Notes on Path Integral Methods

$$\approx \partial_\mu A^a_\nu t_a + \frac{1}{g}\partial_\mu\partial_\nu\alpha^a t_a - \underset{\sim\sim\sim}{f_{abc}\partial_\mu\alpha^a A^b_\nu t_c} - f_{abc}\alpha^a\partial_\mu A^b_\nu t_c \; \textcircled{1} -$$

$$-\partial_\nu A^a_\mu t_a - \frac{1}{g}\partial_\nu\partial_\mu\alpha^a t_a + \underline{f_{abc}\partial_\nu\alpha^a A^b_\mu t_c} + f_{abc}\alpha^a\partial_\nu A^b_\mu t_c \; \textcircled{2} -$$

$$-igA^a_\mu A^b_\nu[t_a,t_b] - \underline{iA^a_\mu\partial_\nu\alpha^b[t_a,t_b]} + igA^a_\mu f_{bcd}\alpha^b A^c_\nu[t_a,t_d] \; \textcircled{3} -$$

$$-\underset{\sim\sim\sim}{i\partial_\mu\alpha^a A^b_\nu[t_a,t_b]} + ig f_{abc}\alpha^a A^b_\mu A^d_\nu[t_c,t_d] \; \textcircled{4} =$$

$$= \partial_\mu A^a_\nu t_a - \partial_\nu A^a_\mu t_a + g A^a_\mu A^b_\nu f_{abc} t_c + \textcircled{1}_{\mu\nu} + \textcircled{2}_{\mu\nu} + \textcircled{3}_{\mu\nu} + \textcircled{4}_{\mu\nu} \; \textcircled{=}$$

$$\textcircled{1}_{\mu\nu} \equiv -f_{abc}\alpha^a\partial_\mu A^b_\nu t_c$$
$$\textcircled{2}_{\mu\nu} \equiv f_{abc}\alpha^a\partial_\nu A^b_\mu t_c$$
$$\textcircled{3}_{\mu\nu} \equiv -gA^a_\mu f_{bcd}\alpha^b A^c_\nu f_{ead} t_e$$
$$\textcircled{4}_{\mu\nu} \equiv -g f_{abc}\alpha^a A^b_\mu A^d_\nu f_{ecd} t_e$$

STRUCTURE CONSTANTS

$f_{abc} = f_{[ab]c} =$ [IN APPROPRIATE BASE] $= f_{[abc]}$

JACOBI IDENTITY

$$\underbrace{[[t_a,t_b],t_c]}_{[f_{abd}t_d,\,t_c] = f_{abd}f_{dce}t_e} + \mathrm{CYCL}_{abc} = 0 \longrightarrow f_{abd}f_{dce} + \mathrm{CYCL}_{abc} = 0$$

$$\textcircled{3}_{\mu\nu} + \textcircled{4}_{\mu\nu} = -gA^a_\mu\alpha^b A^c_\nu t_e \underbrace{(f_{bcd}f_{ead} + f_{bad}f_{edc})}$$

$f_{bcd}f_{dea} + f_{bad}f_{dce} = -f_{bcd}f_{dae} - f_{abd}f_{dce} =$ [JACOBI IDENTITY] $= f_{cad}f_{dbe}$

$$\textcircled{1}_{\mu\nu} + \textcircled{2}_{\mu\nu} + \textcircled{3}_{\mu\nu} + \textcircled{4}_{\mu\nu} = -f_{abc}\alpha^a\partial_\mu A^b_\nu t_c + f_{abc}\alpha^a\partial_\nu A^b_\mu t_c -$$

$$\underbrace{-gA^a_\mu\alpha^b A^c_\nu t_e f_{cad}f_{dbe}}_{-f_{abc}\alpha^b g f_{cde} A^d_\mu A^e_\nu t_a} = -f_{abc}\alpha^b F^c_{\mu\nu} t_a$$

$$\textcircled{=}\; F^a_{\mu\nu}t_a - f_{abc}\alpha^b F^c_{\mu\nu} t_a$$

$$F^{a\alpha}_{\mu\nu} \approx F^a_{\mu\nu} - f_{abc}\alpha^b F^c_{\mu\nu}$$

$$F^{a\alpha}_{\mu\nu}F^{a\mu\nu\alpha} \simeq F^{a}_{\mu\nu}F^{a\mu\nu} \underbrace{- F^{a}_{\mu\nu} f_{abc}\alpha^{b} F^{c\mu\nu} - f_{abc}\alpha^{b} F^{c}_{\mu\nu} F^{a\mu\nu}}_{0}$$

$$f_{abc}F^{a}_{\mu\nu}F^{b\mu\nu} = \eta^{\mu\varsigma}\eta^{\nu\sigma} f_{[ab]c} F^{(a}_{\mu\nu}F^{b)}_{\varsigma\sigma} = 0 \longrightarrow 0$$

GAUGE FIXING FOR YANG-MILLS THEORY (FADDEEV-POPOV)

$$\int \mathcal{D}A e^{iS[A]} = \int \Pi_a \mathcal{D}\alpha^a \mathcal{D}A \det\left(\frac{\delta G(A^{a\alpha})}{\delta \alpha^b}\right) \Pi_c \delta(G(A^{c\alpha})) e^{iS[A]} =$$

$$S[A] = S[A^\alpha]$$

$$\int \mathcal{D}A = \int \det\left(\frac{\delta A^a}{\delta A^{b\alpha}}\right) \mathcal{D}A^\alpha = \left[\text{JACOBIAN} = 1\right] =$$

$$A^{a\alpha}t_a = e^{i\alpha^b t_b}\left(A^a t_a + \frac{i}{g}\partial_\mu\right)e^{-i\alpha^c t_c}$$

$$\frac{\delta A^{a\alpha}}{\delta A^b} = e^{i\alpha^c t_c}\frac{\delta A^a}{\delta A^b}e^{-i\alpha^d t_d} =$$

BAKER-CAMPBELL-HAUSDORF FORMULA

$$= \delta^a_b \exp\Big\{ i\alpha^c t_c - i\alpha^d t_d + \frac{1}{2} i(-i)\underbrace{\alpha^c\alpha^d}_{\alpha^{(c}\alpha^{d)}}[t_c, t_d] +$$

$$+ \frac{1}{12} i^2(-i)\alpha^c \overbrace{\alpha^e\alpha^d}^{\alpha^{(e}\alpha^{d)}}[t_c,[t_e,t_d]] + \dots\Big\} = \delta^a_b$$

$$= \int \mathcal{D}A^\alpha$$

THEN RENAME $A^\alpha \longrightarrow A^\alpha|_{\alpha=0} = A$

$$= \int \Pi_a \mathcal{D}\alpha^a \mathcal{D}A \det\left(\frac{\delta G(A^{a\alpha})}{\delta \alpha^b}\Big|_{\alpha=0}\right) \Pi_c \delta(G(A^c)) e^{iS[A]} \Rightarrow$$

GAUGE FIXING CONDITION $G(A) = \partial^\mu A^a_\mu - \omega^a(x) = 0$

$$A^{a\alpha}_\mu t_a = A^a_\mu t_a + \frac{1}{g}\partial_\mu \alpha^a t_a - f_{abc}\alpha^b A^c_\mu t_a + \mathcal{O}(\alpha^2)$$

$$G(A^{a\alpha}) = \partial^\mu A^a_\mu + \frac{1}{g}\partial^\mu\partial_\mu\alpha^a + f_{abc}\partial^\mu(A^b_\mu\alpha^c) - \omega^a + \mathcal{O}(\alpha^2)$$

$$\frac{\delta G(A^{a\alpha})}{\delta \alpha^b} = \frac{1}{g}\partial^\mu\partial_\mu\delta_{ab} + f_{adb}\partial^\mu(A^d_\mu \cdot) + \mathcal{O}(\alpha) =$$

$$= \frac{1}{g}\delta_{ab}\Box - f_{abc}\partial^\mu A^c_\mu - f_{abc}A^c_\mu\partial^\mu + \mathcal{O}(\alpha)$$

$$\Rightarrow \int \Pi_a \mathcal{D}\alpha^a \mathcal{D}A \det\left(\frac{1}{g}\delta_{ab}\Box - f_{abc}\partial^\mu A^c_\mu - f_{abc}A^c_\mu\partial^\mu\right)\cdot$$

$$\cdot \Pi_c \delta\left(\partial^\mu A^c_\mu - \omega^c\right) e^{iS[A]} = \left[\begin{array}{l}\text{DOES NOT DEPEND}\\ \text{ON } \omega^a\end{array}\right] =$$

$$= \frac{\int \Pi_d \mathcal{D}\omega^d \, \Pi_a \mathcal{D}\alpha^a \mathcal{D}A \, e^{-i\int d^4x \frac{\omega^{c2}}{2\xi_c}} \det(\ldots)\, \Pi_c \delta(\partial^\mu A^c_\mu - \omega^c) e^{iS[A]}}{\int \Pi_a \mathcal{D}\omega^a \, e^{-i\int d^4x \frac{\omega^{a2}}{2\xi_a}}} =$$

$$= \underbrace{\frac{\left(\int \Pi_a \mathcal{D}\alpha^a\right)}{\int \Pi_a \mathcal{D}\omega^a e^{-i\int d^4x \frac{\omega^{a2}}{2\xi_a}}}}_{\text{INFINITE CONSTANT, OBSERVABLES DO NOT DEPEND ON IT}} \int \mathcal{D}A \det(\ldots) \exp\left\{ i\left[S[A] \underbrace{- \sum_a \frac{1}{2\xi_a}\int d^4x \left(\partial^\mu A^a_\mu\right)^2}_{S_{g.f.}[A]}\right]\right\} =$$

MULTIDIMENSIONAL COMPLEX GRASSMANN INTEGRAL

$$\int d^M\xi^* d^M\xi \, e^{-\xi^\dagger A \xi} = \det A$$

$$\det\left(\frac{1}{g}\delta_{ab}\Box - f_{abc}\partial^\mu A^c_\mu - f_{abc}A^c_\mu\partial^\mu\right) =$$

$$= \frac{1}{(-ig)^n} \det\left[i\left(-\delta_{ab}\Box + g f_{abc}\partial^\mu A^c_\mu + g f_{abc} A^c_\mu \partial^\mu\right)\right] \rightarrow$$

$$\rightarrow \left(\lim_{n\to\infty} \frac{1}{(-ig)^n}\right) \int \mathcal{D}c \mathcal{D}\bar{c} \exp\left\{ i\int d^4x \, \bar{c}^a\left(-\delta_{ab}\Box + \right.\right.$$

$$\left.\left. + g f_{abc}\partial^\mu A^c_\mu + g f_{abc} A^c_\mu \partial^\mu\right) c^b \right\}$$

$$= \frac{\left(\int \Pi_a \mathcal{D}\alpha^a\right)\left(\lim_{n\to\infty}\frac{1}{(-ig)^n}\right)}{\int \Pi_a \mathcal{D}\omega^a e^{-i\int d^4x \frac{\omega^{a2}}{2\xi_a}}} \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \, e^{i\left(S[A] + S_{g.f.}[A] + S_{ghost}[c,\bar{c},A]\right)}$$

GRASSMANN (FERMIONIC) FIELDS
FADDEEV-POPOV GHOSTS

$$S_{g.f.}[A] = -\sum_a \frac{1}{2\xi_a}\int d^4x \left(\partial^\mu A^a_\mu\right)^2$$

$$S_{ghost}[c,\bar{c},A] = \int d^4x \Big(\underbrace{-\bar{c}^a \Box c^a}_{\text{P.P. } \eta^{\mu\nu}\bar{c}^a_{,\mu} c^a_{,\nu}} + g\,\bar{c}^a f_{abc}\left(\partial^\mu A^c_\mu\right) c^b + g\,\bar{c}^a f_{abc} A^c_\mu \partial^\mu c^b \Big)$$

$$\int \mathcal{D}A\, e^{iS[A]} = (\propto \text{CONST}) \int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar{c}\, \exp\Big\{ i \int d^4x \Big(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) + \eta^{\mu\nu} (\partial_\mu \bar{c}^a)(\partial_\nu c^a) + g f_{abc}\, c^a (\partial^\mu A^c_\mu + A^c_\mu \partial^\mu) c^b \Big) \Big\}$$

ADJOINT REPRESENTATION

$$(\mathrm{ad}_{t_a})_{bc} \equiv (t^{(A)}_a)_{bc} = i f_{bac}$$

$$(D_\mu)_{ab} = \delta_{ab} \partial_\mu - ig\, A^{(A)c}_\mu (t^{(A)}_c)_{ab} = \delta_{ab}\partial_\mu + g f_{acb} A^c_\mu$$

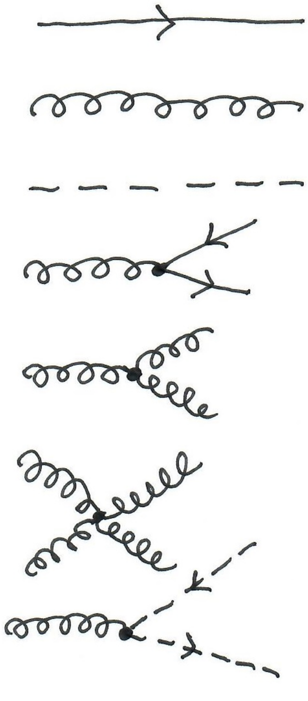
$$\mathcal{L}_{YM} = \bar{\Psi}^a (i \delta_{ab} \not{\partial} - m \delta_{ab}) \Psi^b + i g f_{abc} \bar{\Psi}^a \not{A}^b \Psi^c + \mathcal{L}_{A,c,\bar{c}}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

$$\begin{aligned} F^a_{\mu\nu} F^{a\mu\nu} = [\text{P.P.}] = {} & 2 A^a_\mu (\partial^\mu \partial^\nu - \eta^{\mu\nu} \Box) A^a_\nu + \\ & + 2 g f_{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} + \\ & + g^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \end{aligned}$$

$$-\frac{1}{2\xi} (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) = [\text{P.P.}] = \frac{1}{2\xi} A^a_\mu \partial^\mu \partial^\nu A^a_\nu$$

$$\begin{aligned} \mathcal{L}_{YM} = {} & \bar{\Psi}^a (i\delta_{ab} \not{\partial} - m \delta_{ab}) \Psi^b + \\ & + \frac{1}{2} A^a_\mu \Big(\eta^{\mu\nu} \Box + \Big(\frac{1}{\xi} - 1\Big) \partial^\mu \partial^\nu \Big) \delta_{ab} A^b_\nu + \\ & + (\partial_\mu \bar{c}^a) \eta^{\mu\nu} (\partial_\nu c^b) + \\ & + i g f_{abc} \bar{\Psi}^a \not{A}^b \Psi^c - \\ & - \frac{1}{2} g f_{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \\ & - \frac{1}{4} g^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} - \\ & - g f_{abc} \bar{c}^a (\partial^\mu A^b_\mu + A^b_\mu \partial^\mu) c^c \end{aligned}$$



GHOST PROPAGATOR ← GHOST PART OF GENERATING FUNCTIONAL

$$\int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\Big\{ i\int d^4x \int d^4y \left[-\bar{c}^a(x)\,\Box_{(x)}\,\delta(x-y)\,\delta_{ab}\,c^b(y)\right] +$$

$$+ i\int d^4x \left[\bar{\xi}_a c^a + \bar{c}^a \xi_a\right] \Big\} =$$

$$= \exp\left\{ -i\int d^4x \int d^4y\, \bar{\xi}_a(x) \left[-\Box_{(x)}\delta(x-y)\delta_{ab}\right]^{-1} \xi_b(y) \right\} \cdot$$

$$\cdot \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left\{ i\int d^4x \int d^4y\, \bar{c}^a(x)\left[-\Box_{(x)}\delta(x-y)\delta_{ab}\right] c^b(y) \right\} =$$

$$\int d^4z \left[\Box_{(x)}\delta(x-z)\delta_{ac}\right]^{-1}\left[\Box_{(z)}\delta(z-y)\delta_{cb}\right] = \delta(x-y)\delta_{ab}$$

$$\int d^4z\, \underbrace{\Box_{(x)}\left[\Box_{(x)}\delta(x-z)\delta_{ac}\right]^{-1}}_{\overset{!}{=}\,\delta(x-z)\delta_{ac}}\left[\Box_{(z)}\delta(z-y)\delta_{cb}\right] = \Box_{(x)}\delta(x-y)\delta_{ab}$$

$$\Rightarrow \Box_{(x)} \underbrace{\left[\Box_{(x)}\delta(x-y)\right]^{-1}}_{\int \frac{d^4k}{(2\pi)^4} B^{-1}{}_k\, e^{-ik\cdot(x-y)}} \overset{!}{=} \delta(x-y)$$

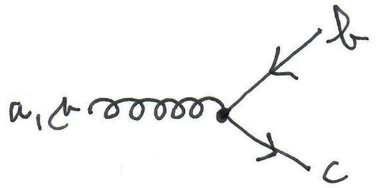
$$(-ik)^2 B^{-1}{}_k = 1 \qquad B^{-1}{}_k = -1/k^2$$

$$= \exp\left\{ -\int d^4x \int d^4y\, \bar{\xi}_a(x) \int \frac{d^4k}{(2\pi)^4} \frac{i\delta_{ab}}{k^2} e^{-ik\cdot(x-y)} \xi_b(y) \right\} \cdot$$

$$\cdot \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left\{ i\int d^4x \int d^4y\, \bar{c}^a(x)\left[-\Box_{(x)}\delta(x-y)\delta_{ab}\right] c^b(y) \right\}$$

$$\Rightarrow \quad a \; - - \overset{p}{\rightarrow} - - \; b \quad \ldots \quad \frac{i\delta_{ab}}{p^2}$$

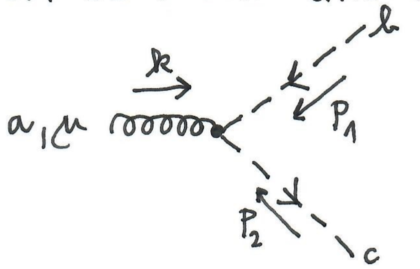
GAUGE BOSON-FERMIONS INTERACTION VERTEX



$$\leftarrow \; i g f_{bac} \bar{\Psi}^b A^a \Psi^c$$

$$\ldots \quad -g f_{bac} \gamma^\mu$$

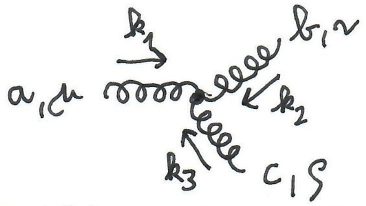
GAUGE BOSON-GHOSTS VERTEX



$$\longleftarrow -g f_{bac}\bar{c}^a(\partial^\mu A^b_\mu + A^b_\mu\partial^\mu)c^c$$

$$\left.\begin{array}{l} \dots -ig f_{bac}[(-ik^\mu)+(-ip_2^\mu)] \\ \dots\ g f_{bac}\, p_1^\mu \end{array}\right\} k+p_1+p_2=0$$

GAUGE BOSONS VERTICES



$$[ef] \xrightarrow{\text{ANTISYM.}} [\alpha\beta]$$

$$\longleftarrow -\frac{1}{2} g f_{def}(\partial_\alpha A^d_\beta - \partial_\beta A^d_\alpha)A^{e\alpha}A^{f\beta} =$$

$$= g f_{def}(\partial_\beta A^d_\alpha)A^{e\alpha}A^{f\beta}$$

THIS VERTEX COMES FROM THE PART OF GENERATING FUNCTIONAL

$$\dots \exp\left\{i\int d^4x\, g f_{def}\left(\frac{\partial}{\partial x^\beta}\frac{1}{i}\frac{\delta}{\delta J^\alpha_d(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{e\alpha}(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{f\beta}(x)}\right)\right\}\dots e^{iS_{FREE}}$$

DERIVING FEYNMAN DIAGRAMS FROM GENERATING FUNCTIONAL THEN GIVES 3! TERMS CORRESPONDING TO NUMBER OF WAYS k_1, k_2, k_3 CAN BE ASSIGNED TO $A^d_\alpha, A^{e\alpha}, A^{f\beta}$ AND SINCE IN FOURIER IMAGE $\partial_\beta \to -ik_\beta$ AND ALSO $g f_{def}(\partial_\beta A^d_\alpha)A^{e\alpha}A^{f\beta} = g f_{def}\eta^{\alpha\gamma}(\partial^\beta A^d_\alpha)A^e_\gamma A^f_\beta$ ALL 3! TERMS LEAD TO TERMS WHICH CAN BE WRITTEN AS

$$3!\, g f_{def}\eta^{\alpha\gamma}(-ik_1^\beta)A^d_{k_1\alpha}A^e_{k_2\gamma}A^f_{k_3\beta} = -ig A^a_{k_1\mu}A^b_{k_2\nu}A^c_{k_3\rho} f_{abc}\big($$

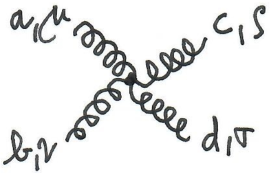
$$\eta^{\mu\nu}k_1^\rho - \eta^{\mu\rho}k_1^\nu - k_2^\rho\eta^{\mu\nu} + k_2^\mu\eta^{\nu\rho} + k_3^\nu\eta^{\mu\rho} - k_3^\mu\eta^{\nu\rho}\big) =$$

$\eta^{\mu\nu}k_1^\rho$	$\eta^{\mu\rho}k_1^\nu$	$k_2^\rho\eta^{\mu\nu}$	$k_2^\mu\eta^{\nu\rho}$	$k_3^\nu\eta^{\mu\rho}$	$k_3^\mu\eta^{\nu\rho}$
$d=a$	$d=a$ RENAME	$d=b\to a$	$d=b\to a$	$d=c\to a$	$d=c\to a$
$e=b$	$e=c\to b$	$e=a\to b$	$e=c\to b$	$e=a\to b$	$e=b$
$f=c$	$f=b\to c$	$f=c$	$f=a\to c$	$f=b\to c$	$f=a\to c$
	GIVES MINUS SIGN DUE TO $f_{a[bc]}$				

$$= -ig f_{abc}\left[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu\right]A^a_{k_1\mu}A^b_{k_2\nu}A^c_{k_3\rho}$$

FACTOR FOR THE VERTEX IS THEN

$$g f_{abc}\left[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu\right]$$



$$\leftarrow -\frac{1}{4} g^2 f_{fgh} f_{fij} A^g_\lambda A^h_\varkappa A^{i\lambda} A^{j\varkappa}$$

AS PREVIOUSLY THE GENERATING FUNCTIONAL PRODUCES FEYNMAN DIAGRAMS WITH VERTICES ↑ WITH FACTORS CORRESPONDING TO

$$\underbrace{4!\left(-\frac{1}{4}\right)}_{-3!} g^2 f_{fgh} f_{fij} \eta^{\lambda\alpha} \eta^{\varkappa\beta} A^g_{k_1\lambda} A^h_{k_2\varkappa} A^i_{k_3\alpha} A^j_{k_4\beta} =$$

$$= -g^2 A^a_{k_1\mu} A^b_{k_2\nu} A^c_{k_3\varsigma} A^d_{k_4\sigma} \left[f_{fab} f_{fcd} \left(\eta^{\mu\varsigma}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\varsigma} \right) + \right.$$

$g = a$
$h = b$
$i = c$
$j = d$

1, RENAME $c \to d$, $d \to c$ 2, MINUS SIGN $f_{fdc} = -f_{fcd}$ 3, RENAME $\varsigma \to \sigma$, $\sigma \to \varsigma$

$$+ f_{fac} f_{fbd} \left(\eta^{\mu\nu}\eta^{\varsigma\sigma} - \eta^{\mu\sigma}\eta^{\nu\varsigma} \right) +$$

1, RENAME $b \to c$, $c \to b$ 2, RENAME $\nu \to \varsigma$, $\varsigma \to \nu$

1, RENAME $c \to d$, $d \to c$ 2, MINUS SIGN $f_{fdc} = -f_{fcd}$ 3, RENAME $\varsigma \to \sigma$, $\sigma \to \varsigma$ 4, 5, THE SAME AS HERE

$$\left. + f_{fad} f_{fbc} \left(\eta^{\mu\nu}\eta^{\varsigma\sigma} - \eta^{\mu\varsigma}\eta^{\nu\sigma} \right) \right]$$

1, RENAME $b \to c$, $c \to b$ 2, RENAME $\nu \to \varsigma$, $\varsigma \to \nu$ 3, RENAME $c \to d$, $d \to c$ 4, RENAME $\varsigma \to \sigma$, $\sigma \to \varsigma$

1, RENAME $b \to d$, $d \to b$ 2, MINUS SIGN $f_{fcb} = -f_{fbc}$ 3, RENAME $\nu \to \sigma$, $\sigma \to \nu$

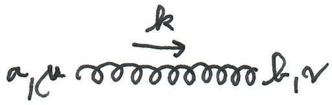
$\Rightarrow$ FACTOR FOR THIS VERTEX IS

$$-ig^2 \left[f_{abe} f_{cde} \left(\eta^{\mu\varsigma}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\varsigma} \right) + \right.$$
$$+ f_{ace} f_{bde} \left(\eta^{\mu\nu}\eta^{\varsigma\sigma} - \eta^{\mu\sigma}\eta^{\nu\varsigma} \right) +$$
$$\left. + f_{ade} f_{bce} \left(\eta^{\mu\nu}\eta^{\varsigma\sigma} - \eta^{\mu\varsigma}\eta^{\nu\sigma} \right) \right]$$

PATH INTEGRAL QUANTIZATION OF YANG-MILLS THEORY → FEYNMAN RULES

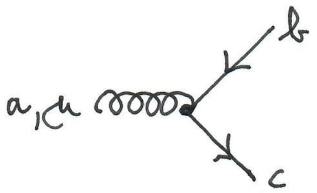
PROPAGATORS:

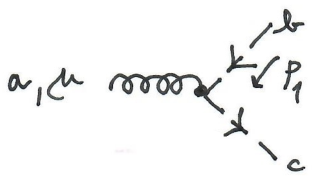
FERMIONS  $\ldots \quad \dfrac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}\,\delta_{ab}$

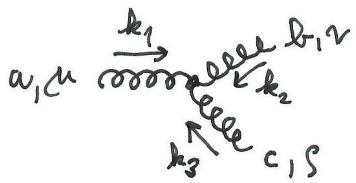
GAUGE BOSONS  $\ldots \quad \dfrac{-i}{k^2 + i\varepsilon}\left[\eta_{\mu\nu} - (1-\xi)\dfrac{k_\mu k_\nu}{k^2}\right]\delta_{ab}$

GHOSTS  $\ldots \quad \dfrac{i}{p^2 + i\varepsilon}\,\delta_{ab}$

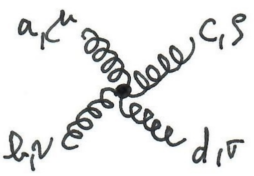
VERTICES:

 $\ldots \quad -g f_{bac}\gamma^\mu$

 $\ldots \quad g f_{bac}\, p_1^\mu$



$$\ldots \quad g f_{abc}\left[\eta^{\mu\nu}(k_1 - k_2)^\rho + \eta^{\nu\rho}(k_2 - k_3)^\mu + \eta^{\rho\mu}(k_3 - k_1)^\nu\right]$$



$$\ldots \quad -ig^2\left[f_{abe}f_{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + f_{ace}f_{bde}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + f_{ade}f_{bce}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}\right)\right]$$

WARD-TAKAHASHI IDENTITY IN QED

QED GENERATING FUNCTIONAL

$$Z[\eta,\bar\eta,J^\mu]=\int\mathcal{D}\bar\Psi\mathcal{D}\Psi\mathcal{D}A\,\exp\left\{i\left[S_0+S_{g.f.}+\int d^4x\left(\bar\Psi\eta+\bar\eta\Psi+J^\mu A_\mu\right)\right]\right\}$$

$$S_0=\int d^4x\left[\bar\Psi(i\not{D}-m)\Psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\qquad S_{g.f.}=\int d^4x\,\frac{-1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

INFINITESIMALLY SMALL GAUGE TRANSFORM

$$\underline{\begin{aligned}A^\alpha_\mu(x)&=A_\mu+\frac{1}{e}\partial_\mu\alpha(x)\\ \psi^\alpha(x)&=(1-i\alpha(x))\psi(x)\end{aligned}}$$

GAUGE SYMMETRY $S_0^\alpha=S_0$

$$S^\alpha_{g.f.}=S_{g.f.}+\int d^4x\,\frac{-1}{2\xi}\,2(\partial_\mu A^\mu)\frac{1}{e}(\partial_\nu\partial^\nu\alpha)+\mathcal{O}(\alpha^2)=[2\times\text{P.P.}]=$$

$$=S_{g.f.}-\frac{1}{\xi e}(-1)^2\int d^4x\,\alpha\,\Box\,\partial_\mu A^\mu+\mathcal{O}(\alpha^2)$$

$$Z^\alpha[\eta,\bar\eta,J^\mu]=\int\mathcal{D}\bar\Psi\mathcal{D}\Psi\mathcal{D}A\,\exp\left\{i\left[S_0+S_{g.f.}-\frac{1}{\xi e}\int d^4x\,\alpha\,\Box\,\partial_\mu A^\mu+\right.\right.$$

$$\left.\left.+\int d^4x\left((1+i\alpha)\bar\Psi\eta+\bar\eta(1-i\alpha)\Psi+\underbrace{J^\mu\left(A_\mu+\frac{1}{e}\partial_\mu\alpha\right)}_{[\text{P.P.}]}\right)\right]\right\}+\mathcal{O}(\alpha^2)=$$

$$=\int\mathcal{D}\bar\Psi\mathcal{D}\Psi\mathcal{D}A\left\{1+i\int d^4x\,\alpha\left[-\frac{1}{\xi e}\Box\,\partial_\mu A^\mu+i\bar\Psi\eta-i\bar\eta\Psi-\frac{1}{e}\partial_\mu J^\mu\right]\right\}\cdot$$

$$\cdot\exp\left\{i\left[S_0+S_{g.f.}+\int d^4x\left(\bar\Psi\eta+\bar\eta\Psi+J^\mu A_\mu\right)\right]\right\}+\mathcal{O}(\alpha^2)$$

GAUGE TRANSFORM CANNOT CHANGE THE GENERATING FUNCTIONAL

HENCE $\underbrace{Z^\alpha[\eta,\bar\eta,J^\mu]-Z[\eta,\bar\eta,J^\mu]}=0$

$$0+\mathcal{O}(\alpha^2)=\int\mathcal{D}\bar\Psi\mathcal{D}\Psi\mathcal{D}A\;i\int d^4x\,\alpha\left(-\frac{1}{\xi e}\Box\,\partial_\mu A^\mu\overbrace{+i\bar\Psi\eta}^{-i\eta\bar\Psi\;\leftarrow\;\text{GRASSMANN}}-i\bar\eta\Psi-\frac{1}{e}\partial_\mu J^\mu\right)\cdot$$

$$\cdot\exp\left\{i\left[S_0+S_{g.f.}+\int d^4x\left(\bar\Psi\eta+\bar\eta\Psi+J^\mu A_\mu\right)\right]\right\}\Longrightarrow$$

$$\Rightarrow \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}A\, i\int d^4x\, \alpha(x)\Big(-\frac{1}{\xi e}\Box\partial_\mu \frac{1}{i}\frac{\delta}{\delta J_\mu(x)} - i\eta\frac{-1}{i}\frac{\delta}{\delta\eta(x)} -$$

WHEN VARIATIONS HERE ACT ON EXPONENTIAL HERE WE OBTAIN THE EXPRESSION FROM PREVIOUS PAGE

$$-i\bar{\eta}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)} - \frac{1}{e}\partial_\mu J^\mu\Big)\exp\Big\{i\Big[S_0+S_{g.f.}+\int d^4x\,(\bar{\Psi}\eta+\bar{\eta}\Psi+J^\mu A_\mu)\Big]\Big\} =$$

INTEGRAL WITH VARIATIONS CAN BE PULLED OUT FROM THE PATH INTEGRAL

$$= i\int d^4x\,\alpha\underbrace{\Big(\frac{i}{\xi e}\Box\partial_\mu\frac{\delta}{\delta J_\mu}+\eta\frac{\delta}{\delta\eta}-\bar{\eta}\frac{\delta}{\delta\bar{\eta}}-\frac{1}{e}\partial_\mu J^\mu\Big)Z[\eta,\bar{\eta},J^\mu]}_{=0 \,\Leftarrow\, \text{FOR ALL } \alpha} \overset{\uparrow}{=} \mathcal{O}(\alpha^2)$$

WE PASS TO 1PI GENERATING FUNCTIONAL

1, $Z = e^W$ 2, LEGENDRE TRANSFORM

$$\Gamma[\Psi,\bar{\Psi},\mathcal{A}_\mu] = -i\int d^4x\Big(J^\mu\frac{\delta W}{i\delta J^\mu}+\eta\frac{\delta W}{i\delta\eta}+\bar{\eta}\frac{\delta W}{i\delta\bar{\eta}}\Big)+W[\eta,\bar{\eta},J^\mu]$$

$$\mathcal{A}_\mu = \frac{\delta W}{i\delta J^\mu} \qquad \Psi = \frac{\delta W}{i\delta\bar{\eta}} \qquad \bar{\Psi} = -\frac{\delta W}{i\delta\eta}$$

GRASSMANN

$$J^\mu = -\frac{\delta\Gamma}{i\delta\mathcal{A}_\mu} \qquad \bar{\eta} = \frac{\delta\Gamma}{i\delta\Psi} \qquad \eta = -\frac{\delta\Gamma}{i\delta\bar{\Psi}}$$

$$\frac{\delta Z}{\delta J^\mu} = e^W\frac{\delta W}{\delta J^\mu} = Zi\mathcal{A}_\mu,\ \frac{\delta Z}{\delta\eta} = -i\bar{\Psi}Z,\ \frac{\delta Z}{\delta\bar{\eta}} = i\Psi Z$$

$$\Big(\frac{i}{\xi e}\Box\partial_\mu\frac{\delta}{\delta J_\mu}+\eta\frac{\delta}{\delta\eta}-\bar{\eta}\frac{\delta}{\delta\bar{\eta}}-\frac{1}{e}\partial_\mu J^\mu\Big)Z = 0$$

$$\frac{i}{\xi e}\Box\partial_\mu i\mathcal{A}^\mu Z+\Big(-\frac{\delta\Gamma}{i\delta\bar{\Psi}}\Big)(-i\bar{\Psi}Z)-\frac{\delta\Gamma}{i\delta\Psi}i\Psi Z -$$

$$-\frac{1}{e}\partial_\mu\Big(-\frac{\delta\Gamma}{i\delta\mathcal{A}_\mu}\Big)Z = 0$$

$$-\frac{1}{\xi e}\Box\partial_\mu\mathcal{A}^\mu+\frac{\delta\Gamma}{\delta\bar{\Psi}}\bar{\Psi}-\frac{\delta\Gamma}{\delta\Psi}\Psi-\frac{i}{e}\partial_\mu\frac{\delta\Gamma}{\delta\mathcal{A}_\mu} = 0$$

$$-\frac{1}{\xi e}\Box_{(x)}\partial^{(x)}_{\mu}A^{\mu}(x)+\frac{\delta\Gamma}{\delta\bar{\Psi}(x)}\bar{\Psi}(x)-\frac{\delta\Gamma}{\delta\Psi(x)}\Psi(x)-\frac{i}{e}\partial^{(x)}_{\mu}\frac{\delta\Gamma}{\delta A_{\mu}(x)}=0$$

$$\Big/\frac{\delta}{\delta\bar{\Psi}(y)}\frac{\delta}{\delta\Psi(z)}$$

$$\frac{\delta}{\delta\bar{\Psi}(y)}\frac{\delta}{\delta\Psi(z)}\left(\frac{\delta\Gamma}{\delta\bar{\Psi}(x)}\bar{\Psi}(x)-\frac{\delta\Gamma}{\delta\Psi(x)}\Psi(x)\right)-\frac{i}{e}\frac{\partial}{\partial x^{\mu}}\frac{\delta^{3}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(z)\delta A_{\mu}(x)}=0$$

GRASSMANN

$$\frac{\delta^{2}\Gamma}{\delta\Psi(z)\delta\bar{\Psi}(x)}(-1)^{2}\frac{\delta}{\delta\bar{\Psi}(y)}\bar{\Psi}(x)-\frac{\delta^{2}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(x)}(-1)\frac{\delta}{\delta\Psi(z)}\Psi(x)+$$

$$+\left[\text{TERMS WITH }\frac{\delta^{3}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(z)\delta\bar{\Psi}(x)}=0,\ \frac{\delta^{3}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(z)\delta\Psi(x)}=0\right]-$$

$$-\frac{i}{e}\frac{\partial}{\partial x^{\mu}}\frac{\delta^{3}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(z)\delta A_{\mu}(x)}=0\qquad\left[\frac{\delta\bar{\Psi}(x)}{\delta\bar{\Psi}(y)}=\delta(x-y),\ \frac{\delta\Psi(x)}{\delta\Psi(z)}=\delta(x-z)\right]$$

$$\frac{\delta^{2}\Gamma}{\delta\Psi(z)\delta\bar{\Psi}(x)}\delta(x-y)+\frac{\delta^{2}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(x)}\delta(x-z)-\frac{i}{e}\frac{\partial}{\partial x^{\mu}}\frac{\delta^{3}\Gamma}{\delta\bar{\Psi}(y)\delta\Psi(z)\delta A_{\mu}(x)}=0$$

1PI 2&3-POINT FUNCTIONS

$$\Gamma^{(2)}_{\bar{\Psi}\Psi}(x,y)\equiv\frac{\delta}{\delta\bar{\Psi}(x)}(-1)\frac{\delta}{\delta\Psi(y)}\Gamma[\Psi,\bar{\Psi},A_{\mu}]\Big|_{\Psi=0,\bar{\Psi}=0,A_{\mu}=0}$$

$$\Gamma^{(3)\mu}_{A\bar{\Psi}\Psi}(x,y,z)\equiv\frac{\delta}{\delta A_{\mu}(x)}\frac{\delta}{\delta\bar{\Psi}(y)}(-1)\frac{\delta}{\delta\Psi(z)}\Gamma[\Psi,\bar{\Psi},A_{\mu}]\Big|_{\Psi=0,\bar{\Psi}=0,A_{\mu}=0}$$

$$-(-1)\Gamma^{(2)}_{\bar{\Psi}\Psi}(x,z)\delta(x-y)-\Gamma^{(2)}_{\bar{\Psi}\Psi}(y,x)\delta(x-z)-\frac{i}{e}\frac{\partial}{\partial x^{\mu}}(-1)\Gamma^{(3)\mu}_{A\bar{\Psi}\Psi}(x,y,z)=0$$

FOURIER IMAGE

$$\int\frac{d^{4}p_{1}}{(2\pi)^{4}}e^{-ip_{1}\cdot(x-y)}\int\frac{d^{4}p_{2}}{(2\pi)^{4}}\int\frac{d^{4}p_{3}}{(2\pi)^{4}}\Gamma^{(2)}_{\bar{\Psi}\Psi}(p_{2},p_{3})e^{-ip_{2}\cdot x}e^{-ip_{3}\cdot z}-$$

$$-\int\frac{d^{4}q_{1}}{(2\pi)^{4}}e^{-iq_{1}\cdot(x-z)}\int\frac{d^{4}q_{2}}{(2\pi)^{4}}\int\frac{d^{4}q_{3}}{(2\pi)^{4}}\Gamma^{(2)}_{\bar{\Psi}\Psi}(q_{2},q_{3})e^{-iq_{2}\cdot y}e^{-iq_{3}\cdot x}+$$

$$+\frac{i}{e}\frac{\partial}{\partial x^{\mu}}\int\frac{d^{4}k_{1}}{(2\pi)^{4}}\int\frac{d^{4}k_{2}}{(2\pi)^{4}}\int\frac{d^{4}k_{3}}{(2\pi)^{4}}\Gamma^{(3)\mu}_{A\bar{\Psi}\Psi}(k_{1},k_{2},k_{3})e^{-ik_{1}\cdot x}e^{-ik_{2}\cdot y}e^{-ik_{3}\cdot z}=0$$

$$\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\,\Gamma^{(2)}_{\bar{\Psi}\Psi}(p_2,p_3)\,e^{-i[(p_1+p_2)\cdot x-p_1\cdot y+p_3\cdot z]}\;-$$

$$-\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\,\Gamma^{(2)}_{\bar{\Psi}\Psi}(q_2,q_3)\,e^{-i[(q_1+q_3)\cdot x+q_2\cdot y-q_1\cdot z]}\;+$$

$$+\frac{i}{e}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}\,\Gamma^{(3)}_{A\bar{\Psi}\Psi}(k_1,k_2,k_3)^{\mu}(-ik_{1\mu})\,e^{-ik_1\cdot x-ik_2\cdot y-ik_3\cdot z}=0$$

$$\left\{\begin{array}{l} p_1+p_2=k_1\;,\;-p_1=k_2\;,\;p_3=k_3\;;\;q_1+q_3=k_1\;,\;q_2=k_2\;,\;-q_1=k_3 \\ p_1=-k_2\;,\;p_2=k_1+k_2\;,\;p_3=k_3\;;\;q_1=-k_3\;,\;q_2=k_2\;,\;q_3=k_1+k_3 \end{array}\right\}$$

$$\Gamma^{(2)}_{\bar{\Psi}\Psi}(k_1+k_2,k_3)-\Gamma^{(2)}_{\bar{\Psi}\Psi}(k_2,k_1+k_3)+\frac{1}{e}k_{1\mu}\Gamma^{(3)}_{A\bar{\Psi}\Psi}(k_1,k_2,k_3)^{\mu}=0$$

AFTER USING CONSERVATION OF 4-MOMENTA $k_1+k_2+k_3=0$

$$\frac{1}{e}k_{1\mu}\Gamma^{(3)}_{A\bar{\Psi}\Psi}(k_1,k_2,\bullet)^{\mu}=\Gamma^{(2)}_{\bar{\Psi}\Psi}(k_2,\bullet)-\Gamma^{(2)}_{\bar{\Psi}\Psi}(k_1+k_2,\bullet)$$