Phases of fuzzy field theories

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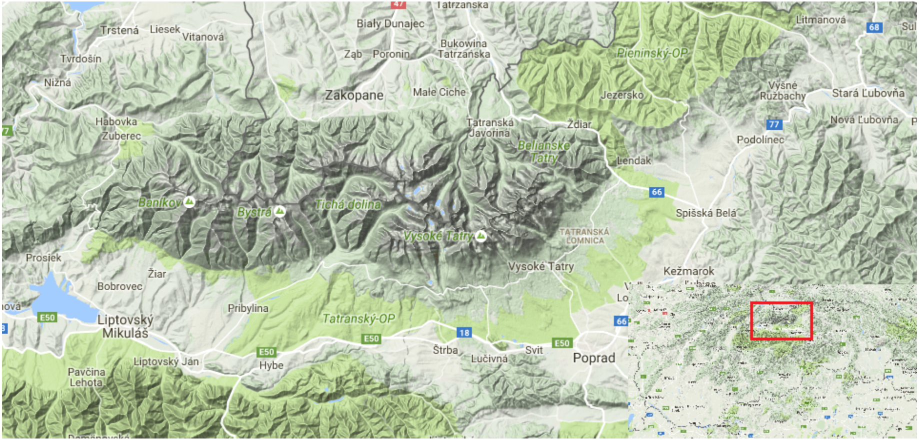
Quantum Structure of Spacetime

Physics in fuzzy spaces, Stará Lesná, 26.9.2016

[1510.07496 [hep-th]], [1512.00689 [hep-th]], [1601.05628 [hep-th]], work in progress



Tatranská
Biały Dunajec
Litmanová
Trstená
Liesek
Vitanová
Bukowina Tatrzańska
Ząb
Poronin
Pieninský-OP
Tvrdošín
Nižná
Zakopane
Małe Ciche
Jezersko
Vyšné Ružbachy
Stará Ľubovňa
Habovka
Zuberec
Tatranská Javorina
Ždiar
Nová Ľubovňa
Podolínec
Belianske Tatry
Lendak
Baníkov
Bystrá
Tichá dolina
Spišská Belá
Vysoké Tatry
TATRANSKÁ LOMNICA
Prosiek
Žiar
Bobrovec
Kežmarok
Pribylina
Tatranský-OP
Liptovský Mikuláš
Štrba
Lučivná
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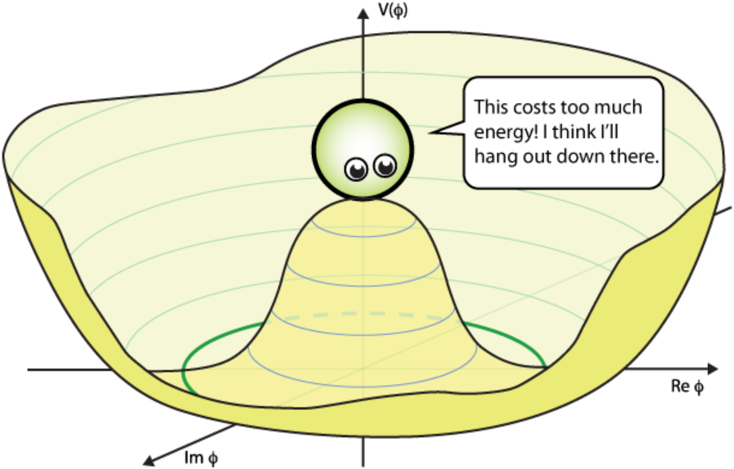
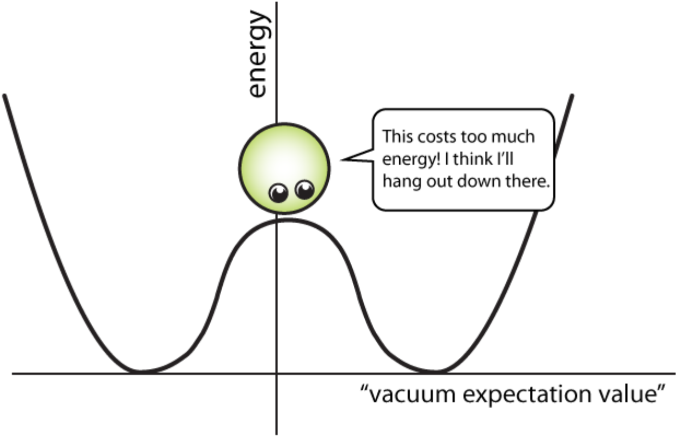
The ABEGHHK'tH mechanism



Outline

- What do we mean by phase structure and why is it important?
- Lightning fast recapitulation of fuzzy field theories.
- Matrix model description of fuzzy field theories.
- Implications for their phase structure.

Real scalar ϕ^4 field on plane



Real scalar ϕ^4 field on plane

- The commutative field theory has two phases in the phase diagram, disorder and uniform order phases.
  Glimm, Jaffe '74; Glimm, Jaffe, Spencer '75; Chang '76
  Loinaz, Willey '98; Schaich, Loinaz '09
- In disorder phase the field oscillates around the value $\phi = 0$.
- In uniform order phase the field oscillates around the a nonzero value which minimim of the potential.

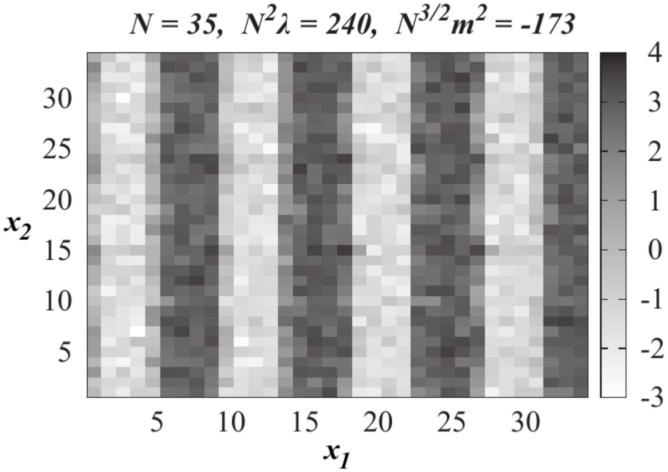
Real scalar ϕ^4 field on plane

Real scalar ϕ^4 field noncommutative theories

- The phase diagram of noncommutative field theories has one more phase in the phase diagram. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; G.-H. Chen and Y.-S. Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; W. Bietenholz, F. Hofheinz, H. Mejía-Díaz, M. Panero '14; H. Mejía-Díaz, W. Bietenholz, M. Panero '14; J. Medina, W. Bietenholz, D. O'Connor '07

Real noncommutative scalar ϕ^4 field on plane

Mejía-Díaz, Bietenholz, Panero '14



UV/IR mixing

- This phase is a result of the nonlocality of the theory. Mermin-Wagner Theorem : no spontaneous symmetry breaking in local 2-dimensional theories.
- This phase survives the commutative limit of the noncommutative theory! Result of the UV/IR mixing.
- The trademark property of the noncommutative field theories, which arises as a consequence of the non-locality of the theory. Causes interplay of long and short distance phenomena. Minwalla, Van Raamsdonk, Seiberg '00; Chu, Madore, Steinacker '01
- The commutative limit of such noncommutative theory is (very) different than the commutative theory we started with.

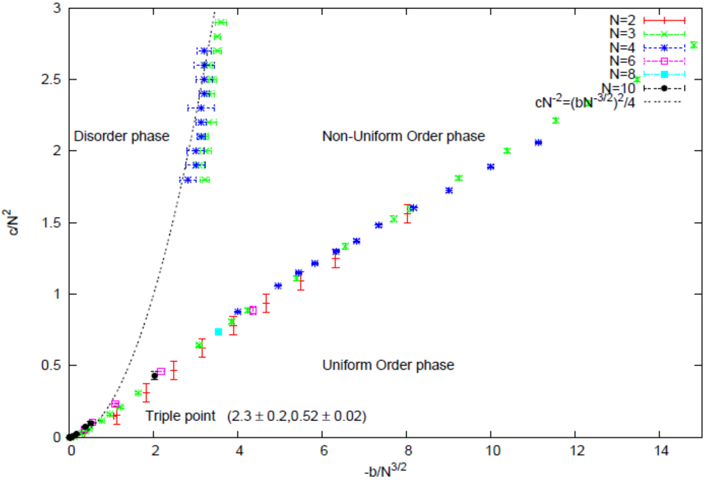
UV/IR mixing

- For a noncommutative theory with no UV/IR mixing, the extra phase should not be present in the commutative limit of the phase diagram.
    - B.P. Dolan, D. O'Connor and P. Prešnajder [arXiv:0109084],
    - H. Grosse and R. Wulkenhaar [arXiv:0401128],
    - R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa [arXiv:0802.0791].
- Understanding the phase diagram of such theories, especially mechanism of the departure the striped phase could teach us a lot technically and conceptually.
- We need tools to analyze the phase diagrams of the noncommutative theories. A good idea is to start with the simplest case.

Phase diagram for ϕ^4 on S_F^2

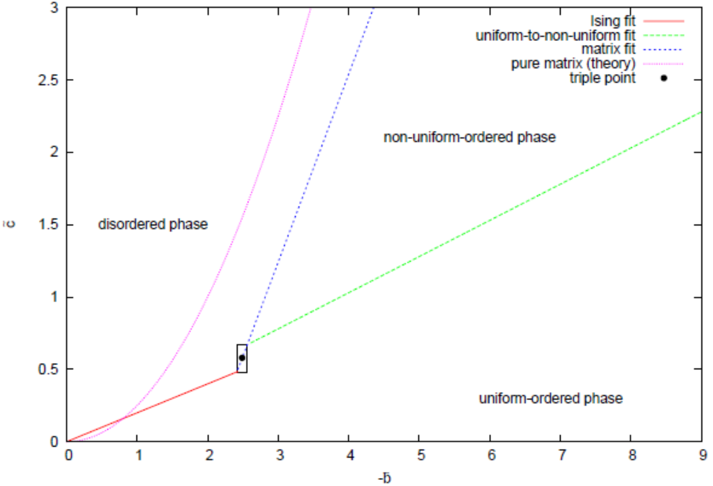
Numerical phase diagram for S_F^2

For the fuzzy sphere, the following is numerically obtained phase diagram
García Flores, Martin, O'Connor '09



Numerical phase diagram for S^2_F

For the fuzzy sphere, the following is numerically obtained phase diagram
Ydri '14



Numerical phase diagram for S_F^2

(Published) Numerical results agree on the critical value of the coupling

$$g_c \approx (0.12, 0.17)$$

The scalar field theory on a fuzzy space is a multitrace matrix model.

The scalar field theory on **a fuzzy space** is a multi-trace matrix model.

Fuzzy sphere Madore '92; Grosse, Klimcik, Presnajder '90s

- The fuzzy sphere S^2_F is a space, which has the algebra of functions generated by

$$x_i x_i = \rho^2 \quad , \quad x_i x_j - x_j x_i = i\theta\varepsilon_{ijk} x_k$$

- This can be realized as a $N = 2j + 1$ dimensional representation of the $SU(2)$

$$x_i = \frac{2r}{\sqrt{N^2 - 1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2 - 1}} \quad , \quad \rho^2 = \frac{4r^2}{N^2 - 1} j(j+1) = r^2$$

- The coordinates x_i still carry an action of $SU(2)$ and thus the space still has the symmetry of the sphere.
- Limit of large N reproduces the original sphere S^2.
- x_i are $N \times N$ matrices and functions on S^2_F are combinations of products $\rightarrow$ a herminitian matrix M

Fuzzy sphere

- Finite mode approximation to the regular round sphere S^2.
- Introduces minimal length (minimal area).
- Still posses the original symmetry.
- Functions become $N \times N$-matrices, where

$$\theta \sim \frac{1}{N} \ .$$

- Derivatives $\rightarrow$ commutators with L_i.
  Integrals $\rightarrow$ traces.

The scalar field theory on a fuzzy space is a multi-trace matrix model.

Scalar field on S_F^2

- We can write an Euclidean field theory action

$$S(M) = -\frac{4\pi R^2}{N}\mathrm{Tr}\left(\frac{1}{2R^2}[L_i, M][L_i, M] + \frac{1}{2}rM^2 + V(M)\right) =$$
$$= \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + V(M)\right)$$

- The theory is given by functional correlation functions

$$\langle F \rangle = \frac{\int dM\, F(M)e^{-S(M)}}{\int dM\, e^{-S(M)}}$$

Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03

The scalar field theory on a fuzzy space is a multitrace **matrix model.**

Hermitian matrix model

- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}r\mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

- What kind of matrix model is this? We diagonalize $M = U\Lambda U^\dagger$ for some $U \in SU(N)$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, the integration measure becomes

$$dM = dU\left(\prod_{i=1}^N d\lambda_i\right) \times \prod_{i<j}(\lambda_i - \lambda_j)^2$$

and we are to compute integrals like

$$\langle F\rangle \sim \int\left(\prod_{i=1}^N d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[\frac{1}{2}r\sum\lambda_i^2 + g\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$\times\int dU\, e^{-N^2\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

(minus the red Brezin, Itzykson, Parisi, Zuber '78)

Hermitian matrix model

- Fuzzy field theory = matrix model with

$$S(M)=\frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right)+\frac{1}{2}r\mathrm{Tr}\left(M^2\right)+g\mathrm{Tr}\left(M^4\right)$$

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and we are to compute integrals like

$$\langle F\rangle\sim\int\left(\prod_{i=1}^N d\lambda_i\right)F(\lambda_i)\,e^{-N^2\left[S_{eff}(\lambda_i)+\frac{1}{2}r\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$e^{-N^2S_{eff}(\lambda_i)}=\int dU\,e^{-N^2\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Steinacker '05

The scalar field theory on a fuzzy space is a **multitrace** matrix model.

Multitraces in the effective action

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. O'Connor, Sämann '07; Sämann '10

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2 \varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- The most recent result is Sämann '15

$$\begin{aligned} S_{eff}(M) =& \frac{1}{2}\left[\varepsilon \frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2 \frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4 \frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ & - \varepsilon^4 \frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ & - \varepsilon^3 \frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \end{aligned}$$

where

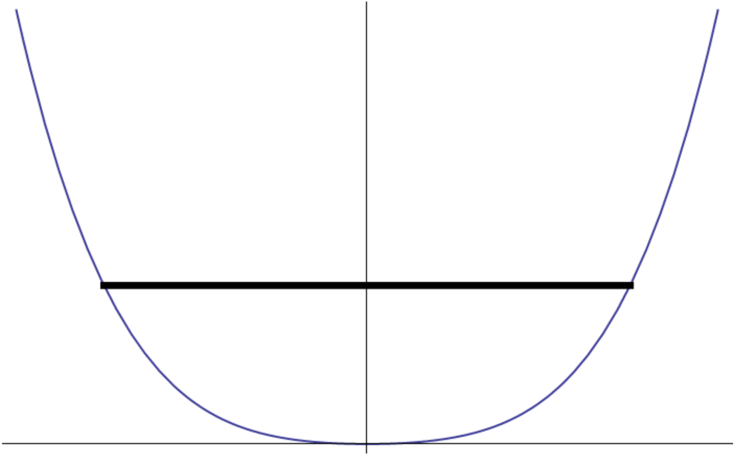
$$c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

Some basics of single trace matrix models

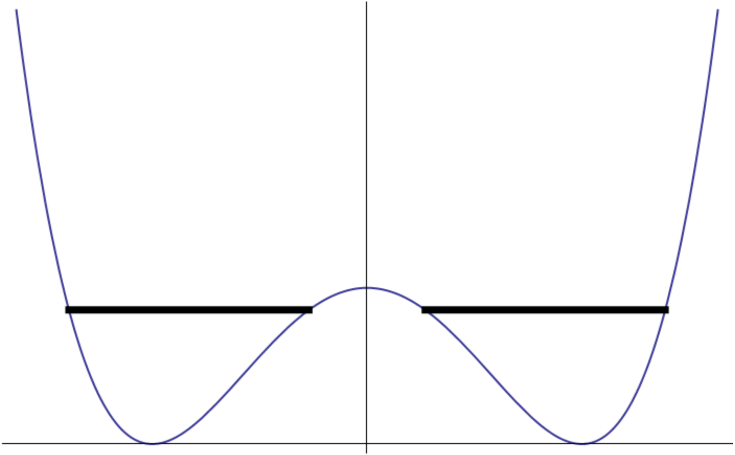
Single trace matrix models

- Coulomb gas description.
- Eigenvalue repulsion.

Single trace matrix models



Single trace matrix models

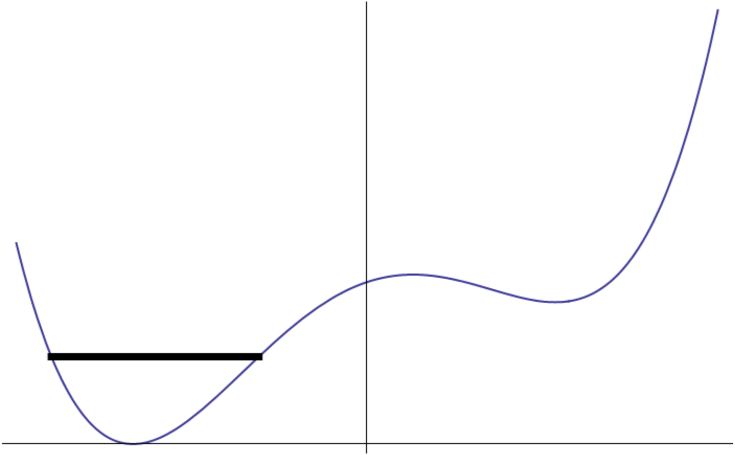


Some basics of asymmetric single trace matrix models

Asymmetric single trace matrix models

- Nothing new in principle, just equations not solvable analytically.

Asymmetric single trace matrix models

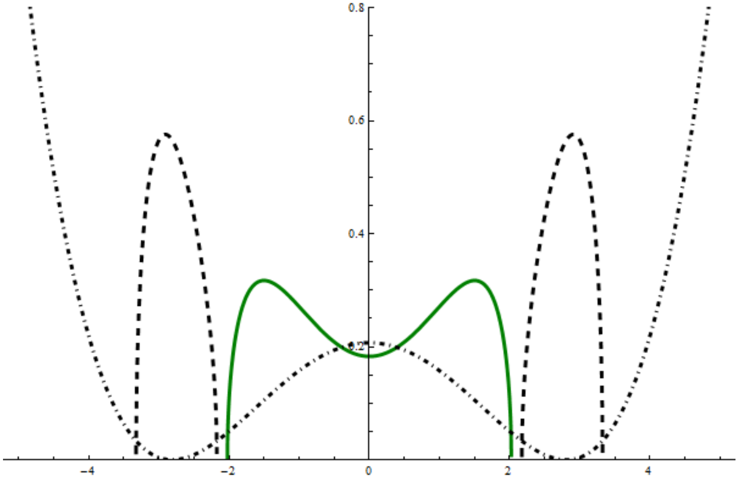


Some basics of multitrace matrix models

Some basics of multitrace matrix models

- Further interaction among the eigenvalues.

Some basics of multitrace matrix models



$g = 0.1, r = -3.152$

Perturbative model

Perturbative effective action

- The model

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon \frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2 \frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4 \frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4 \frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3 \frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

Perturbative effective action

- The model

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

  is not good. (Or at least its symmetric regime)

- Self interaction which is introduced is way too strong in the important region.

Non-perturbative considerations I

Effective action

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2}F(c_2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}$$

- Recall the perturbative action

$$\begin{aligned} S_{eff}(M) =& \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ & - \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ & - \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \end{aligned}$$

these are exactly the first terms of the small c_2 expansion with $c_2 \to c_2 - c_1^2$.

Effective action

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. Steinacker '05
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$$S_{eff} = \frac{1}{2}F(c_2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}$$

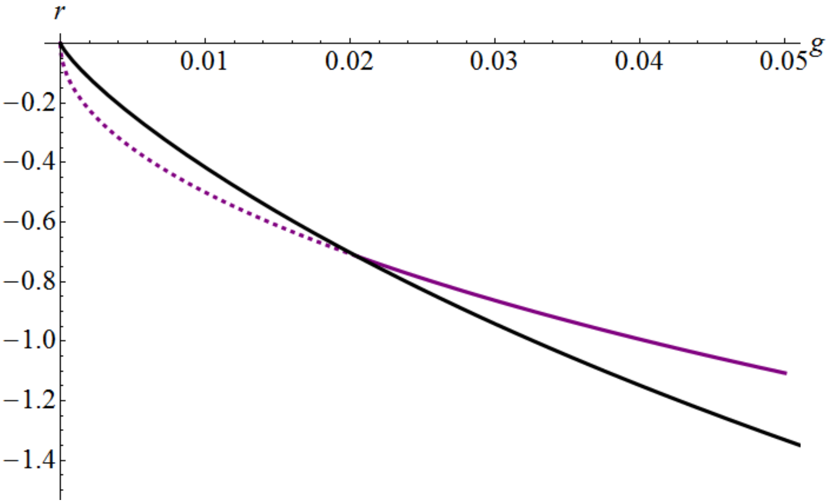
- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ and interaction we obtain a matrix model

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}\mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right) \quad , \quad F(t) = \log\left(\frac{t}{1-e^{-t}}\right)$$

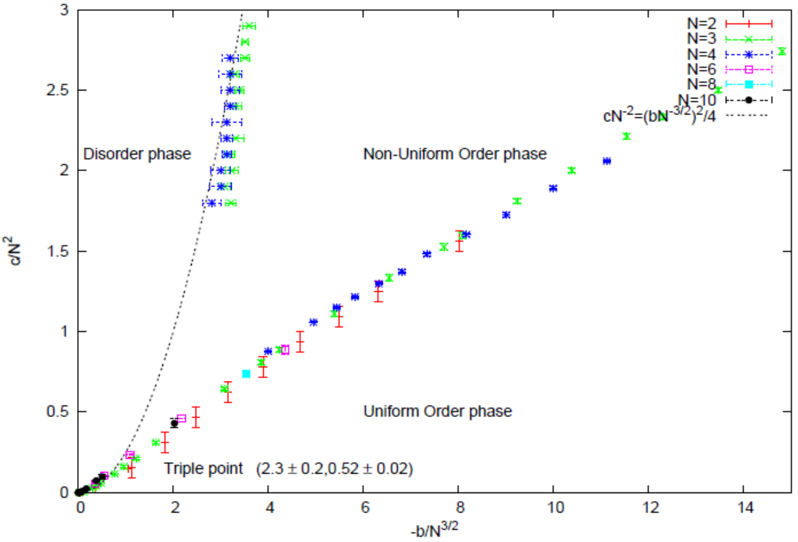
Polychronakos '13; JT '14, JT '15

Phase diagram

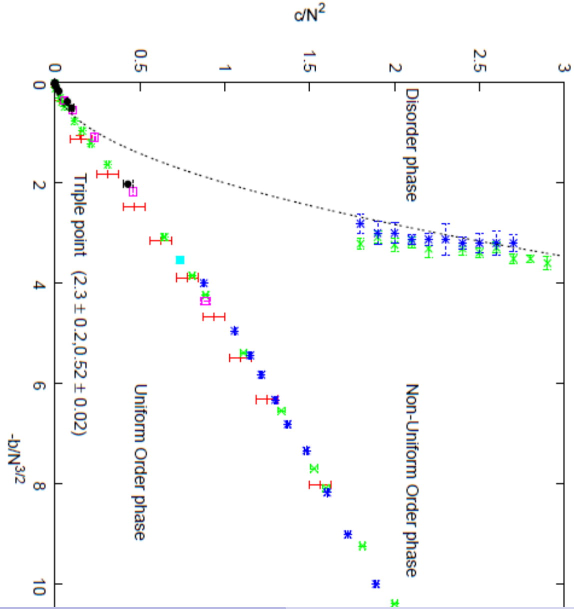
The phase diagram



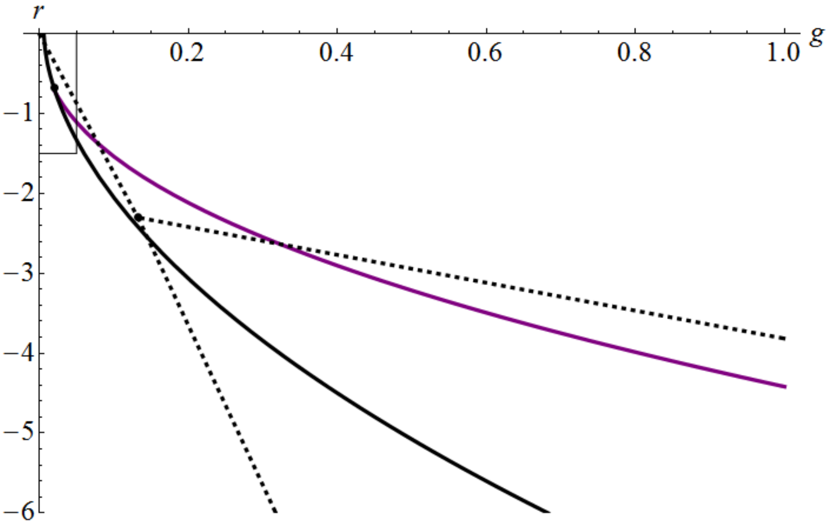
Phase diagram



Phase diagram



Phase diagram



Non-perturbative considerations II

Beyond the second moment

- To progress, we need somehow include further terms from the perturbative effective action

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

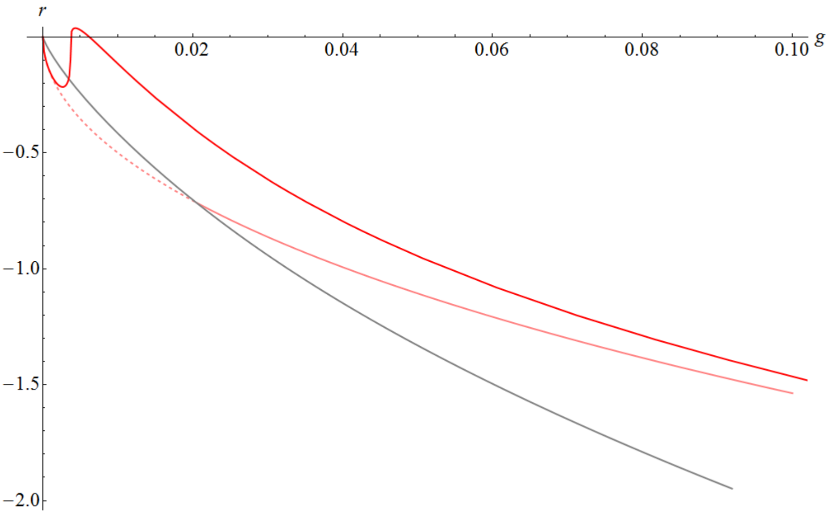
in a way that is well behaved close to the triple point.

- There is a class of models which does this consistently with the new numerical data. JT '16

$$S_{eff}(M) = \frac{1}{2}F(c_2) + \left[1 + \frac{n(c_4 - 2c_2^2)^2}{3456}\right]^{-1/n} - 1$$

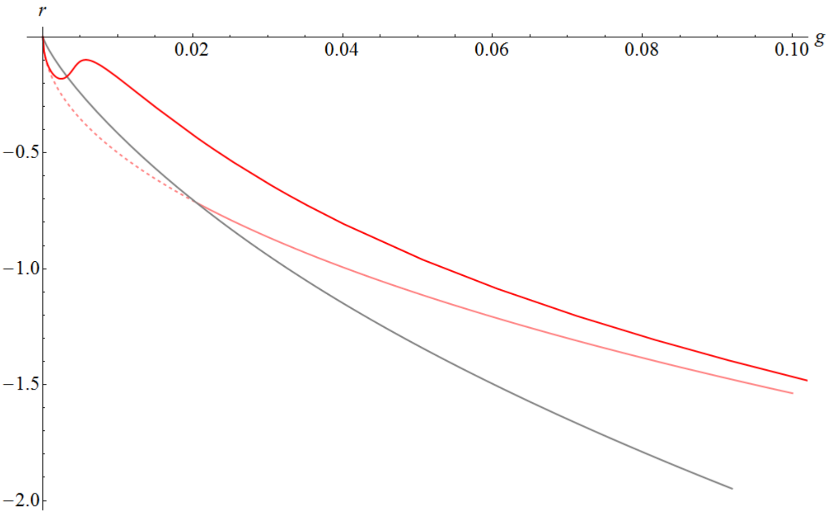
Beyond the second moment

Symmetric regime for $n = 1$



Beyond the second moment

Symmetric regime for $n = 2$



Renormalizable (UV/IR-free) theory

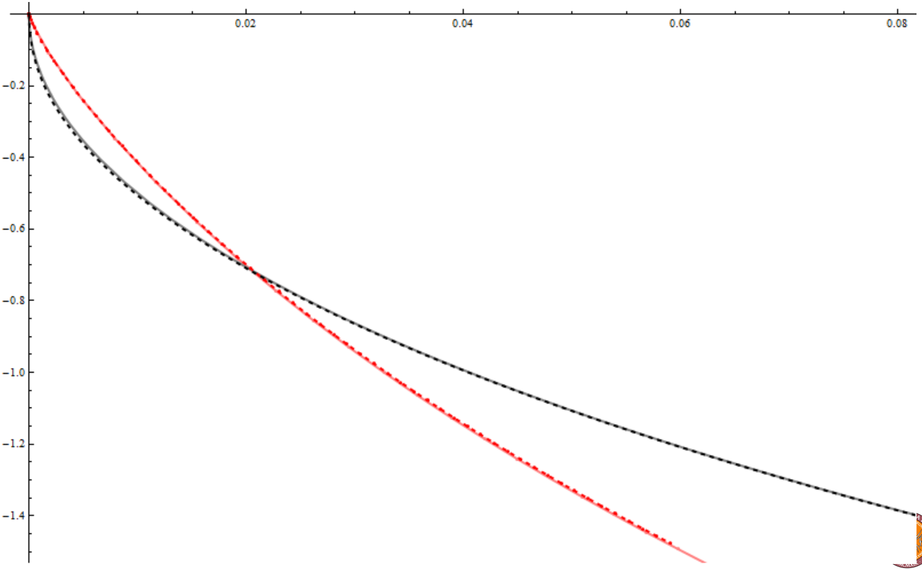
Renormalizable (UV/IR-free) theory

- It has been suggested that models with kinetic term

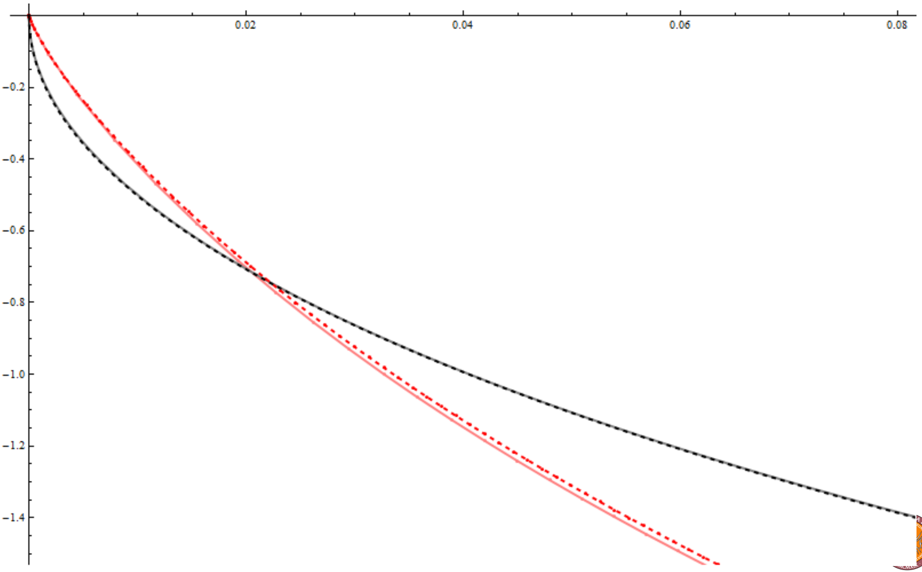
$$\frac{1}{2}MC_2M + aMC_2C_2M \ , \ \frac{1}{2}MC_2M + agMC_2C_2M$$

  with changing parameter a could represent a UV/IR-free theory on S_F^2. O'Connor, Sämann '07.
- This modifies the effective action F in the presented approach.
- Some preliminary results look promising, extra term moves the triple point away from the origin.

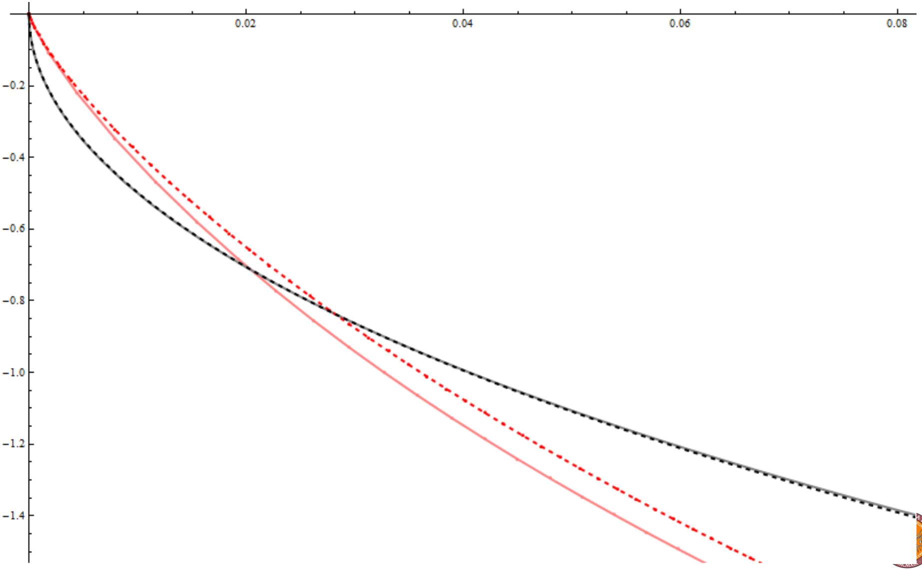
Renormalizable (UV/IR-free) theory



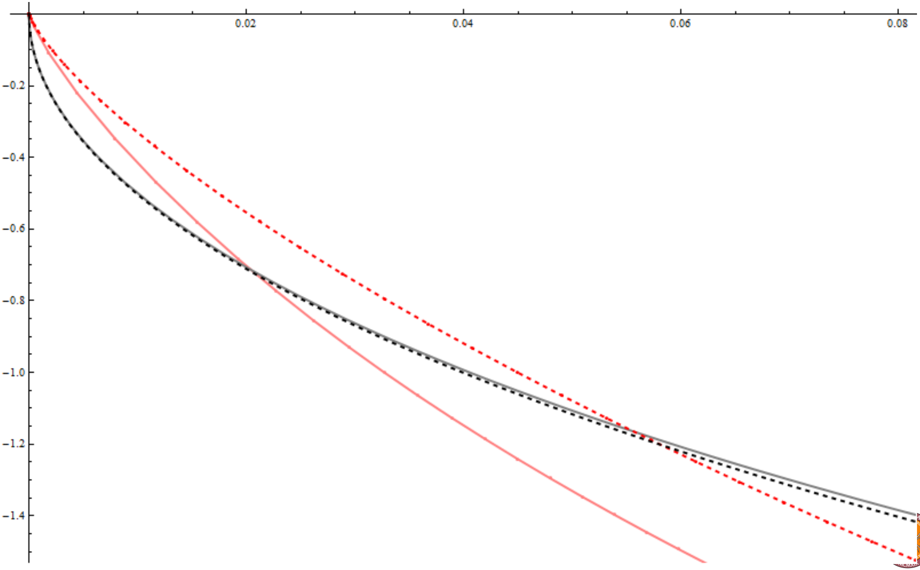
Renormalizable (UV/IR-free) theory



Renormalizable (UV/IR-free) theory



Renormalizable (UV/IR-free) theory



Conclusion and outlook

Matrix models are a powerful tool to analyze the phase structure of fuzzy field theories, which is an appealing window into their properties.

As we have seen, interesting questions left to be answered include

- Finding a way how to evaluate the angular integral.
- Investigating the phase structure of the UV/IR-free theories analytically, mainly beyond S_F^2.
- Investigating the phase structure of the UV/IR-free theories numerically.

Thank you for your attention!