Teória skalárneho poľa na nekomutatívnej sfére

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Introduction and outline

quantum theory + general relativity

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?!?!?!

Introduction and outline

quantum theory + general relativity

$\Downarrow$

some nontrivial short distance structure of space

Doplicher, Fredenhagen, Robersts '95

Introduction and outline

- To measure an event of spatial extent Δx, we need a particle with a similar wavelength. According to de Broglie, this particle has energy

$$E \sim \frac{1}{\Delta x} \ .$$

- As we lower Δx beyond a certain point, the concentration of energy will create a black hole. The result of the measurement will be hidden under the event horizon of this black hole and we can not obtain the information we were after.
- Rather unsurprisingly, this will happen at the Planck scale

$$R_S = \frac{2GM}{c^2} \ , \ E = Mc^2 \ , \ E = \frac{hc}{\lambda} \ \Rightarrow \ L = \sqrt{2} \underbrace{\sqrt{\frac{hG}{c^3}}}_{l_{pl}} \ .$$

Introduction and outline

quantum theory + general relativity

$\Downarrow$

some nontrivial short distance structure of space

$\Uparrow$

space noncommutatitvity

Introduction and outline

In this short talk, we will

- describe the construction of the fuzzy sphere S_F^2, as the trademark example of a noncommutative space,
- show how the noncommutativity introduces a short distance structure without any loss of symmetry of the space,
- mention some most interesting properties of scalar field theory defined on such spaces,
- conclude with other applications of noncommutative spaces in physics.

Fuzzy sphere

Fuzzy sphere

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

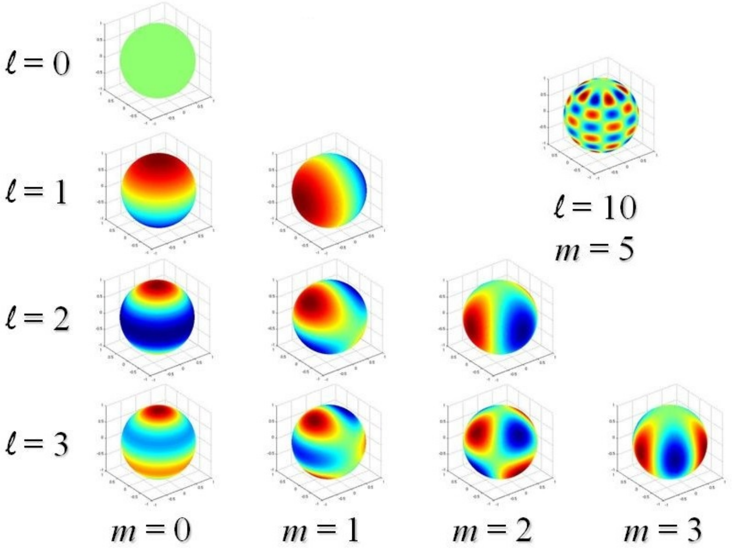
where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- They form an (commutative) algebra and all the information about the sphere is encoded in this algebra (Gelfand, Naimark).
- To describe features at a length scale δx we need functions Y_m^l with

$$l\sim\frac{1}{\delta x}\ .$$

Fuzzy sphere



Obrazok z http://principles.ou.edu/mag/earth.html

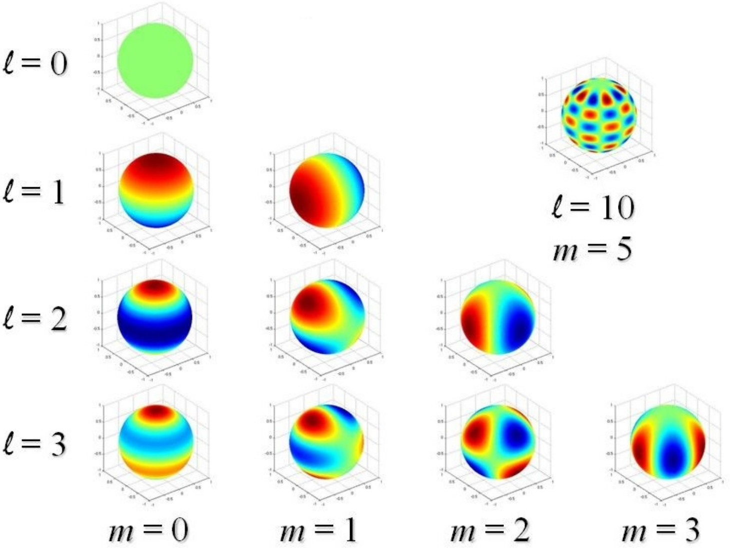
Fuzzy sphere

- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

  we will not be able to see any features of functions at length scales under $\sim 1/L$.
- Points on the sphere (as δ-functions) ceased to exist.
- The problem is, that the expressions defined in this way are not closed under multiplication.

Fuzzy sphere



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Fuzzy sphere

- Number of independent functions with $l \leq L$ is

$$\sum_{l=0}^{L} \sum_{m=-l}^{l} 1 = (L+1)^2 = N^2 \ .$$

This is the same as the number of $N \times N$ hermitian matrices

$$N + 2\sum_{n=1}^{N} (n-1) = N^2 \ .$$

The idea is to map the former on the latter and borrow a closed product from there.

- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \dots \end{array}$$

Fuzzy sphere

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} * Y_{l'm'} := \varphi^{-1} \left(\varphi \left(Y_{lm} \right) \varphi \left(Y_{l'm'} \right) \right) \ .$$

- We have obtained a short distance structure, but the prize we had to pay was a noncommutative product $*$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full roational symmetry.
- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy sphere - an alternative construction

Fuzzy sphere

- The regualar sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ ,$$

which generate the following algebra of functions

$$f = \left\{ \sum_{k \in \mathbb{N}^3} \left(a_{k_1 k_2 k_3} \prod_{i=1}^{3} x_i^{k_i} \right) \Big| x_i x_i = R^2 \right\} \ ,$$

which is by definition commutative.

- Information about the sphere is again hidden in this algebra.

Fuzzy sphere

- For the fuzzy sphere S_F^2 we define

$$x_i x_i = \rho^2 \quad , \quad x_i x_j - x_j x_i = i\theta\varepsilon_{ijk} x_k \ .$$

- Such x_i's generate a different, noncommutative algebra and S_F^2 is an object, which has this algebra as an algebra of functions.
- The conditions can be realized as an $N = 2j+1$ dimensional representation of $SU(2)$

$$x_i = \frac{2r}{\sqrt{N^2-1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1} j(j+1) = r^2 \ .$$

- The group $SU(2)$ still acts on x_i's and this space enjoys a full rotational symmetry.
- And again, in the limit $N \to \infty$ we recover the original sphere.

Fuzzy sphere

- x_i's are $N\times N$ matrices, functions on S_F^2 are combinations of all their possible products and thus hermitian matrices M.
- Such $N\times N$ matrix can be decomposed into

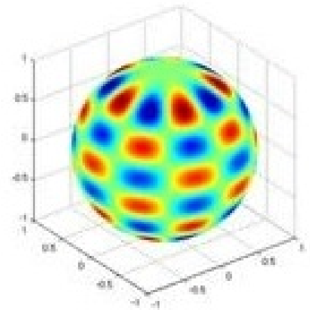
$$M=\sum_{l=0}^{N-1}\sum_{m=-l}^{m=l}c_{lm}T_{lm}\ .$$

where matrices T^l_m are called polarization tensors and

$$\begin{aligned}T^l_m &= \varphi(Y^l_m)\ ,\\ \mathrm{Tr}\,(T_{lm}T_{l'm'}) &= \delta_{ll'}\delta_{mm'}\ ,\\ [L_i,[L_i,T_{lm}]] &= l(l+1)T_{lm}\ .\end{aligned}$$

Fuzzy sphere - conclusion

- Either way, we have divided the sphere into N cells. Function on the fuzzy sphere is given by a matrix M and the eigenvalues of M represent the values of the function on these cells.



- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Noncommutative spaces

- Noncommutative spaces are (by definition) spaces, which correspond to noncommutative algebras

$$[x_i, x_j] = i\Theta_{ij} \ .$$

- The choice of Θ determines the space, $|\Theta|$ determines the scale of the spatial structure.
- Recall the phase space of the QM.

Scalar field on S_F^2

Scalar field on S_F^2

- In an analogy to the commutative theory we define a scalar field action

$$S(M) = -\frac{4\pi R^2}{N}\mathrm{Tr}\left(\frac{1}{2R^2}[L_i, M][L_i, M] + \frac{1}{2}rM^2 + V(M)\right) =$$
$$= \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + V(M)\right)$$

and the theory is given by the functional correlation functions

$$\langle F\rangle = \frac{\int dM\, F(M)e^{-S(M)}}{\int dM\, e^{-S(M)}}\ .$$

Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03

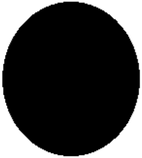
- One can compute these using the noncommutative version of the Feynmnan diagramatic rules with the propagator

$$\langle c_{lm}c_{l'm'}\rangle = \frac{\delta_{ll'}\delta_{mm'}}{l(l+1)+r}\ .$$

UV/IR mixing

- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.

  Minwalla, Van Raamsdonk, Seiberg '00; Chu, Madore, Steinacker '01
- Quanta can not be compressed into an arbitrarily small volume. If we try to squeze a packet in one direction, it will spread out in a different one. Processes with large momentum contribute to processes at small momentum.



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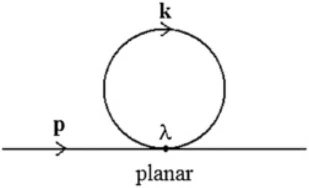
UV/IR mixing

- In terms of diagrams different properties of planar and non-planar ones.



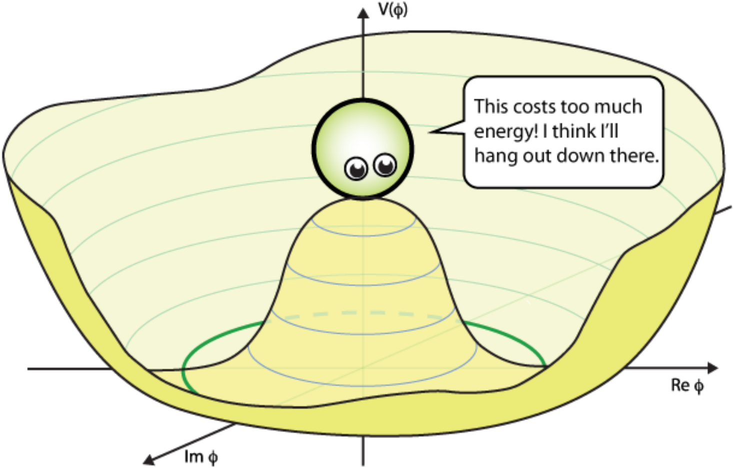
- There is no clear separation of scales and the theory is no longer renormalizable.
- This effect survives the commutative limit.

UV/IR mixing

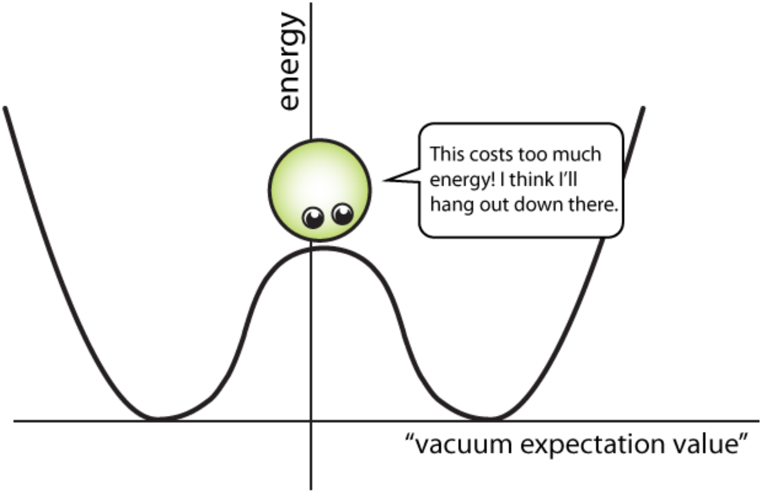
- The commutative limit of a noncommutative theory is very different from the commutative theory we started with.
- The space (geometry) forgets where it came from but the field theory (physics) remembers its fuzzy origin.

Spontaneous symmetry breaking in NC theories

The ABEGHHK'tH mechanism



Real scalar ϕ^4 field on plane



Real scalar ϕ^4 field on plane

- The commutative field theory has two phases in the phase diagram, disorder and uniform order phases.
  Glimm, Jaffe '74; Glimm, Jaffe, Spencer '75; Chang '76
  Loinaz, Willey '98; Schaich, Loinaz '09
- In disorder phase the field oscillates around the value $\phi = 0$.
- In uniform order phase the field oscillates around the a nonzero value which minimim of the potential.

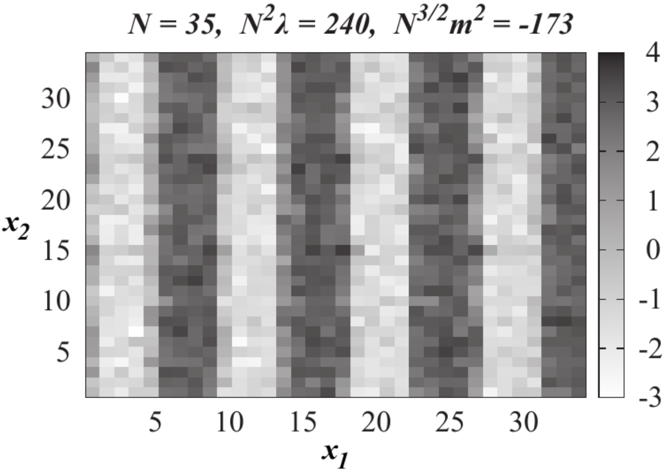
Real scalar ϕ^4 field on plane

Real scalar ϕ^4 field noncommutative theories

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; G.-H. Chen and Y.-S. Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; W. Bietenholz, F. Hofheinz, H. Mejía-Díaz, M. Panero '14; H. Mejía-Díaz, W. Bietenholz, M. Panero '14; J. Medina, W. Bietenholz, D. O'Connor '07

Real noncommutative scalar ϕ^4 field on plane

Mejía-Díaz, Bietenholz, Panero '14



UV/IR mixing

- This phase is a result of the nonlocality of the theory.
  Mermin-Wagner Theorem : no spontaneous symmetry breaking in local 2-dimensional theories.
- This phase survives the commutative limit of the noncommutative theory! Result of the UV/IR mixing.
- The commutative limit of such noncommutative theory is (very) different than the commutative theory we started with.

Noncommutative spaces in physics

Noncommutative spaces in physics

- Regularization of infinities in the standard QFT.
  Heisenberg ∼'30; Snyder '47, Yang '47
- Regularization of field theories for numerical simulations.
  Panero '16
- An effective description of the open string dynamics in a magnetic background in the low energy limit.
  Seiberg Witten '99; Douglas, Nekrasov '01
- Solutions of various matrix formulations of the string theory.
  Abe '01
- Geometric unification of the particle physics and theory of gravity.
  van Suijlekom '15
- An effective description of various systems in a certain limit (eg. QHE).
  Karabali, Nair '06

Conclusions

In this short talk, we have

- described the construction of the fuzzy sphere S_F^2,
- shown how the noncommutativity introduced a short distance structure without any loss of symmetry of the space,
- mentioned two (related) properties of NC scalar field theories,
- seen some other applications of noncommutative spaces in physics.

Thank you for your attention!

Interesting questions to ask

- Quantitative, analytical analysis of the NC field theory phase diagram. Opposing to qualitative or numerical analysis.
- One can formulate a modified field theory which is free of the UV/IR mixing. What is the phase structure of such theory?

Matrix models of NC scalar field theories

Hermitian matrix model

- As we have seen, the NC scalar field theory is given by

$$\langle F\rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}}\ .$$

- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2} r \mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

- Standard techniques turn these into integrals over eigenvalues

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[\frac{1}{2} r \frac{1}{N}\sum \lambda_i^2 + g\frac{1}{N}\sum \lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$\times \int dU\, e^{-N^2 \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

(minus the red Brezin, Itzykson, Parisi, Zuber '78)

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- Standard techniques turn these into integrals over eigenvalues

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) F(\lambda_i)\, e^{-N^2\left[S_{eff}(\lambda_i) + \frac{1}{2} r \frac{1}{N}\sum \lambda_i^2 + g\frac{1}{N}\sum \lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2 \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Steinacker '05