR FERMIONS THEY SHOULD BE ANTISYMMETRIC

⇒ FERMIONIC FIELDS AND CORRESPONDING SOURCES MUST BE GRASSMANN VALUED

GRASSMANN NUMBERS : $xy = -yx$, $x^2 = 0$ ⇒ TAYLOR EXPANSION OF ANY FUNCTION MUST BE CUT OFF $f(x) = a + bx$

GRASSMANN PARITY $g.p.(x) = \begin{cases} 1 \dots \text{GRASSMANN} \\ 0 \dots \text{ORDINARY REAL/COMPLEX} \end{cases}$

WORKS LIKE $(\mathbb{Z},+) \leftrightarrow \{\text{EVEN}, \text{ODD}\} \equiv \mathbb{Z}_2$

$$\underbrace{xy}_{a}\,\underbrace{zw}_{b} = -wxyz = \underbrace{zw}_{b}\,\underbrace{xy}_{a}$$

$$x_1x_2x_3y_1y_2 = y_2x_1x_2x_3y_1 = y_1y_2x_1x_2x_3$$

$$x_1x_2x_3y_1y_2y_3 = -y_3x_1x_2x_3y_1y_2 = y_2y_3x_1x_2x_3y_1 = -y_1y_2y_3x_1x_2x_3$$

$$ab = \left((-1)^{g.p.(b)-1+g.p.(a)}\right)^{g.p.(a)} ba =$$

$$= (-1)^{g.p.(a)\cdot g.p.(b)}\, ba\, \underbrace{(-1)^{-g.p.(a)+(g.p.(a))^2}}_{(-1)^0 = 1}$$

INTEGRALS

$$\int_{-\infty}^{\infty} dx\, f(x) = \int_{-\infty}^{\infty} dx\,(a+bx) = \begin{bmatrix} \text{SUBSTITUTION} \\ x = y - y_0 \end{bmatrix} = \int_{-\infty}^{\infty} dy\,(\overbrace{a - by_0}^{a'} + by)$$

⇒ INTEGRAL DEPENDS ONLY ON b AND MUST BE LINEAR IN IT

$$\Rightarrow \qquad \int_{-\infty}^{\infty} dx\,(a+bx) = b$$

DERIVATIVES $\frac{d}{dx}\, y x = (-1)^{g.p.(y)}\, y$

MULTIDIMENSIONAL INTEGRALS $\int dx \underbrace{\int dy\, y x}_{z} = \int dx\, z x = z \equiv \int dy\, y = 1$

$$\int dx_1 \ldots \int dx_m \underbrace{x_m \ldots x_1}_{(-1)^{m-1} x_1 x_m \ldots x_2 = \ldots = \overbrace{(-1)^{m-1}(-1)^{m-2} \ldots (-1)}^{(-1)^{\frac{m(m-1)}{2}}} x_1 \ldots x_m} = 1$$

$$\int dx_1 \ldots \int dx_m\, x_1 \ldots x_m = (-1)^{m(m-1)/2}$$

$$\int dx_1 \ldots \int dx_m\, x_i \ldots x_j = (-1)^{m(m-1)/2}\, \varepsilon_{i \ldots j}$$

COMPLEX GRASSMANN NUMBERS $(xy)^* = y^* x^* = -x^* y^*$

$$x x^* = -x^* x$$

$$\overbrace{\int dx^* x^*}^{1} \overbrace{\int dx\, x}^{1} \xrightarrow{(-1)^2} = (-1)^2 \int dx^* \int dx\; x x^* = 1$$

USEFUL INTEGRAL ($g.p.(x_i) = 1$, $g.p.(A) = 0$, A - SYMMETRIC MATRIX)

$$\int dx_1^* \int dx_1 \ldots \int dx_m^* \int dx_m\, e^{-x^\dagger A x} = \left[\begin{array}{l}\text{ONLY THE } m\text{-TH TERM IN TAYLOR} \\ \text{EXPANSION OF } e^{\cdots} \text{ CONTRIBUTES}\end{array}\right] =$$

$$= \frac{1}{m!} \int dx_1^* \int dx_1 \ldots \int dx_m^* \int dx_m \left[\sum_{ij} (-x_i^* A_{ij} x_j)\right]^m =$$

$$= \frac{1}{m!} \int dx_1^* \int dx_1 \ldots \int dx_m^* \int dx_m \sum_{i_1 j_1} \ldots \sum_{i_m j_m} (-x_{i_1}^* A_{i_1 j_1} x_{j_1}) \ldots (-x_{i_m}^* A_{i_m j_m} x_{j_m}) =$$

$$= \frac{(-1)^m}{m!} \sum_{i_1 j_1} \ldots \sum_{i_m j_m} A_{i_1 j_1} \ldots A_{i_m j_m} (\pm 1) \int dx_1^* \ldots \int dx_m^* \int dx_1 \ldots \int dx_m (\pm 1) x_{i_1}^* \ldots x_{i_m}^* x_{j_1} \ldots x_{j_m} =$$

THE SAME

$$(-1)^{m+m^2} = 1$$

$$= \frac{(-1)^m}{m!} \sum_{i_1 j_1} \ldots \sum_{i_m j_m} A_{i_1 j_1} \ldots A_{i_m j_m} (-1)^{m^2} \underbrace{\int dx_1^* \ldots \int dx_m^*\, x_{i_1}^* \ldots x_{i_m}^*}_{(-1)^{\frac{m(m-1)}{2}} \varepsilon_{i_1 \ldots i_m}} \underbrace{\int dx_1 \ldots \int dx_m\, x_{j_1} \ldots x_{j_m}}_{(-1)^{\frac{m(m-1)}{2}} \varepsilon_{j_1 \ldots j_m}} =$$

$$= \frac{1}{m!} \sum_{i_1 j_1} \ldots \sum_{i_m j_m} A_{i_1 j_1} \ldots A_{i_m j_m}\, \varepsilon_{i_1 \ldots i_m}\, \varepsilon_{j_1 \ldots j_m} = \det A$$

ORDINARY INTEGRAL $\int dx_1 \dots \int dx_m \, e^{-x^T A x} = \sqrt{\frac{\pi^m}{\det A}}$

COMPLEX GRASSMANN INTEGRAL $\int dx_1^* \int dx_1 \dots \int dx_m^* \int dx_m \, e^{-x^+ A x} = \det A$

VERY USEFUL INTEGRAL ($g.p.(\xi_i) = g.p.(\eta_j) = 1$, $g.p.(B) = 0$, B – HERMITIAN MATRIX)

$$\frac{\int d^m\xi^* \int d^m\xi \, e^{i\xi^+ B \xi + i\eta^+ \xi + i\xi^+ \eta}}{\int d^m\xi^* \int d^m\xi \, e^{i\xi^+ B\xi}} = \begin{bmatrix} \text{SUBSTITUTION} \\ \chi = \xi + B^{-1}\eta \\ \text{JACOBIAN} = 1 \end{bmatrix} =$$

$$\left\{ \begin{aligned} & i\xi^+ B\xi + i\eta^+\xi + i\xi^+\eta = i(\chi^+ - \eta^+ B^{-1}) B (\chi - B^{-1}\eta) + \\ & + i\eta^+(\chi - B^{-1}\eta) + i(\chi^+ - \eta^+ B^{-1})\eta = i\chi^+ B\chi - \\ & - i\chi^+\eta - i\eta^+\chi + i\eta^+ B^{-1}\eta + i\eta^+\chi - i\eta^+ B^{-1}\eta + i\chi^+\eta - \\ & - i\eta^+ B^{-1}\eta = i\chi^+ B\chi - i\eta^+ B^{-1}\eta \end{aligned} \right\}$$

$$= \frac{\int d^m\chi^* \int d^m\chi \, e^{i\chi^+ B\chi - i\eta^+ B^{-1}\eta}}{\int d^m\xi^* d^m\xi \, e^{i\xi^+ B\xi}} = e^{-i\eta^+ B^{-1}\eta}$$

QFT WITH BISPINOR FERMIONS

$g.p.(\Psi) = 1$ & BISPINORS

$\mathcal{L}(\bar\Psi, \Psi) = \bar\Psi (i\not\partial - m)\Psi$

$\not a = \gamma^\mu a_\mu = \gamma^0 a_0 - \vec\gamma \cdot \vec a$

$\bar\Psi = \Psi^+ \gamma^0$

DIRAC MATRICES IN DIRAC BASIS

$$\gamma^0 = \begin{pmatrix} \hat 1 & 0 \\ 0 & -\hat 1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

CLIFFORD ALGEBRA $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\hat 1$

CLASSICAL Eq. of M.

$$\delta \int d^4x \, \mathcal{L}(\bar\Psi, \Psi) = \int d^4x \left[\delta\bar\Psi (i\not\partial - m)\Psi + \bar\Psi (i\not\partial - m)\delta\Psi \right] =$$

$$\left\{ \begin{aligned} & \bar\Psi \not\partial \delta\Psi \xrightarrow{P.P.} -(\not\partial \bar\Psi)\delta\Psi \stackrel{g.p.}{=} +\delta\Psi \not\partial \bar\Psi \\ & \bar\Psi \delta\Psi \stackrel{g.p.}{=} -\delta\Psi \bar\Psi \end{aligned} \right\}$$

$$= \int d^4x \left[\delta\bar\Psi (i\not\partial - m)\Psi + \delta\Psi (i\not\partial + m)\bar\Psi \right] \stackrel{!}{=} 0 \Rightarrow \text{DIRAC EQUATION}$$

DIRAC EQUATION
$$(i\not{\partial} - m)\psi = 0$$
$$(i\not{\partial} + m)\bar{\psi} = 0$$

GENERATING FUNCTIONAL

$$Z[\eta,\bar{\eta}] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ i\int d^4x \left[\bar{\psi}(i\not{\partial} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta \right] \right\}$$

$$\int d^4x\, \bar{\psi}(i\not{\partial} - m)\psi = \int d^4x \int d^4y\, \bar{\psi}(x) B(x,y)\psi(y)$$
$$B(x,y) = (i\not{\partial}_x - m)\delta(x-y)$$

$$Z[\eta,\bar{\eta}] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i\bar{\psi}B\psi + i\bar{\eta}\psi + i\bar{\psi}\eta} = Z[0,0]\, e^{-i\bar{\eta}B^{-1}\eta}$$

$$Z[\eta,\bar{\eta}] = Z[0,0] \exp\left\{ i\int d^4x \int d^4y\, \bar{\eta}(x) B^{-1}(x,y)\eta(y) \right\} =$$

$$\int d^4z\, B^{-1}(x,z)B(z,y) = \delta(x-y) \quad / \ (i\not{\partial}_x - m)$$
$$\int d^4z\, (i\not{\partial}_x - m)B^{-1}(x,z)B(z,y) = B(x,y)$$
$$\to (i\not{\partial}_x - m)\underbrace{B^{-1}(x,z)}_{\int \frac{d^4k}{(2\pi)^4} B^{-1}_k e^{-ik\cdot(x-z)}} = \delta(x-z)$$
$$[i(-i\not{k}) - m]B^{-1}_k = 1 \to B^{-1}_k = \frac{1}{\not{k} - m}$$

$$= Z[0,0]\exp\left\{ -\int d^4x \int d^4y\, \bar{\eta}(x) \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} - m} e^{-ik\cdot(x-y)} \eta(y) \right\}$$

GREEN FUNCTIONS

$$\langle 0 | T\{ \ldots \psi(x_a) \ldots \bar{\psi}(x_b) \ldots \} | 0 \rangle =$$
$$= \frac{1}{Z[0,0]} \left[\ldots \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x_a)} \ldots \frac{-1}{i}\frac{\delta}{\delta\eta(x_b)} \ldots Z[\eta,\bar{\eta}] \right]_{\eta=\bar{\eta}=0}$$

QUANTUM ELECTRODYNAMICS

$$\mathcal{L}_{QED} = \bar\Psi(i\not D - m)\Psi - \frac14 F_{\mu\nu}F^{\mu\nu} =$$

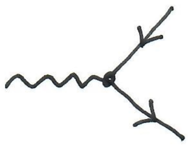
$$\left\{ \begin{aligned} D_\mu &= \partial_\mu + ieA_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \right\}$$

$$= \bar\Psi(i\not\partial - m)\Psi - \frac14 F_{\mu\nu}F^{\mu\nu} - e\bar\Psi\gamma^\mu\Psi A_\mu$$

ELECTRONS/POSITRONS

PHOTONS

INTERACTION



GAUGE SYMMETRY $\Psi, \bar\Psi, A_\mu \longrightarrow \Psi^\alpha, \bar\Psi^\alpha, A^\alpha_\mu$

$$\Psi^\alpha(x) = e^{i\alpha(x)}\Psi(x), \quad A^\alpha_\mu(x) = A_\mu(x) - \frac1e \partial_\mu\alpha(x)$$

$$\bar\Psi^\alpha(i\not D^\alpha - m)\Psi^\alpha = e^{-i\alpha}\bar\Psi(i\not\partial - e\not A + \not\partial\alpha - m)e^{i\alpha}\Psi =$$
$$= e^{-i\alpha}\bar\Psi(-\not\partial\alpha\, e^{i\alpha}\Psi + ie^{i\alpha}\not\partial\Psi - e\not A e^{i\alpha}\Psi + \not\partial\alpha\, e^{i\alpha}\Psi -$$
$$- m e^{i\alpha}\Psi) = \bar\Psi(i\not\partial - e\not A - m)\Psi = \bar\Psi(i\not D - m)\Psi$$

$$F^\alpha_{\mu\nu}F^{\alpha\mu\nu} = (A^\alpha_{\nu,\mu} - A^\alpha_{\mu,\nu})(A^{\alpha\nu,\mu} - A^{\alpha\mu,\nu}) =$$
$$= \left(A_{\nu,\mu} - \frac1e\alpha_{,\nu\mu} - A_{\mu,\nu} + \frac1e\alpha_{,\mu\nu}\right)\left(A^{\nu,\mu} - \frac1e\alpha^{,\nu\mu} -\right.$$
$$\left. - A^{\mu,\nu} + \frac1e\alpha^{,\mu\nu}\right) = (A_{\nu,\mu} - A_{\mu,\nu})(A^{\nu,\mu} - A^{\mu,\nu}) = F_{\mu\nu}F^{\mu\nu}$$

PHOTON PROPAGATOR

$$F_{\mu\nu}F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) =$$
$$= \eta^{\mu\alpha}\eta^{\nu\beta}\left[(\partial_\mu A_\nu)(\partial_\alpha A_\beta) - (\partial_\mu A_\nu)(\partial_\beta A_\alpha) - (\partial_\nu A_\mu)(\partial_\alpha A_\beta) + (\partial_\nu A_\mu)(\partial_\beta A_\alpha)\right] \xrightarrow{P.P.}$$

$$\xrightarrow{P.P.} \eta^{\mu\alpha}\eta^{\nu\beta}\left[-(\partial_\mu\partial_\alpha A_\nu)A_\beta + (\partial_\mu\partial_\beta A_\nu)A_\alpha + (\partial_\nu\partial_\alpha A_\mu)A_\beta - (\partial_\nu\partial_\beta A_\mu)A_\alpha\right] =$$
$$= -(\partial_\mu\partial^\mu A_\nu)A^\nu + (\partial_\mu\partial^\nu A_\nu)A^\mu + (\partial_\nu\partial^\mu A_\mu)A^\nu - (\partial_\nu\partial^\nu A_\mu)A^\mu =$$
$$= 2A_\mu(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)A_\nu$$

$$\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \frac{1}{2}\int d^4x \int d^4y\, A_\mu(x)\left(\eta^{\mu\nu}\Box_{(x)} - \partial^\mu_{(x)}\partial^\nu_{(x)}\right)\delta(x-y)A_\nu(y) =$$

$$\left\{ A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} A_{k\mu} e^{-ik\cdot x} \ ,\ \delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \right\}$$

$$= \frac{1}{2}\int d^4x \int d^4y \int \frac{d^4k_1}{(2\pi)^4} A_{k_1\mu} e^{-ik_1\cdot x}\left(\eta^{\mu\nu}\Box_{(x)} - \partial^\mu_{(x)}\partial^\nu_{(x)}\right)\int \frac{d^4k_2}{(2\pi)^4} e^{-ik_2\cdot(x-y)} \int \frac{d^4k_3}{(2\pi)^4} A_{k_3\nu} e^{-ik_3\cdot y} =$$

$$= \frac{1}{2}\int d^4x \int d^4y \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} A_{k_1\mu}\left(-\eta^{\mu\nu}k_2^2 + k_2^\mu k_2^\nu\right)A_{k_3\nu} e^{-i(k_1+k_2)\cdot x} e^{-i(-k_2+k_3)\cdot y} =$$

$$= \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} A_{k_1\mu}\left(-\eta^{\mu\nu}k_2^2 + k_2^\mu k_2^\nu\right)A_{k_3\nu}(2\pi)^4\delta(k_1+k_2)(2\pi)^4\delta(-k_2+k_3) =$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_{k\mu}\left(k^\mu k^\nu - \eta^{\mu\nu}k^2\right)A_{-k\nu} =$$

$$= \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} A_{k_1\mu}\left(k_1^\mu k_1^\nu - \eta^{\mu\nu}k_1^2\right)\delta(k_1+k_2)A_{k_2\nu} \equiv \frac{1}{2}A^T B A$$

IN THIS CASE B IS SINGULAR MATRIX AND CANNOT BE INVERTED

$$p_\mu B^{\mu\nu} = p_\mu\left(p^\mu p^\nu - \eta^{\mu\nu}p^2\right) = p^2p^\nu - p^\nu p^2 = 0$$

THERE ARE MANY FIELDS A_μ NOT CONTRIBUTING TO LAGRANGIAN DENSITY

$$A_{k\mu} = k_\mu f(k) \Rightarrow \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = 0 \Rightarrow$$ CORRESPONDING CONTRIBUTION TO PATH INTEGRAL IS INFINITE $\int \mathcal{D}A\, e^{iS} = \int \mathcal{D}A\, 1 = \infty$

WE SEPARATE THIS INFINITY BY FIXING THE GAUGE

$$\left\{ \int d^m x\, \delta^{(m)}(\vec{a}(\vec{x})) = \left[\vec{y} = \vec{a}(\vec{x}) \quad d^m x = \left|\frac{\partial x_i}{\partial y_j}\right| d^m y \quad \text{(JACOBIAN)}\right] = \right.$$

$$= \int d^m y \left|\frac{\partial x_i}{\partial a_j}\right| \delta^{(m)}(\vec{y})$$

$$\left. \int d^m x \left|\frac{\partial a_i}{\partial x_j}\right| \delta^{(m)}(\vec{a}(\vec{x})) = \int d^m y\, \delta^{(m)}(\vec{y}) = 1 \right\}$$

$$\int \mathcal{D}\alpha(x)\, \delta\left(G(A^\alpha)\right) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = 1$$

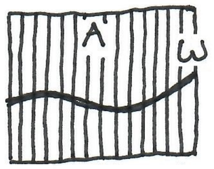
$$\int \mathcal{D}A\, e^{iS[A]} = \int \mathcal{D}\alpha\, \mathcal{D}A\, \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\delta\left(G(A^\alpha)\right) e^{iS[A]} =$$

GAUGE FIXING $G(A) = \partial^\mu A_\mu - \omega(x)$

($\omega = 0 \leftrightarrow$ LORENTZ GAUGE)

$$G(A^\alpha) = \partial^\mu A_\mu + \partial^\mu \frac{1}{e}\partial_\mu \alpha - \omega$$

$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{e}\partial^\mu\partial_\mu = \frac{1}{e}\Box$$



$$= \int \mathcal{D}\alpha\,\mathcal{D}A\, \det\left(\frac{1}{e}\Box\right)\delta\left(\partial^\mu A^\alpha_\mu - \omega\right) e^{iS[A]} = \left[\begin{array}{l} \int \mathcal{D}A \overset{\text{JACOBIAN}=1}{=} \int \mathcal{D}A^\alpha \\ S[A] = S[A^\alpha] \end{array}, \begin{array}{l} \text{RENAME} \\ A^\alpha \to A \end{array}\right] =$$

$$= \int \mathcal{D}\alpha\,\mathcal{D}A\, \det\left(\frac{1}{e}\Box\right)\delta\left(\partial^\mu A_\mu - \omega\right) e^{iS[A]} = \left[\begin{array}{l} \text{DOES NOT DEPEND} \\ \text{ON } \omega(x) \end{array}\right] =$$

[IF] $\int dx\, f(x,y)$ [DOES NOT DEPEND ON y THEN IT] $=$

$$= \frac{\int dy\, e^{-y^2} \int dx\, f(x,y)}{\int dy\, e^{-y^2}} = \frac{\int dx \int dy\, e^{-y^2} f(x,y)}{\int dy\, e^{-y^2}}$$

$$= \frac{\int \mathcal{D}\omega\,\mathcal{D}\alpha\,\mathcal{D}A\, e^{-i\int d^4x \frac{\omega^2}{2\xi}} \det\left(\frac{1}{e}\Box\right)\delta\left(\partial^\mu A_\mu - \omega\right) e^{iS[A]}}{\int \mathcal{D}\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}}} =$$

$$= \underbrace{\frac{\det\left(\frac{1}{e}\Box\right)\left(\int \mathcal{D}\alpha\right)}{\int \mathcal{D}\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}}}} \int \mathcal{D}A\, \exp\left\{ i \left[S[A] \underbrace{- \frac{1}{2\xi}\int d^4x \left(\partial^\mu A_\mu\right)^2}_{S_{g.f.}[A]} \right] \right\}$$

INFINITE CONSTANT, OBSERVABLES DO NOT DEPEND ON IT

$\Rightarrow$ GAUGE FIXING TERM MUST BE ADDED TO ACTION

$$S_{g.f.}[A] = -\frac{1}{2\xi}\int d^4x \left(\partial^\mu A_\mu\right)^2 \qquad \begin{cases} \xi = 0 & \text{LANDAU} \\ \xi = 1 & \text{FEYNMAN} \end{cases}$$

$$\mathcal{L}_{QED} = \overline{\Psi}(i\not{\partial} - m)\Psi \underbrace{- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2}_{\text{PHOTON PROPAGATOR AFTER GAUGE FIXING}} - e\overline{\Psi}\gamma^\mu\Psi A_\mu$$

$$\int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)(\partial^\nu A_\nu)\right] = \left[\begin{array}{c} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \text{\& PER PARTES}\end{array}\right] =$$

$$= \int d^4x \int d^4y \, \frac{1}{2} A_\mu(x) \underbrace{\left[\eta^{\mu\nu}\Box_{(x)} + \left(\frac{1}{\xi}-1\right)\partial^\mu_{(x)}\partial^\nu_{(x)}\right]}_{B^{\mu\nu}(x,y) \xrightarrow{()^{-1}} \text{PROPAGATOR}} \delta(x-y) A_\nu(y)$$

$$\int d^4z \, B^{-1}_{\mu\nu}(x,z) B^{\nu\rho}(z,y) = \delta(x-y)\,\delta^\rho_\mu \quad / \quad \eta^{\mu\sigma}\Box_{(x)} + \left(\frac{1}{\xi}-1\right)\partial^\mu_{(x)}\partial^\sigma_{(x)}$$

$$\int d^4z \underbrace{\left[\eta^{\mu\sigma}\Box_{(x)} + \left(\frac{1}{\xi}-1\right)\partial^\mu_{(x)}\partial^\sigma_{(x)}\right] \underbrace{B^{-1}_{\mu\nu}(x,z)}_{\to \int \frac{d^4k}{(2\pi)^4} B^{-1}_{k\,\mu\nu} e^{-ik\cdot(x-z)}}}_{\overset{!}{=} \delta(x-z)\,\delta^\sigma_\nu} B^{\nu\rho}(z,y) = B^{\mu\sigma}(x,y)\,\delta^\rho_\mu$$

$$\left[\eta^{\mu\sigma}(-k^2) + \left(\frac{1}{\xi}-1\right)(-k^\mu k^\sigma)\right] B^{-1}_{k\,\mu\nu} = \delta^\sigma_\nu$$

$$B^{-1}_{k\,\mu\nu} = -\frac{1}{k^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$$-\frac{1}{k^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\left[\eta^{\nu\rho}(-k^2) + \left(\frac{1}{\xi}-1\right)(-k^\nu k^\rho)\right] =$$

$$= \eta_{\mu\nu}\eta^{\nu\rho} + \left(\frac{1}{\xi}-1\right)\frac{k_\mu k^\rho}{k^2} - (1-\xi)\frac{k_\mu k^\rho}{k^2} - (1-\xi)\left(\frac{1}{\xi}-1\right)\frac{k_\mu k^2 k^\rho}{k^4} =$$

$$\left\{\frac{1}{\xi} - 1 - (1-\xi) - (1-\xi)\left(\frac{1}{\xi}-1\right) = \frac{1}{\xi} - 2 + \xi - \frac{1}{\xi} + 1 + 1 - \xi = 0\right\}$$

$$= \delta^\rho_\mu \quad \text{O.K.}$$

QED GENERATING FUNCTIONAL

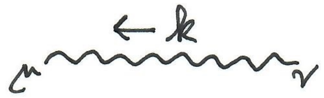
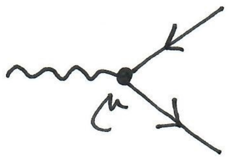
$$Z[\eta, \bar\eta, J^\mu] = Z[0,0,0] \cdot \exp\left\{i(-e\gamma^\mu)\int d^4x \, \frac{-1}{i}\frac{\delta}{\delta\eta(x)}\frac{1}{i}\frac{\delta}{\delta\bar\eta(x)}\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right\} \cdot$$

$$\left\{ \not{k}^2 = \gamma^\mu k_\mu \gamma^\nu k_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) k_\mu k_\nu = \eta^{\mu\nu}k_\mu k_\nu = k^2 \right\}$$

$$\cdot \exp\left\{-\int d^4x \int d^4y \, \bar\eta(x) \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)} \eta(y)\right\} \cdot \left\{\frac{1}{\not{k}-m} = \frac{\not{k}+m}{k^2-m^2}\right\}$$

$$\cdot \exp\left\{-\frac{1}{2}\int d^4x \int d^4y \, J^\mu(x) \int \frac{d^4k}{(2\pi)^4} \frac{-i\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\varepsilon} e^{-ik\cdot(x-y)} J^\nu(y)\right\}$$

QED FEYNMAN RULES

FERMION PROPAGATOR		$\dfrac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$
PHOTON PROPAGATOR		$\dfrac{-i\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\varepsilon}$
INTERACTION VERTEX		$-ie\gamma^\mu$

YANG-MILLS THEORY

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_m(x) \end{pmatrix}, \quad m \text{ - DIMENSION OF LIE GROUP}$$

$D_\mu = \partial_\mu - ig A^a_\mu(x) t_a$, t_a – BASE IN CORRESPONDING LIE ALGEBRA

STRUCTURE CONSTANTS $[t_a, t_b] = i f_{abc} t_c$

DEF. $F^a_{\mu\nu}$: $[D_\mu, D_\nu] = -ig F^a_{\mu\nu} t_a$

LAGRANGIAN DENSITY $\mathcal{L}_{YM} = \overline{\psi}(i\not D - m)\psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$

GAUGE SYMMETRY $\psi, A^a_\mu \longrightarrow \psi^\alpha, A^{a\alpha}_\mu$

$$\psi^\alpha(x) = e^{i\alpha^a(x) t_a}\psi(x)$$

$$A^{a\alpha}_\mu(x) t_a = e^{i\alpha^b(x) t_b}\left(A^a_\mu(x) t_a + \frac{i}{g}\partial_\mu\right) e^{-i\alpha^c(x) t_c}$$

TECHNICAL DETAILS:

$$[D_\mu, D_\nu]\phi = (\partial_\mu - ig A^a_\mu t_a)(\partial_\nu - ig A^b_\nu t_b)\phi - (\mu \leftrightarrow \nu) =$$

$$= \partial_\mu\partial_\nu\phi - \partial_\nu\partial_\mu\phi - ig\partial_\mu A^b_\nu t_b\phi + ig\partial_\nu A^a_\mu t_a\phi -$$

$$- ig A^a_\mu t_a \partial_\nu\phi + ig A^b_\nu t_b \partial_\mu\phi -$$

$$- g^2 A^a_\mu t_a A^b_\nu t_b \phi + g^2 A^b_\nu t_b A^a_\mu t_a \phi =$$

$$= -ig\left(A^a_{\nu,\mu} + A^a_\nu\partial_\mu - A^a_{\mu,\nu} - A^a_\mu\partial_\nu + A^a_\mu\partial_\nu - A^a_\nu\partial_\mu\right)\phi\, t_a -$$

$$- g^2 A^a_\mu A^b_\nu \underbrace{(t_a t_b - t_b t_a)}_{[t_a, t_b] = i f_{abc} t_c}\phi \quad \Rightarrow$$

$$\boxed{=} \ -ig\left(A^a_{\nu,\mu} - A^a_{\mu,\nu} + g f_{abc} A^b_\mu A^c_\nu\right) t_a \phi$$

$$\Rightarrow \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

CHECKING GAUGE SYMMETRY (ONLY UP TO FIRST ORDER IN α)

$$\psi^\alpha(x) = e^{i\alpha^a(x) t_a}\psi(x) \approx (1 + i\alpha^a t_a)\psi$$

$$A^{a\alpha}_\mu(x) t_a = e^{i\alpha^b(x) t_b}\left(A^a_\mu(x) t_a + \frac{i}{g}\partial_\mu\right) e^{-i\alpha^c(x) t_c} \approx$$

$$\approx (1 + i\alpha^b t_b)\left(A^a_\mu t_a + \frac{i}{g}\partial_\mu\right)(1 - i\alpha^c t_c) \approx$$

$$\approx A^a_\mu t_a + i\alpha^b A^a_\mu t_b t_a - i A^a_\mu \alpha^c t_a t_c + \frac{i}{g}(-i)\partial_\mu \alpha^c t_c =$$

$$= A^a_\mu t_a + \frac{1}{g}\partial_\mu \alpha^a t_a + i\alpha^a A^b_\mu [t_a, t_b] =$$

$$= A^a_\mu t_a + \frac{1}{g}\partial_\mu \alpha^a t_a - f_{abc}\alpha^a A^b_\mu t_c$$

$$D^\alpha_\mu = \partial_\mu - ig A^{a\alpha}_\mu t_a \approx \partial_\mu - ig A^a_\mu t_a - i\partial_\mu \alpha^a t_a + ig f_{abc}\alpha^a A^b_\mu t_c$$

$$\bar\psi^\alpha D^\alpha_\mu \psi^\alpha \approx \bar\psi (1 - i\alpha^a t_a)(\partial_\mu - ig A^a_\mu t_a - i\partial_\mu \alpha^a t_a + ig f_{abc}\alpha^a A^b_\mu t_c)(1 + i\alpha^a t_a)\psi \approx$$

$$\approx \bar\psi (1 - i\alpha^a t_a)\left(i\partial_\mu \alpha^a t_a + i\alpha^a t_a \partial_\mu + \partial_\mu - ig A^a_\mu t_a + g A^a_\mu \alpha^b t_a t_b - \right.$$
$$\left. - i\partial_\mu \alpha^a t_a + ig f_{abc}\alpha^a A^b_\mu t_c\right)\psi \approx$$

$$\approx \bar\psi\left(i\alpha^a t_a \partial_\mu + \partial_\mu - ig A^a_\mu t_a + g A^a_\mu \alpha^b t_a t_b + ig f_{abc}\alpha^a A^b_\mu t_c - \right.$$
$$\left. - i\alpha^a t_a \partial_\mu + g\alpha^a A^b_\mu t_a t_b\right)\psi = \bar\psi(\partial_\mu - ig A^a_\mu t_a)\psi = \bar\psi D_\mu \psi$$

$$F^{a\alpha}_{\mu\nu} t_a = \partial_\mu A^{a\alpha}_\nu t_a - \partial_\nu A^{a\alpha}_\mu t_a + \underbrace{g f_{abc} A^{b\alpha}_\mu A^{c\alpha}_\nu t_a}_{-ig[A^{b\alpha}_\mu t_b, A^{c\alpha}_\nu t_c]} \approx$$

$$\approx \partial_\mu\left(A^a_\nu t_a + \frac{1}{g}\partial_\nu \alpha^a t_a - f_{abc}\alpha^a A^b_\nu t_c\right) -$$

$$- \partial_\nu\left(A^a_\mu t_a + \frac{1}{g}\partial_\mu \alpha^a t_a - f_{abc}\alpha^a A^b_\mu t_c\right) -$$

$$- ig\left[A^a_\mu t_a + \frac{1}{g}\partial_\mu \alpha^a t_a - f_{abc}\alpha^a A^b_\mu t_c ,\ A^a_\nu t_a + \frac{1}{g}\partial_\nu \alpha^a t_a - f_{abc}\alpha^a A^b_\nu t_c\right] \boxed{\approx}$$